On $R^2$ Corrections for 5D Black Holes

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Abstract

We study higher order corrections to extremal black holes/black string in five dimensions. These higher order corrections are due to supersymmetric completion of $R^2$ term in five dimensions. By making use of the results we extend the notion of very special geometry when higher derivative terms are also taken into account. This can be used to make a connection between total bundle space of near horizon wrapped M2’s and wrapped M5’s in the presence of higher order corrections. We also show how the corrected geometry removes the singularity of a small black hole.
1 Introduction

In recent years our understanding of corrections to the black hole entropy has increased considerably. In a gravitational theory using the Wald entropy formula [1] one can find the contribution of higher order corrections to the tree level Bekenstein-Hawking area law formula. For extremal black holes in string theory taking into account the higher derivative terms the corrected entropy has been evaluated in several papers including [2–4] where it was shown that the results are in agreement with microscopic description of entropy coming from microstate counting in string theory.

Although at the beginning the corrections have mainly been studied for the supersymmetric black holes where the attractor mechanism [5] was observed, recently it was generalized to non-supersymmetric cases as well (for example see [6–21]).

An extremal black hole in four dimensions has $AdS_2 \times S^2$ near horizon geometry, while in five dimensions the near horizon geometry could be either $AdS_2 \times S^3$ or $AdS_3 \times S^2$. To compute the contributions of higher order corrections to a black hole with near horizon geometry $AdS_2 \times S^3$ one may simplify the Wald formula leading to the entropy function formalism [22]. While for those with $AdS_3 \times S^2$ near horizon geometry it is useful to work within the c-extremization framework [23]. Using these mechanisms one can find corrections to the Bekenstein-Hawking area law formula coming from higher derivative terms. These corrections have to be compare with the microstate counting in string theory/M-theory.

As far as the five dimensional black holes are concerned, We note that although the microscopic origin of $\mathcal{N} = 8, 4$ five dimensional rotating black holes has been understood for a decade [24, 25], the origin of the entropy for five dimensional $\mathcal{N} = 2$ black hole has not been fully understood yet. Recently the microscopic accounting of the five dimensional rotating black hole arising from wrapped M2-branes in Calabi-Yau compactification of M-theory has been studied in [26] where the authors established a connection between this black hole and a well understood one by making use of an embedding of space-time in the total space of the $U(1)$ gauge bundle over near horizon geometry of the black holes. Of course it was done in an specific case, namely near zero-entropy, zero-temperature and maximally rotating limits.

This is the aim of this article to further study higher derivative corrections to $\mathcal{N} = 2$ five dimensional black holes. To do this we will work with the full 5D supersymmetry invariant four-derivative action, corresponding to the supersymmetric completion of the four-derivative Chern-Simons term which has recently been obtained in [27].

The article is organized as follows. In section 2 we shall fix our notation where we present the five dimensional action obtained in [27]. In section 3 we will consider a five dimensional black string at $R^2$ level. In section 4 we will generalize the notion of very special geometry in the presence of higher derivative terms using the results of section 3. This generalization can be used to extend the consideration of [26] to the case where higher order corrections are also taken into account. In section 5 we
shall study higher order corrections to a five dimensional extremal black hole using the fully supersymmetrized higher derivative terms. We then compare the result with the corrections coming from the bosonic Gauss-Bonnet action. We shall also see how the higher derivative terms remove the singularity of a small black hole. The last section is devoted to discussions.

2 Basic setup

In this section we will fix our notation. We would like to study $\mathcal{N} = 2$ supergravity in five dimensions. This model is usually studied in the context of very special geometry. We note, however, that sometime it is useful to work with a more general context, namely the superconformal approach. This approach, in particular, is useful when we want to write the explicit form of the action. In this approach we start with a five dimensional theory which is invariant under a larger group that is superconformal group and therefore we construct a conformal supergravity. Then by imposing a gauge fixing condition, one breaks the conformal supergravity to standard supergravity model.

The representation of superconformal group includes Weyl, vector and hyper multiples. The bosonic part of the Weyl multiplet contains the vielbein $e_{\mu}^a$, two-form auxiliary field $v_{ab}$, and a scalar auxiliary field $D$. The bosonic part of the vector multiplet contains one-form gauge field $A^I$ and scalar fields $X^I$, where $I = 1, \cdots, n_v$ labels generators of a gauge group. The hypermultiplet contains scalar fields $A_i^\alpha$, where $i = 1, 2$ is $SU(2)$ doublet index and $\alpha = 1, \cdots, 2r$ refers to $USp(2r)$ group. Although we won’t couple the theory to matters, we shall consider the hyper multiplet to gauge fix the dilataional symmetry reducing the action to the standard $\mathcal{N} = 2$ supergravity action.

