Exact Master Equation and Quantum Decoherence of Two Coupled Harmonic Oscillators in a General Environment

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In this paper we derive an exact master equation for two coupled quantum harmonic oscillators interacting via bilinear coupling with a common environment made up of many harmonic oscillators at arbitrary temperature for a general spectral density function. We first show a simple derivation based on the observation that the two harmonic oscillator model can be effectively mapped into that of a single harmonic oscillator in a general environment plus a free harmonic oscillator. Since the exact one harmonic oscillator master equation is known \cite{Hu, Paz and Zhang, Phys. Rev. D 45, 2843 (1992)}, the exact master equation with all its coefficients for this two harmonic oscillator model can be easily deduced from the known results of the single harmonic oscillator case. In the second part we give an influence functional treatment of this model and provide explicit expressions for the evolution operator of the reduced density matrix which are useful for the study of decoherence and disentanglement issues. We show a simple application of this master equation by examining the decoherence and disentanglement of two harmonic oscillators due to their interaction with a common environment under Markovian approximation. This model and its generalization to $N$ harmonic oscillators are expected to be useful for the analysis of quantum coherence, entanglement, fluctuations and dissipation of mesoscopic and macroscopic objects.

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1. INTRODUCTION

Macroscopic quantum coherence phenomena (MQP) manifested in double slit experiments, micromechanical resonators, Bose-Einstein condensates, Josephson junction circuits, mesoscopic systems, or even mirrors (see, e.g., [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15]) is a subject of both basic theoretical and practical application interest. Theoretically it focuses on issues at the intersection of two trunk lines of important inquires in physics: the relation between the microscopic and the macroscopic world on the one hand, and the relation between the quantum and the classical on the other. Rapid recent advances in precision measurements with high degree of control and adaptability in atomic-optical, electro-mechanical, opto-mechanical, nano-material, magnetic-spin and low temperature systems have provided the rationale and substance for such theoretical investigations, and in some emergent areas where high goals are set, such as the quest for quantum information processing, even with some sense of urgency.

The issues of interest in MQP include quantum dissipation, entanglement, teleportation, decoherence, noise, correlation and fluctuations. A familiar model which one could use to address many of these issues is the quantum Brownian motion (QBM) [16, 17, 18, 19, 20] and its dynamics described by the master equation or the associated Langevin or Fokker-Planck equations. But since the systems of interest to MQP necessarily involve many microscopic or mesoscopic constituents, a many-body generalization of QBM is needed. In addition, since most of these systems involve non-negligible correlations amongst their components, quantum memory (non-Markovian) effects cannot be ignored. Even for the well-studied single harmonic oscillator (1HO) QBM, Markovian approximation is valid only for a high temperature Ohmic bath [17]. Fortunately an exact master (HPZ) equation [19] for the 1HO with bilinear coupling to a general environment has been found via several techniques ranging from the influence functional [19] and Wigner function [20] to quantum trajectories [21]. The 1HO master equation for the QBM is complex enough to encompass non-Markovian dynamics yet simple enough to yield exact solutions. The new challenge is to find the master equation for $N$ oscillators in a general environment good for the analysis of these issues in mesoscopic physics.

In this paper we show the derivation of such an equation for two coupled harmonic oscillators (2HO). A key observation is that this problem can be mapped into that of a single harmonic oscillator in a general environment plus a free harmonic oscillator. Since the master equation with all its coefficients for the 1HO QBM is known [19, 20] one can derive the master equation for the 2HO QBM easily from them. As an application of this model, we can deduce the decoherence
properties of the 2HO system following the similar pattern of the 1HO. As another example, we show explicitly how, in some parameter choice, under the Markovian limit, an entangled state evolves into a separable state in a finite time.

The results derived in this paper may be deduced by intuitive reasoning, but we are not aware of any theoretical study which yields our results. Our aim here is to provide a proof, or at least a plausibility argument, to the effect that the center of mass coordinate is the one most sensitive to the environmental influence. This model and its generalization to $N$ harmonic oscillators are expected to be useful for the analysis of quantum coherence, entanglement, fluctuations and dissipation of mesoscopic and macroscopic objects.