In this notation at leading order the bosonic part of the action is [27]

$$I = \frac{1}{16\pi G_5} \int d^5x L_0,$$

with

$$L_0 = \partial_a A^i_a \partial^a A^a_i + (2\nu + A^2) \frac{D}{4} + (2\nu - 3A^2) \frac{R}{8} + (6\nu - A^2) \frac{v^2}{2} + 2\nu_I F^I_{ab} v^{ab}$$

$$+ \frac{1}{4} \nu_{IJ} (F^I_{ab} F^J_{cd} + 2\partial_a X^I \partial^a X^J) + \frac{g^{-1}}{24} C_{IJK} \epsilon^{abcde} A^I_{a} F^J_{be} F^K_{de},$$

where $A^2 = A^i_{ab} A^a_i$, $v^2 = v_{ab} v^{ab}$ and

$$\nu = \frac{1}{6} C_{IJK} X^I X^J X^K, \quad \nu_I = \frac{1}{2} C_{IJK} X^J X^K, \quad \nu_{IJ} = C_{IJK} X^K.$$

To fix the gauge it is convenient to set $A^2 = -2$. Then integrating out the auxiliary fields by making use of their equations of motion one finds

$$L_0 = R - \frac{1}{2} G_{IJ} F^I_{ab} F^J_{ab} - G_{ij} \partial_a \phi^i \partial^a \phi^j + \frac{g^{-1}}{24} \epsilon^{abcde} C_{IJK} F^I_{a} F^J_{be} F^K_{de}.$$
The parameters in the action (2.4) are defined by

\[
G_{IJ} = -\frac{1}{2} \partial_i \partial_J \log \nu |_{\nu=1}, \quad G_{ij} = G_{IJ} \partial_i X^I \partial_j X^J |_{\nu=1},
\]

where \( \partial_i \) refers to a partial derivative with respect to the scalar fields \( \phi^i \). In fact doing this, we recover the very special geometry underlines the theory in the leading order.

We can also find higher derivative terms in the action using the superconformal language. Actually, the supersymmetrized higher order action with four-derivative has recently been obtained in [27]. The corresponding action is

\[
\mathcal{L}_1 = \frac{c_{2I}}{24} \left( \frac{1}{16} g^{-1} \epsilon_{abcd} A_I^{ab} C^{bcfg} C^{de} f_{g} + \frac{1}{8} X^I C^{abcd} C_{abcd} + \frac{1}{12} X^I D^2 + \frac{1}{6} F_{Iab} v_{ab} D \right.
\]

\[
- \frac{1}{3} X^I C_{abcd} v^{ab} v^{cd} - \frac{1}{2} F_{Iab} C_{abcd} v^{cd} + \frac{8}{3} X^I v_{ab} \hat{D}^b \hat{D}_c v^{ac} \right.
\]

\[
+ \frac{4}{3} X^I \hat{D}_a v^{bc} \hat{D}_a v_{bc} + \frac{4}{3} X^I \hat{D}_a v^{bc} \hat{D}_b v_{ca} - \frac{2}{3} \epsilon^{-1} X^I \epsilon_{abcd} v^{bc} v^{cd} \hat{D}_f \epsilon^f
\]

\[
+ \frac{2}{3} \epsilon^{-1} F_{Iab} \epsilon_{abcd} v^{cd} \hat{D}_f \epsilon^f + \epsilon^{-1} F_{Iab} \epsilon_{abcd} v^c \hat{D}_f v^d
\]

\[
- \frac{4}{3} F_{Iab} v_{ac} v^{cd} v_{db} = \frac{1}{3} F_{Iab} v_{ab} v^2 + 4 X^I v_{ab} v^{bc} v_{cd} v^{da} - X^I (v_{ab} v^{ab})^2 \right)
\]

where \( C_{abcd} \) is the Weyl tensor defined as

\[
C^{ab}_{\, \, cd} = R^{ab}_{\, \, cd} + \frac{1}{6} R \delta^{[a}_{[c} \delta^{b]}_{d]} - \frac{4}{3} \delta^{[a}_{[c} R_{d]}^{b]} .
\]

The double covariant derivative of \( v_{ab} \) has curvature contributions given by

\[
v_{ab} \hat{D}^b \hat{D}_c v^{ac} = v_{ab} \hat{D}^b \hat{D}_c v^{ac} + \frac{2}{3} v_{ac} v_{db} R^b_a + \frac{1}{12} v_{ab} v^{ab} R .
\]

In general it is quite difficult to solve the equations of motion coming from the action contains both \( \mathcal{L}_0 \) and \( \mathcal{L}_1 \). Nevertheless as long as the supersymmetric solutions are concerned, it is useful to look at the supersymmetry transformations which could give a simple way to find explicit solutions. The supersymmetry variations of the fermions in Weyl, vector and hyper multiplets are

\[
\delta \psi^i_\mu = \frac{1}{2} \psi^i_\mu + \frac{1}{2} v^{ab} \gamma_{\mu ab} \hat{v}^i - \gamma_\mu \hat{\eta}^i,
\]

\[
\delta \chi^i = D \hat{v}^i - 2 \gamma^{ab} \hat{v}_{ab} v^i + 2 \gamma^a v^i \epsilon_{abcd} v^{bc} v^{de} + 4 \gamma \cdot \nu \hat{\eta}^i,
\]

\[
\delta \Omega^I = -\frac{1}{4} \gamma \cdot F^{I} \hat{v}^i - \frac{1}{2} \gamma^a \partial_a X^I - X^I \hat{\eta}^i,
\]

\[
\delta \zeta^a = \gamma^a \partial_a \hat{\eta}_j - \gamma \cdot \nu \epsilon \hat{A}_j^a + 3 \hat{A}_j^a \hat{\eta}^a .
\]

where garavitino \( \psi^i_\mu \) and the auxiliary Majorana spinor \( \chi^i \) come from the Weyl multiple, while the gaugino \( \Omega^I \) and \( \zeta^a \) come from vector and hyper multiplets, respectively.
3 Black string and c-extremization

In this section we will consider a five dimensional extremal black string solution whose near horizon geometry is $\text{AdS}_3 \times S^2$. Using the symmetry of the near horizon geometry one may start with the following ansatz for near horizon solution

$$ds = l_A^2 ds_{\text{AdS}_3}^2 + l_S^2 ds_{S^2}^2, \quad X^I = \text{cont.} \quad F_{\theta\phi}^I = \frac{p^I}{2} \sin \theta. \quad (3.1)$$

By making use of the c-extremization [23] we can fix the parameters $l_A, l_S$ and $X_I$ as follows. In this method we will first define c-function whose critical points correspond to the solutions of the equations of motion. Then evaluating the c-function at critical points gives the average of the left and right moving central charges of the associated CFT.

In our model at leading order the c-function is given by

$$c = -6 l_A^3 l_S^2 \mathcal{L}, \quad (3.2)$$

where $\mathcal{L}$ is the Lagrangian evaluated on the above ansatz. The parameters of the ansatz are obtained by extremizing this function with respect to them

$$\frac{\partial c}{\partial l_A} = 0, \quad \frac{\partial c}{\partial l_S} = 0, \quad \frac{\partial c}{\partial X^I} = 0. \quad (3.3)$$

For the five dimensional action (2.4) the above equations can be solved leading to

$$l_A = 2l_S = \left(\frac{1}{6} C_{IJK} p^I p^J p^K \right)^{1/3}, \quad X^I = \frac{p^I}{\left(\frac{1}{6} C_{IJK} p^I p^J p^K \right)^{1/3}}. \quad (3.4)$$

Plugging these into (3.2), one finds the central charge for the black string at two-derivative level as $c = C_{IJK} p^I p^J p^K$.

Let us now redo c-extremization for the black string solution (3.1) in the presence of higher derivative terms. This correction has recently been studied in [28]. In general one should evaluate the Lagrangian (2.2) and (2.6) for our ansatz (3.1) and then extremize it with respect to the parameters. We note, however, that it is in general difficult to solve the equations of motion explicitly. Nevertheless one may use the supersymmetry transformations (2.9) to fix some of the parameters. The remaining parameters can then be found by equation of motion of the auxiliary field $D$ [28]. To be precise, from the supersymmetry transformations (2.9) for our ansatz (3.1) one finds

$$D = \frac{12}{l_S^2}, \quad p^I = -\frac{8}{3} V X^I, \quad V = -\frac{3}{8} l_A, \quad l_A = 2l_S. \quad (3.5)$$

On the other hand the equation of motion of auxiliary field $D$ is

$$\frac{1}{6} C_{IJK} X^I X^J X^K + \frac{c_{2I}}{l_S^2} \left( DX^I + \frac{V p^I}{l_S^2} \right) = 1, \quad (3.6)$$
which, by making use of (3.5), can be recast to the following form

$$\frac{1}{6} C_{IJK}^{} X^I X^J X^K + \frac{1}{12 l_A^2} c_{2l} X^I = 1. \quad (3.7)$$

Setting $c_{2l} = 0$, it reduces to $\nu = 1$ where one can define very special geometry underlines the theory at leading order. Therefore we would like to interpret this expression as a generalization of $\nu = 1$ when the $R^2$ correction is also taken into account. This is analogous to the one loop correction to the prepotential of four dimensional $\mathcal{N} = 2$ supergravity where it is given by $F = \frac{1}{6} C_{IJK}^{} X^I X^J X^K + \Lambda^2 c_{2l} X^I$.

It is then natural to define the dual coordinates $X_I$ as

$$X_I = \frac{1}{6} C_{IJK}^{} X^J X^K + \frac{1}{12 l_A^2} c_{2l}, \quad (3.8)$$

such that $X_I X^I = 1$. From the supersymmetry conditions (3.5) one has $X^I = \frac{p^I}{l_A}$.