The paper is organized as follows: in Section II we consider the dynamics of two harmonic oscillators coupled to a common heat bath. By employing the center of mass and relative coordinates we show how to derive the master equations of the two coupled Brownian particles. In Section III the coefficients of the master equation are given explicitly, so are the exact form of the propagators for the reduced density matrices. The details are expected to be useful for studying quantum decoherence and disentanglement issues. In Section IV as applications of this master equation we discuss both quantum decoherence and disentanglement in the Markovian limit. We summarize and conclude in Section V. Technical details are relegated to the two appendices.

II. THE MODEL AND THE EXACT MASTER EQUATION

Quantum Brownian motion (QBM) of a damped harmonic oscillator bilinearly coupled to a bath of harmonic oscillators has been studied for decades, notably by Feynman-Vernon and Caldeira-Leggett using path integral techniques [16, 17]. For such a model an exact master equation can be deduced without making the Markovian approximation [19]. The purpose of this section is to extend the well-known Brownian motion model into the case where the system of interest contains two coupled harmonic oscillators.

A. The Model

The Hamiltonian of the total system consisting of a system (sys) of two mutually coupled harmonic oscillators of equal mass $M$ and frequency $\Omega$ interacting with a bath (bath) of $N_B$ harmonic oscillators of masses $m_n$ and frequencies $\omega_n$ in an equilibrium state at a finite temperature
$T$ can be formally written as,

$$H_{\text{tot}} = H_{\text{sys}} + H_{\text{bath}} + H_{\text{int}},$$  \hspace{1cm} (1)$$

where

$$H_{\text{sys}} = \frac{P_1^2}{2M} + \frac{1}{2}M\Omega^2x_1^2 + \frac{P_2^2}{2M} + \frac{1}{2}M\Omega^2x_2^2 + \kappa(x_1 - x_2)^k$$  \hspace{1cm} (2)$$
is the system Hamiltonian for the two system oscillators of interest, with $(x_1, x_2)$ displacements, conjugate momenta $(P_1, P_2)$ and coupling constant $\kappa$,

$$H_{\text{bath}} = \sum_{n=1}^{N_B} \left( \frac{P_n^2}{2m_n} + \frac{1}{2}m_n\omega_n^2q_n^2 \right)$$  \hspace{1cm} (3)$$
is the bath Hamiltonian with displacement $q_n$ for the $n^{th}$ oscillator and conjugate momentum $p_n$ and

$$H_{\text{int}} = (x_1 + x_2) \sum_{n=1}^{N_B} C_nq_n$$  \hspace{1cm} (4)$$
is the interaction Hamiltonian between the system and the bath. Here for simplicity, we have assumed that the two harmonic oscillators are coupled with the same coupling constants $C_n$ to the bath oscillators.

Our primary focus in this paper is to derive an exact master equation for the two coupled harmonic oscillators. Since the two harmonic oscillators interact with a common thermal bath, there will be induced coupling between the two harmonic oscillators even when initially they are uncoupled. Thus, the master equation for 2HO QBM is not simply the addition of the two master equations for 1HO QBM. It must account for the mutual interactions between the two Brownian particles introduced by their coupling to the common heat bath. Of interest is a comparison with the model that consists of 2HO each in its own heat bath. In our model, the coupling to a common heat bath can give rise to several new features, of particular interest here is the generation of entanglement between the two Brownian particles due to the back-action of the heat bath on the system [23, 24, 25, 26].

However, as is well-known for classical mechanics, the dynamics of an N body quantum open system can be made simpler by changing the N body coordinates to that of their center of mass (cm) and relative (rel) coordinates. Here, the difference is that the N harmonic oscillators (NHO) are coupled with an environment and we seek a quantum mechanical treatment. A quantum mechanical theory of N body dynamics forms the theoretical basis for treating MQP. In this paper
we treat the 2HO case. We will show in what follows that the exact master equation for the two coupled harmonic oscillators can be obtained directly from the master equation for the single harmonic oscillator, known as the Hu-Paz-Zhang (HPZ) master equation.