Plugging this into the above expressions we get $l_A^3 = \frac{1}{6} C_{IJK}^{} p^I p^J p^K + \frac{1}{12} c_{2l} l_A$ and therefore

$$X^I = \frac{p^I}{(\frac{1}{6} C_{IJK}^{} p^I p^J p^K + \frac{1}{12} c_{2l} l_A)^{1/3}}, \quad X_I = \frac{\frac{1}{6} C_{IJK}^{} p^I p^K + \frac{1}{12} c_{2l}}{(\frac{1}{6} C_{IJK}^{} p^I p^J p^K + \frac{1}{12} c_{2l} l_A)^{2/3}}. \quad (3.9)$$

We will give more comments on this structure in the next section.

4 Very special geometry and black holes at $R^2$ level

In the previous section we have considered the black string solution in the presence of higher derivative terms. We have observed that adding higher derivative terms will change the feature of the very special geometry in such a way that the leading order constraint $\nu = 1$ is not satisfied any more. Therefore the standard method of very special geometry is not applicable beyond tree level and we will have to solve the equations of motion coming from c-extremization directly.

In this section we would like to generalize the notion of very special geometry when higher order corrections are also taken into account. It is possible to do that due to new progresses which have recently been made in computing the higher order correction using the fully supersymmetrized higher derivative terms. We note, however, that the notion of generalized very special geometry depends on the explicit solution we are considering. To be specified we will consider a black string solution in five dimensions in the presence of supersymmetric higher derivative terms and we define the generalized very special geometry for this case. Then we will be able to reproduce all results of the previous section in this framework. We will give a comment on how to generalize it for an arbitrary ansatz later in the discussions section.
Since the leading order constraint $\nu = 1$ defining the very special geometry is, indeed, the equation of motion of the auxiliary field $D$, we will also define the constraint for the generalized very special geometry by making use of the equation of motion of field $D$ in the presence of higher derivative terms. Using the black string ansatz of the previous section and taking into account the supersymmetry constraints (3.5) the corresponding equation can be recast to the following form

$$\frac{1}{6} C_{IJK} X^I X^J X^K + \frac{1}{12 l_A^2} c_{2I} X^I = 1.$$  

Now the main point to define the generalized very special geometry is as follows. When we are considering the supersymmetrized action, in the ansatz we are choosing there is an auxiliary two-form field $v_{\mu \nu}$ which could be treated as an additional gauge field in the theory with charge $p^0$. Accordingly, we could introduce new scalar field $X^0$ such that in the near horizon geometry one may set $X^0 = \frac{p^0}{l_A}$. Using this notation the above constraint may be written as

$$\frac{1}{6} C_{IJK} X^I X^J X^K + \frac{1}{12} c_{2I}(X^0)^2 X^I_{|p^0=1} = 1.$$  

One may also define $F^0_{\theta \phi} = \frac{p^0}{2} \sin \theta$ such that $v_{\theta \phi} = -\frac{3}{l_A} F^0_{\theta \phi}$. Obviously working with this notation all expressions reduce to those in section 3 for $p^0 = 1$, though it is not a solution for $p^0 \neq 1$. Nevertheless we will work with $p^0 \neq 1$ with the understanding that the solution is obtained by setting $p^0 = 1$. Now we shall demonstrate how the solutions of previous section can be reproduced in this framework.

To start, let us first introduce new indices $A, B, \cdots$ such that they take their values over $0$ and $I, J, \cdots$ by which the equation (4.2) can be recast to the following form

$$\frac{1}{6} C_{ABC} X^A X^B X^C = 1.$$  

where $C_{ABC} = C_{IJK}$ for $A, B, C = I, J, K$ and $C_{00I} = \frac{c_{2I}}{6}$ and the other components are zero. Thus, it is natural to define $X_A$ and the metric $C_{AB}$ as follows

$$X_A = \frac{1}{6} C_{ABC} X^B X^C, \quad C_{AB} = \frac{1}{6} C_{ABC} X^C.$$  

More explicitly one has

$$X_I = \frac{1}{6} C_{IJK} X^I X^J X^K + \frac{c_{2I}}{36} (X^0)^2, \quad X_0 = \frac{c_{2I}}{18} X^I,$$

and

$$C_{I0} = \frac{c_{2I}}{36} X^0, \quad C_{00} = \frac{c_{2I}}{36}.$$  

It is easy to verify that

$$X_A X^A = 1, \quad X_A = C_{AB} X^B, \quad C_{AB} X^A X^B = 1.$$  

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Following the standard notion of very special geometry, the magnetic central charge may be defined as $Z_m = X_A p^A$ and the near horizon parameters are fixed by extremizing it, i.e. $\partial_i Z_m = \partial_i X_A p^A = 0$. Using (4.7) a solution would be $X^A = p^A / Z_m$. Plugging this into (4.3) we find

$$Z_m = \left( \frac{1}{6} C_{ABC} p^A p^B p^C \right)^{1/3} = \left( \frac{1}{6} C_{IJK} p^I p^J p^K + \frac{(p^0)^2}{12} c_{2I} p^I \right)^{1/3}. \quad (4.8)$$

To get our results in the previous section one needs to set $p^0 = 1$ at the end of the computations.