Let us first rewrite the total Hamiltonian in terms of a set of new variables $X, x, P, p$ defined as

\[ X = \frac{1}{2}(x_1 + x_2), \quad x = x_1 - x_2, \]

\[ P = P_1 + P_2, \quad p = \frac{1}{2}(P_1 - P_2), \]

and the new masses $M_1 = 2M, M_2 = M/2$. In terms of these new variables the Hamiltonian takes the following form:

\[ H_{\text{sys}} = H_{\text{cm}} + H_{\text{rel}} \]

where

\[ H_{\text{cm}} = \frac{P^2}{2M_1} + \frac{1}{2}M_1\Omega^2X^2, \]

\[ H_{\text{rel}} = \frac{P^2}{2M_2} + \frac{1}{2}M_2\Omega^2x^2 + \kappa x^k, \]

and

\[ H_{\text{int}} = (x_1 + x_2)\sum_{n=1}^{N_B}C_nq_n = 2X\sum_{n=1}^{N_B}C_nq_n = X\sum_{n=1}^{N_B}\tilde{C}_nq_n \]

where $\tilde{C}_n = 2C_n$ are modified coupling constants. Since (5) and (6) are canonical transformations, all the commutators are preserved, and it is easy to check that

\[ [X, P] = [x, p] = i\hbar, \quad [P, x] = [p, X] = [X, x] = [P, p] = 0. \]

We see that the fictitious particle with mass $M_2$ and dynamical variables $x, p$ has no interaction with either the cm particle with mass $M_1$ with canonical variables $X, P$ or the oscillators of the heat bath with canonical variables $q_n$.

The total Hamiltonian $H_{\text{tot}}$ in (1) can now be written as $H_{\text{tot}} = H'_{\text{tot}} + H_{\text{rel}}$ with a new effective total Hamiltonian

\[ H'_{\text{tot}} = H_{\text{cm}} + H_{\text{int}} + H_{\text{bath}} \]

\[ = \frac{P^2}{2M_1} + \frac{1}{2}M_1\Omega^2X^2 + X\sum_{n=1}^{N_B}\tilde{C}_nq_n + \sum_{n=1}^{N_B}\left(\frac{p^2}{2m_n} + \frac{1}{2}m_n\Omega_n^2q_n^2\right). \]
This Hamiltonian is formally the same as the Hamiltonian for the single harmonic oscillator in cm variables \((X, P)\) coupled to the heat bath with coupling constants \(\tilde{C}_n\). Note that for this case the spectral density \(\tilde{I}(\omega)\) is given by:

\[
\tilde{I}(\omega) = \pi \sum_{n=1}^{N_B} \frac{\tilde{C}_n^2}{2m_n\omega_n} \delta(\omega - \omega_n),
\]

which differs from the original spectral density \(I(\omega)\) by a numerical factor 4.

### B. Density Matrix

We now consider the dynamics of two coupled harmonic oscillators interacting with a common heat bath. The density matrix \(\rho\) evolves in time under the unitary operator:

\[
\rho(t) = \exp \left[ -i \frac{H_{\text{tot}} t}{\hbar} \right] \rho(0) \exp \left[ i \frac{H_{\text{tot}} t}{\hbar} \right].
\]

From (12), it is easy to see that this evolution can be decomposed into two parts, a dissipative evolution of the center of mass system,

\[
\tilde{\rho}(t) = \exp \left[ -i \frac{H_{\text{tot}} t}{\hbar} \right] \rho(0) \exp \left[ i \frac{H_{\text{tot}} t}{\hbar} \right],
\]

and the unitary evolution of the free harmonic oscillator with mass \(M_1\),

\[
\rho(t) = \exp \left[ -i \frac{H_{\text{rel}} t}{\hbar} \right] \tilde{\rho}(t) \exp \left[ i \frac{H_{\text{rel}} t}{\hbar} \right],
\]

where \(H_{\text{rel}}\) is the Hamiltonian for the 1HO system with reduced mass \(M_2 = M/2\) and \(x, p\) variables:

\[
H_{\text{rel}} = \frac{p^2}{2M_2} + \frac{1}{2} M_2 \omega^2 x^2 + \kappa x^k.
\]

For technical simplicity we make the usual assumption that the initial state of the total system is uncorrelated,

\[
\rho(0) = \rho_{\text{sys}}(0) \times \rho_{\text{bath}}(0),
\]

and that the heat bath is in a thermal equilibrium state at temperature \(T\).

### C. Exact Master Equation

If we are interested in the detailed dynamics of the system but only the coarse-grained effect of the bath we can work with the reduced density matrix obtained by tracing \(\rho\), the density matrix of the total system described by (1), over the bath variables [27, 28]:

\[
\rho_r = \text{Tr}_{\text{bath}} \rho(t).
\]
The reduced density operator for the center of mass system is obtained in a similar way,

\[ \tilde{\rho}_r = \text{Tr}_{\text{bath}} \tilde{\rho}(t). \]  

(20)

where \( \tilde{\rho} \) defined in (15) is the density operator for the effective total system (12). The relationship between \( \tilde{\rho}_r \) and \( \rho_r \) is given by

\[ \rho_r(t) = \exp \left[ -i \frac{H_{\text{rel}}}{\hbar} t \right] \tilde{\rho}_r(t) \exp \left[ i \frac{H_{\text{rel}}}{\hbar} t \right]. \]  

(21)

Tracing over the heat bath variables in (15) leads us to a HPZ type master equation for the center of mass variables \( X, P \):

\[ \dot{\tilde{\rho}}_r = \frac{1}{i \hbar} [H_{\text{cm}}, \tilde{\rho}_r] + \frac{a(t)}{2i \hbar} [X^2, \tilde{\rho}_r] + \frac{b(t)}{2i \hbar} [X, \{P, \tilde{\rho}_r\}] + \frac{c(t)}{\hbar^2} [X, [P, \tilde{\rho}_r]] - \frac{d(t)}{\hbar^2} [X, [X, \tilde{\rho}_r]]. \]  

(22)

Note here that \( H_{\text{cm}} \) defined in (9) is the Hamiltonian for the center of mass variables \( X, P \) only. This is the exact master equation for \( X, P \) interacting with a thermal heat bath with the spectral density \( \tilde{I}(\omega) \) rather than \( I(\omega) \). As a consequence, the coefficients \( a, b, c, d \) in the above master equation satisfy the same types of equations given by [19] (or [20]), only the coupling constants and mass are different here.

From the evolution equation (16), the required master equation for the reduced density matrix \( \rho_r(t) \) is thus obtained,

\[ \dot{\rho}_r = \frac{1}{i \hbar} [H_{\text{sys}}, \rho_r] + \frac{a(t)}{2i \hbar} [X^2, \rho_r] + \frac{b(t)}{2i \hbar} [X, \{P, \rho_r\}] + \frac{c(t)}{\hbar^2} [X, [P, \rho_r]] - \frac{d(t)}{\hbar^2} [X, [X, \rho_r]]. \]  

(23)

The only difference between Eq. (23) and Eq. (22) is that the unitary evolution is modified by the fictitious harmonic oscillator \( x, p \).

In terms of the original variables \( x_1, x_2, P_1, P_2 \), we get

\[ \dot{\rho}_r = \frac{1}{i \hbar} [H_{\text{sys}}, \rho_r] + \frac{a(t)}{8i \hbar} [(x_1 + x_2)^2, \rho_r] + \frac{b(t)}{4i \hbar} [x_1 + x_2, \{P_1 + P_2, \rho_r\}] \]

\[ + \frac{c(t)}{2i \hbar^2} [x_1 + x_2, [P_1 + P_2, \rho_r]] - \frac{d(t)}{4i \hbar^2} [x_1 + x_2, [x_1 + x_2, \rho_r]]. \]  

(24)

This is the exact master equation for the two coupled harmonic oscillators. In the coordinate representation,

\[ \rho_r(x_1, x_2, y_1, y_2) \equiv \langle x_1, x_2 | \rho_r | y_1, y_2 \rangle, \]  

(25)
the master equation can be easily written as:

\[
\frac{i\hbar}{\partial t} \frac{\partial \rho_r}{\partial t} = -\frac{\hbar^2}{2M} \left( \frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial y_1^2} + \frac{\partial^2}{\partial x_2^2} - \frac{\partial^2}{\partial y_2^2} \right) \rho_r + \frac{1}{2} M \Omega^2 (x_1^2 - y_1^2 + x_2^2 - y_2^2) \rho_r \\
+ \frac{1}{2} M \delta \Omega^2(t) (x_1 - y_1 + x_2 - y_2) \frac{1}{2} (x_1 + y_1 + x_2 + y_2) \rho_r \\
- i \hbar \Gamma(t) (x_1 - y_1 + x_2 - y_2) \frac{1}{2} \left( \frac{\partial}{\partial x_1} - \frac{\partial}{\partial y_1} + \frac{\partial}{\partial x_2} - \frac{\partial}{\partial y_2} \right) \rho_r \\
- i M \Sigma(t) (x_1 - y_1 + x_2 - y_2)^2 \rho_r \\
+ \hbar \Delta(t) (x_1 - y_1 + x_2 - y_2) \left( \frac{\partial}{\partial x_1} + \frac{\partial}{\partial y_1} + \frac{\partial}{\partial x_2} + \frac{\partial}{\partial y_2} \right) \rho_r.
\]