It is almost obvious, but worth to mention, that having had the analogous to very special geometry does not mean that the action can also be given just by (2.2) with the generalized constraint (4.3). Of course ultimately it might probably be possible to extend the action in the context of the generalized very special geometry, though we are not going to do this in the present article.

In the rest of this section we would like to use the generalized very special geometry to extend the consideration of [26] while higher derivative terms are also taken into account. To do this, consider the black string solution obtained in section 3. At $R^2$ level the solution can be written as follows

$$\begin{align*}
 ds^2 &= \frac{l_A^2}{4} (dx^2 - 2rdxdr + \frac{dr^2}{r^2} + d\theta^2 + \sin^2 \theta \, d\phi^2), \quad X^I = \frac{p^I}{l_A}, \\
 A^I_\theta &= -\frac{p^I}{2} \sin \theta \, d\phi, \quad A^0_\theta = -\frac{1}{2} \sin \theta \, d\phi, \quad \text{with} \quad v = -\frac{3}{4} l_A F^0_{\theta \phi}, \quad (4.9)
\end{align*}$$

where $l_A = \left( \frac{1}{6} C_{IJK} p^I p^J p^K + \frac{1}{12} c_{2I} p^I \right)^{1/3}$. In writing the metric we have used the fact that $AdS_3$ can be written as $S^1$ fibered over $AdS_2$. Now we would like to embed this solution into the framework of the generalized very special geometry. In order to do that, we start with the following solution with the understanding of $p^0 = 1$,

$$\begin{align*}
 ds^2 &= \frac{Z_m^2}{4} (dx^2 - 2rdxdr + \frac{dr^2}{r^2} + d\theta^2 + \sin^2 \theta \, d\phi^2), \quad X^A = \frac{p^A}{Z_m}, \\
 A^A_\theta &= -\frac{p^A}{2} \sin \theta \, d\phi, \quad Z^3_m = \frac{1}{6} C_{ABC} p^A p^B p^C. \quad (4.10)
\end{align*}$$

Following [26] we will consider the total space of $U(1)$ bundle over (4.10) to define a six dimensional manifold with the metric

$$ds_6^2 = \frac{Z_m^2}{4} (dx^2 - 2rdxdt + \frac{dr^2}{r^2} + \sigma_1^2 + \sigma_2^2) + (2X_A X_B - C_{AB})(dy^A + A^A)(dy^B + A^B). \quad (4.11)$$

Here $\sigma_i$ are right invariant one-forms such that $\sigma_1^2 + \sigma_2^2 = d\theta + \sin^2 \theta \, d\phi^2$. Let us define new coordinates $z, \psi$ through the following expressions

$$\begin{align*}
 y^A &= z^A + (\sin B - 1)X_A X_B z^B - \frac{1}{2} p^A \psi, \quad x = \frac{2 \cos B}{Z_m} X_A z^A \quad (4.12)
\end{align*}$$
where \( \sin B \) is a constant which its physical meaning will become clear later. We define the coordinates such that the new coordinates have the following identification

\[
\psi \sim \psi + 4\pi m, \quad z^A \sim z^A + 2\pi n^A,
\] (4.13)

where \( m \) and \( n^A \) are integers. Accordingly one can read the identifications of \( y \) and \( x \). Using the new coordinate \( \psi \) one can define the right invariant one-forms by

\[
\sigma_1 = -\sin \psi d\theta + \cos \psi \sin \theta d\phi,
\sigma_2 = \cos \psi d\theta - \sin \psi \sin \theta d\phi,
\sigma_3 = d\psi + \cos \theta d\phi.
\] (4.14)

In terms of the new coordinates the six dimensional metric (4.11) reads

\[
ds^2 = \frac{Z^2}{4} \left[ -(\cos B r dt + \sin B \sigma_3)^2 + \frac{dr^2}{r^2} + \sigma_1^2 + \sigma_2^2 + \sigma_3^2 \right] + (2X_A X_B - C_{AB})(dz^A + \tilde{A}^A)(dz^B + \tilde{A}^B),
\] (4.15)

where \( \tilde{A}^A = -\frac{n^A}{l_A} (\cos B r dt + \sin B \sigma_3) \). The obtained six dimensional manifold can be treated as the total space of a \( U(1) \) bundle over BMPV black hole at \( R^2 \) level. Therefore we can reduce to five dimensions to get BMPV black hole where higher derivative corrections are also taken into account. The resulting five dimensional black hole solution is