(26)

A set of new notations in (26) is introduced to facilitate easy adoption of results from [19]. In particular,

\[
a(t) = M \delta \Omega^2(t), \quad b(t) = 2 \Gamma(t), \\
c(t) = \Delta(t), \quad d(t) = \Sigma(t).
\]

(27) (28)

It is often useful to use the Wigner function defined in phase space, which is related to the reduced density matrix \( \rho_r \) in the following way:

\[
\tilde{W}(x_1, x_2, P_1, P_2, t) = \frac{1}{(2\pi)^2} \int du_1 du_2 \ e^{i(u_1 P_1 + u_2 P_2)/\hbar} \\
\times \rho_r \left( x_1 - \frac{u_1}{2}, x_2 - \frac{u_2}{2}; x_1 + \frac{u_1}{2}, x_2 + \frac{u_2}{2}, t \right).
\]

(29)

In correspondence with (26) the Wigner function satisfies a Fokker-Planck equation:

\[
\frac{\partial \tilde{W}}{\partial t} = - \sum_{i=1,2} \left( \frac{P_i}{M \partial x_i} + M \Omega^2 x_i \frac{\partial \tilde{W}}{\partial P_i} \right) \\
+ M \delta \Omega^2(t) (x_1 + x_2) \left( \frac{\partial}{\partial P_1} + \frac{\partial}{\partial P_2} \right) \tilde{W} + 2 \Gamma(t) \left( \frac{\partial}{\partial P_1} + \frac{\partial}{\partial P_2} \right) [(P_1 + P_2) \tilde{W}] \\
+ \Sigma(t) \left( \frac{\partial}{\partial P_1} + \frac{\partial}{\partial P_2} \right)^2 \tilde{W} + \Delta(t) \left( \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} \right) \left( \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} \right) \tilde{W}.
\]

(30)

In deriving the exact master equation we assumed that the initial state for the two harmonic oscillators is a product of a function of the relative coordinates and a function of the center of mass coordinates. However, it can be easily shown that the derivation is valid for an arbitrary initial state of the system regardless of the condition of separability.

D. Coefficients of the Master Equation

The determination of the coefficients is reasonably standard, so we only provide the explicit forms of those time-dependent functions that will be used later on. As shown in [19], the functions
$\delta \Omega^2(t), \Gamma(t), \Delta(t), \Sigma(t)$ can be constructed in terms of the elementary functions $u_i(s), i = 1, 2$, which satisfy the following homogeneous integro-differential equation:

$$\ddot{f}(s) + \Omega^2 f(s) + \frac{4}{M} \int_0^s d\lambda \eta(s - \lambda) f(\lambda) = 0$$  \hspace{1cm} (31)

with the boundary conditions:

$$u_1(s = 0) = 1, \quad u_1(s = t) = 0,$$  \hspace{1cm} (32)

and

$$u_2(s = 0) = 0, \quad u_2(s = t) = 1.$$  \hspace{1cm} (33)

Here $\eta(t - s)$ is the dissipation kernel given by

$$\eta(s) = -\int_0^\infty d\omega I(\omega) \sin(\omega s),$$  \hspace{1cm} (34)

and $I(\omega)$ is the spectral density of the environment. Note that the numerical factor 4 before the integral in this equation is different from that in [19]. This is the main difference due to the presence of two harmonic oscillators. Although the two harmonic oscillators are not coupled directly, they are connected by the common reservoir, hence they affect each other dynamically.

Let $G_1(s, \tau)$ be the Green function obeying the following equation:

$$\frac{d^2}{ds^2} G_1(s, \tau) + \Omega^2 G_1(s, \tau) + \frac{4}{M} \int_0^s d\tau \eta(s - \tau) G_1(s, \tau) = \delta(s - \tau),$$  \hspace{1cm} (35)

with initial conditions:

$$G_1(s = 0, \tau) = 0, \quad \frac{d}{ds} G_1(s, \tau)|_{s=0} = 0.$$  \hspace{1cm} (36)

The Green function $G_2(s, \tau)$ is defined analogously. The coefficients can then be written as

$$\delta \Omega^2(t) = \frac{2}{M} \int_0^t ds \eta(t - s) \left( u_2(s) - \frac{u_1(s) \dot{u}_2(t)}{\dot{u}_1(t)} \right),$$  \hspace{1cm} (37)

$$\Gamma(t) = \frac{1}{M} \int_0^t ds \eta(t - s) \frac{u_1(s)}{\dot{u}_1(t)},$$  \hspace{1cm} (38)

$$\Delta(t) = \frac{\hbar}{2M} \int_0^t d\lambda G_1(t, \lambda) \nu(t - \lambda)$$

$$- \frac{4\hbar}{M^2} \int_0^t ds \int_0^t d\tau \int_0^t d\lambda \eta(t - s) G_1(t, \lambda) G_2(s, \tau) \nu(\tau - \lambda),$$  \hspace{1cm} (39)
and

\[ \Sigma(t) = \frac{\hbar}{2} \int_{0}^{t} d\lambda G_{1}'(t, \lambda) \nu(t - \lambda) - \frac{4\hbar}{M} \int_{0}^{t} ds \int_{s}^{t} d\tau \int_{0}^{t} d\lambda \eta(t - s) G_{1}'(t, \lambda) G_{2}(s, \tau) \nu(\tau - \lambda), \]

(40)

where \( \nu(s) \) defined as

\[ \nu(s) = \int_{0}^{+\infty} d\omega I(\omega) \coth \left( \frac{1}{2} \frac{\hbar \omega}{\beta} \right) \cos(\omega s), \]

(41)
is the noise kernel of the environment. Here a “prime” denotes taking the derivative with respect to the first variable of \( G_{1}(s, \tau) \).

E. Markov Approximations

The derived master equation (26) is exact, so it is valid in both the Markovian and the non-Markovian regimes. Memory effects due to the environment is encoded in the time-dependent coefficients. In the high temperature ohmic bath limit, the coefficients become constants and the spectral density has the form:

\[ I(\omega) = M_{1} \gamma \omega \exp \left( -\frac{\omega^{2}}{\Lambda^{2}} \right), \]

(42)

where \( \Lambda \) is a cut-off frequency. In the so-called Fokker-Planck limit \( (k_{B} T \gg \hbar \Lambda) \), we have

\[ \nu(s) = \frac{2M_{1} k_{B} T \gamma}{\hbar} \delta(s), \quad \eta(s) = M \gamma \frac{d}{ds} \delta(s). \]

(43)

Hence, \( \delta \Omega^{2} = -2\gamma \delta(0), \Gamma = \gamma, \Delta = 0, \Sigma = 2M_{1} \gamma k_{B} T \). The constant coefficients obtained for such a model give rise to a Markovian master equation. The Wigner function for the center of mass coordinates obeys the Fokker-Planck-Markov equation [29]:

\[ \frac{\partial W_{cm}}{\partial t} = -\frac{P}{M_{1}} \frac{\partial W_{cm}}{\partial X} - M_{1} \Omega^{2} X^{2} \frac{\partial W_{cm}}{\partial P} + 2\gamma \frac{\partial}{\partial P}(PW_{cm}) + 2M_{1} \gamma k_{B} T \frac{\partial^{2}}{\partial P^{2}} W_{cm}, \]

(44)

where \( M_{1} = 2M \) and \( \Omega^{2} = \Omega^{2} + \delta \Omega^{2} \).

III. THE INFLUENCE FUNCTIONAL METHOD

In the last section we showed a simple derivation of the master equation for the reduced density matrix and the Fokker-Planck equation for the Wigner function. In general it is difficult to get
a general analytical solution of the master equation. It turns out that in some cases of interest, one can get analytic solutions of the master equation through the influence functional method \[30\]. Using this method, we can get the evolution operator for the reduced density matrix or the evolution kernel for the exact master equation which will be very useful for the study of quantum decoherence and disentanglement problems.

Because of this, in this subsection, we will outline the key steps in the derivation of the master equation \[26\] via the path integral method.