\[
ds^2 = \frac{l_A^2}{4} \left[ -(\cos B r dt + \sin B \sigma_3)^2 + \frac{dr^2}{r^2} + \sigma_1^2 + \sigma_2^2 + \sigma_3^2 \right], \quad X^I = \frac{p^I}{l_A},
\]

\[
\tilde{A}^I = -\frac{p^I}{2} (\cos B r dt + \sin B \sigma_3), \quad \tilde{A}^0 = -\frac{1}{2} (\cos B r dt + \sin B \sigma_3),
\] (4.16)

and the auxiliary field is given by \( v = -\frac{3}{4} (A \tilde{F}^0) \). From this solution we can identify \( \sin B \) as the angular momentum, \( J \), of the BMPV solution though the relation \( \sin B = \frac{J}{l_A} \). As a result we have demonstrated that the total bundle space of near horizon wrapped M2’s and wrapped M5’s are equivalent up to \( R^2 \) level, generalizing the tree level results of [26]. In fact one may go further to show that both of them can be obtained from quotients of \( AdS_3 \times S^3 \) with a flat \( U(1)^{N-1} \) bundle, much similar to tree level considered in [26]. It is then possible to use this connection to increase our understanding of microstate counting of 5D supersymmetric rotating black hole arising from wrapped M2-branes in Calabi-Yau compactification of M-theory. We hope to come back to this point in our future publication.

5 Black hole solution

In this section we shall consider a five dimensional extremal black hole whose near geometry is \( AdS_2 \times S^3 \). When we are dealing with an extremal black hole with \( AdS_2 \)
near horizon geometry, it is more appropriate to work with entropy function formalism [22]. In fact this approach has been used to study five dimensional extremal black hole in Heterotic string theory in presence of higher derivative terms given by Gauss-Bonnet action [6]. It is shown that this bosonic term was enough to correctly reproduce the microscopic entropy coming from microstate counting in string theory. It is the aim of this section to study the five dimensional extremal black hole in the presence of higher derivative terms which come from supersymmetrized action. We will also compare the result with the case where only bosonic Gauss-Bonnet term is present. In fact it is analogous to the consideration of [29, 30] where four dimensional extremal black hole has been studied in the presence of supersymmetrized action using entropy function formalism.

Let us start with the following ansatz for near horizon geometry

$$ds = l_A^2 ds_{\text{AD}S^2} + l_S^2 ds_{S^3}, \quad X^I = \text{cont.} \quad F_{\mu\nu}^I = e^I, \quad \nu_{rt} = V. \quad (5.1)$$

Then the entropy function is given by $E = 2\pi (e^I q_I - f_0 + f_1)$ where $f_0$ is the leading order contribution coming from quadratic part of the action which is

$$f_0 = \frac{1}{2} l_A^2 l_S^3 \left[ \frac{\nu - 1}{2} D + \frac{\nu + 3}{2} \left( \frac{3}{l_S^2} - \frac{1}{l_A^2} \right) - \frac{2(3\nu + 1)}{l_A^4} V^2 - \frac{4\nu e^I}{l_A^4} V - \frac{\nu_{IJ} e^I e^J}{2l_A^4} \right], \quad (5.2)$$

and the higher order contribution, $f_1$, comes from the four-derivative terms that for our ansatz, is

$$f_1 = \frac{c_1 l_A^2 l_S^3}{48 l_A^3} \left[ \frac{X^I}{4} \left( \frac{1}{l_S^2} - \frac{1}{l_A^2} \right)^2 + \frac{4V^4}{l_A^8} X^I + \frac{4V^3}{3l_A^8} e^I - \frac{DV}{3l_A^4} e^I + \frac{D^2}{12} X^I \right.$$

$$- \frac{2V^2 X^I}{3l_A^5} \left( 3 \frac{l_S^2}{l_A^2} + 5 \right) - \frac{V e^I}{l_A^4} \left( \frac{1}{l_A^2} - \frac{1}{l_S^2} \right) \left]\right. . \quad (5.3)$$

At leading order where only $f_0$ contributes, one may integrate out the the auxiliary fields by extremizing the entropy function with respect to them, arriving at

$$f_0 = \frac{1}{2} l_A^2 l_S^3 \left[ \frac{6}{l_S^2} - \frac{2}{l_A^2} + \frac{1}{2l_A^4} (\nu_I \nu_J - \nu_{IJ}) e^I e^J \right] = \frac{1}{2} l_A^2 l_S^3 \left( \frac{6}{l_S^2} - \frac{2}{l_A^2} + \frac{G_{IJ} e^I e^J}{l_A^4} \right). \quad (5.4)$$

It is easy to extremize the entropy function with respect to the parameters to find $l_A, l_S, X^I$ and $e^I$ as

$$l_S = 2l_A = \left( \frac{1}{6} C^{IJK} q_I q_J q_K \right)^{1/6}, \quad X^I = \frac{1}{2} \frac{C^{IJK} q_I q_J q_K}{\left( \frac{1}{6} C^{IJK} q_I q_J q_K \right)^{2/3}}, \quad e^I = \frac{1}{2} \frac{C^{IJK} q_I q_J q_K}{\left( \frac{1}{6} C^{IJK} q_I q_J q_K \right)^{1/2}}. \quad (5.5)$$

The entropy of the corresponding black hole is $S = 2\pi \sqrt{\frac{1}{6} C^{IJK} q_I q_J q_K}$. 