As before, the density matrix of the total system at any time \( t \) can be written as
\[
\rho(t) = e^{-i \frac{\mu_{\text{tot}} t}{\hbar}} \rho(0) e^{i \frac{\mu_{\text{tot}} t}{\hbar}}. \tag{45}
\]

The reduced density matrix of the system is evolved by the propagator \( J_r \) from time \( t = 0 \) to \( t \) as
\[
\rho_r(x_1, x_2; y_1, y_2, t) = \int dq_0(x_1, x_2, q_n | \rho(t) | y_1, y_2, q_n)
= \int dx_0 dy_0 J_r(x_1, x_2, y_1, y_2, t; x_10, x_20, y_{10}, y_{20}, 0)
\times \rho_r(x_10, x_20; y_{10}, y_{20}; t = 0), \tag{46}
\]
where we have used the collective notation \( dx_0 dy_0 = dx_{10} dx_{20} dy_{10} dy_{20} \).

The evolution propagator \( J_r \) can be written in a path-integral representation as
\[
J_r(x_1, x_2, y_1, y_2, t; x'_1, x'_2, y'_1, y'_2, 0)
= \prod_{k=1}^{2} \int_{x_{ki}}^{x_{kf}} D x_k \int_{y_{ki}}^{y_{kf}} D y_k \exp \left\{ \frac{i}{\hbar} S_\mathcal{S}[x_1, x_2] - \frac{i}{\hbar} S_\mathcal{S}[y_1, y_2] \right\} \times \mathcal{F}[x_1, x_2, y_1, y_2], \tag{47}
\]
where \( \mathcal{F}[x_1, x_2, y_1, y_2] \) is the Feynman-Vernon influence functional defined by
\[
\mathcal{F}[x_1, x_2, y_1, y_2] = \int dq'_n dq''_n dq_n \rho_{\text{bath}}(q'_n, q''_n, 0) \int_{q'_n}^{q_n} D q_n \int_{q''_n}^{q_n} D \tilde{q}_n \exp \left\{ \frac{i}{\hbar} \left( S_I[y_1, x_2, q_n] - S_I[y_1, y_2, \tilde{q}_n] + S_B[q_n] - S_B[\tilde{q}_n] \right) \right\}
= \exp \left\{ \frac{i}{\hbar} (S_{IF}[x_1, x_2, y_1, y_2]) \right\}, \tag{48}
\]
where \( S_{IF} \) is the influence action. For the QBM model we are considering here, the influence action can be written as:
\[
S_{IF}[x_1, x_2, y_1, y_2] = -2 \int_0^t ds_1 \int_0^{s_1} ds_2 [\Delta_1(s_1) + \Delta_2(s_1)] \eta(s_1 - s_2) [\Sigma_1(s_2) + \Sigma_2(s_2)]
+ i \int_0^t ds_1 \int_0^{s_1} ds_2 [\Delta_1(s_1) + \Delta_2(s_1)] \nu(s_1 - s_2) [\Delta_1(s_2) + \Delta_2(s_2)], \tag{49}
\]
where
\[
\Sigma_1 = \frac{1}{2} (x_1 + y_1), \quad \Sigma_2 = \frac{1}{2} (x_2 + y_2), \quad \Delta_1 = x_1 - y_1, \quad \Delta_2 = x_2 - y_2. \tag{50}
\]
Note that the integrand in Eq. (47) is Gaussian, hence the integral can be computed exactly and the explicit form of $J_r$ is,

$$J_r = \tilde{N} \exp \left( \frac{i}{2} S_I - S_R \right),$$  

where the expressions of $S_I$ and $S_R$ can be written in more compact forms with the following notations:

$$x^+_k = x_{1k} + x_{2k}, \quad y^+_k = y_{1k} + y_{2k},$$

$$x^-_k = x_{1k} - x_{2k}, \quad y^-_k = y_{1k} - y_{2k},$$

whence

$$S_I = b_1 (x^+_i + y^+_i)(x^+_i - y^+_i) + b_2 (x^+_0 + y^+_0)(x^+_i - y^+_i)$$

$$- b_3 (x^+_i + y^+_i)(x^+_0 - y^+_0) - b_4 (x^+_0 + y^+_0)(x^+_i - y^+_i)$$

$$+