In general it is difficult to do the same while the higher order corrections are also taking into account. Therefore to proceed, the same as in the previous, we will use the supersymmetry transformations to simplify as much as we can and then using the equation of motion of field \( D \), we find a supersymmetric solution. For our ansatz \( 5.1 \), the supersymmetry conditions \( 2.9 \) lead to

\[
D = -\frac{3}{l_A^2}, \quad e^I = -\frac{4}{3} VX^I, \quad V = -\frac{3}{4} l_A, \quad l_S = 2l_A. \tag{5.6}
\]

Using these relations we may set \( X^I = \frac{e^I}{l_A} \) and defining \( E = \frac{1}{6} C_{1JK} e^I e^J e^K \) one has

\[
\nu = \frac{1}{l_A^3} E, \quad \nu e^I = \frac{3}{l_A^2} E, \quad \nu e^I e^J = \frac{6}{l_A} E. \tag{5.7}
\]

In this notation the equation of motion for auxiliary field \( D \) reads

\[
E - l_A^3 + \frac{l_A^3}{12} c_{2I} \left( \frac{DX^I}{6} - \frac{Ve^I}{3l_A^4} \right) = 0 \tag{5.8}
\]

so that \( l_A = \frac{1}{2} \left( 8E - \frac{2c_{2I}}{6} \right)^{1/3} \). By making use of the expressions for the parameters \( D, V, X^I \) and \( l_S \) given above, the entropy function gets the following simple form

\[
E = 2\pi(q_e^I - 4E + \frac{1}{8} c_{2I} e^I) = 2\pi \left[ (q_I + \frac{1}{8} c_{2I}) e^I - \frac{2}{3} C_{1JK} e^I e^J e^K \right] \tag{5.9}
\]

Extremizing the entropy function with respect to \( e^I \) we get\(^3\)

\[
2C_{1JK} e^J e^K = q_I + \frac{1}{8} c_{2I} \quad \Rightarrow \quad e^I = \frac{1}{2} \left( \frac{1}{6} C_{1JK} q_I^+ q_J^+ q_K^+ \right)^{1/3}. \tag{5.10}
\]

Plugging this into the expressions of \( l_A \) and \( X^I \), we obtain

\[
l_A = \frac{1}{2} \left(\frac{1}{6} C_{1JK} q_I^+ q_J^+ q_K^+ \right)^{1/3}, \quad X^I = \frac{1}{2} \left(\frac{1}{6} C_{1JK} q_I^+ q_J^+ q_K^+ \right)^{-1/6}/(\frac{1}{6} C_{1JK} q_I^+ q_J^+ q_K^+)^{1/3}. \tag{5.11}
\]

Finally the entropy is found to be

\[
S = 2\pi \sqrt{\frac{1}{6} C_{1JK} q_I^+ q_J^+ q_K^+}. \tag{5.12}
\]

It is very interesting in the sense that the entropy of the black hole at \( R^2 \) level has the same form as the tree level except that the charges \( q_I \)'s are replaced by shifted charges \( q_I^+ \)'s.

\(^3\)We use a notion in which \( q_I^+ = q_I + \frac{1}{8} c_{2I} \) and \( \hat{q} = q_I + \frac{1}{24} c_{2I} \).
An interesting observation is that in the case of $c_{2I} \neq 0$ for all $I$, we can still have a non-singular solution even for $q_I = 0$ which is given by

$$l_s = 2l_A = \left( \frac{1}{6} C^{IJK} c_I c_J c_K \right)^{1/6}, \quad X^I = \left( \frac{1}{6} C^{IJK} c_I c_J c_K \right)^{2/3}, \quad e^I = \left( \frac{1}{6} C^{IJK} c_I c_J c_K \right)^{1/2},$$

where $c_I = c_{2I}/8$. The corresponding entropy is also non-zero

$$S = 2\pi \sqrt{\frac{1}{6} C^{IJK} c_I c_J c_K}.$$

The physical significant of this solution is not clear to us. It might be related, upon dimensional reduction to four dimensions, to the single-charge small black hole studied in [31].

We note that in the above consideration we have used the supersymmetry transformations to simplify the computations and therefore the solution would be supersymmetric. Thus it does not exclude other solutions. In fact we would expect to have another solution corresponding to non-BPS solution. Actually one could start from a more general constraint than (5.6) as follows

$$D = \frac{\alpha}{l_A^2}, \quad X^I = \beta e^I/l_A, \quad V = \gamma l_A, \quad l_s = 2l_A$$

and solve the equations for parameters $(\alpha, \beta, \gamma)$. Doing so, for fixed $l_A$ given above, we find two solutions: $(-3, 1, -3/4)$ and $(3, -1, -3/4)$. The first one is the solution we have studied, but the second one which is not supersymmetric leads to the following entropy function

$$\mathcal{E} = 2\pi \left[ (q_I - \frac{3}{8} c_{2I}) e^I - \frac{2}{3} C^{IJK} e^J e^K \right].$$

It is straightforward to extremize the entropy function with respect to $e^I$’s to get the parameters in terms of $q_I$’s, though we won’t do that here.

It is also instructive to compare the results with the case where the higher order corrections are given in terms of the Gauss-Bonnet action. It is known that this term cannot be supersymmetrized, but it is still interesting to see what would be the corresponding corrections. In our notation the Gauss-Bonnet term is given by

$$\mathcal{L}_{GB} = \frac{c_{2I} X^I}{28 \cdot 3 \pi^2} \left( R^{abcd} R_{abcd} - 4 R^{ab} R_{ab} + R^2 \right).$$

It can be shown that with the specific coefficient we have chosen for the Gauss-Bonnet action, the corresponding entropy function, fixing the auxiliary fields as in (5.6) for tree level action, is the same as (5.19) and therefore we get the same results as those in supersymmetrized action.


6 Discussions

In this paper we have studied black string and black hole solutions in the presence of higher derivative terms. In the black string case, where the near horizon geometry is $\text{AdS}_3 \times S^2$, a proper method is c-extremization, whereas in the black hole solution with $\text{AdS}_2 \times S^3$ near horizon geometry the better method is given by the entropy function formalism. In both cases we have seen that adding higher derivative terms will change the feature of the very special geometry in such a way that the leading order constraint $\nu = 1$ is not satisfied any more. Therefore the standard method of very special geometry is not applicable at this level.

We have shown how to generalize the notion of very special geometry in the presence of higher derivative terms. We have observed that the generalized very special geometry depends on the solution we are considering. In particular we have explicitly studied the generalization for the magnetic black string solution where we have reproduced all results obtained from another method, e.g. c-extremization.

For an arbitrary solution, the generalized constraint which defines the very special geometry can be obtained from equation of motion of the auxiliary field $D$. In general we have

$$\frac{1}{6} C_{IJK} X^I X^J X^K + \frac{c_2}{72} (X^I D + F^{I\mu\nu} v_{\mu\nu}) = 1. \quad (6.1)$$

If we are interested in a supersymmetric solution, we can also use the supersymmetry transformations to further simplify the constraint

$$\frac{1}{6} C_{IJK} X^I X^J X^K + \frac{c_2}{54} \left( \frac{3}{2} \epsilon_{abcde} \gamma^a v^{bc} v^{de} - (\gamma \cdot v)^2 - v^2 \right) = 1. \quad (6.2)$$

An interesting application of this construction is to generalize the consideration of [26]. In particular we have shown that the total bundle space of near horizon wrapped M2’s and wrapped M5’s are equivalent up to $R^2$ level, generalizing the tree level results of [26]. Actually we could also show that both of them can be obtained from quotients of $AdS_3 \times S^3$ with a flat $U(1)^{N-1}$ bundle, much similar to tree level considered in [26]. Although we have not pushed this observation any further, one might suspect that using this connection could increase our knowledge of microstate counting of 5D supersymmetric rotating black hole arising from wrapped M2-branes in Calabi-Yau compactification of M-theory.

We have also studied higher order corrections to extremal black holes in five dimensions in which the higher order corrections come from supersymmetric completion of $R^2$ term. We have seen that the corrected entropy can simply be obtained by replacing the electric charges $q_I$’s with the shifted charges $q_I^+$’s defined in footnote 2. In particular this result shows how one can get a smooth solution with non-zero entropy out of a small black hole which in leading order supergravity is singular with vanishing entropy. This might also give an insight about the single-charge small black hole in four dimensions upon dimensional reduction. The reduction can
be done by making use of the fact that the $S^3$ part of the metric can be viewed of $S^1$ fibration over $S^2$. Then reducing along $S^1$ we get a black hole with near horizon geometry of $AdS_2 \times S^2$.

**Note added:** While we were in the final stage of the project we have received the paper [32] where higher order corrections to five dimensional black holes have also been studied in the presence of higher derivative terms given by (2.6).

**Acknowledgments**
I would like to thank Hajar Ebrahim for discussions on the related topic and also comments on the draft of the article.

**References**


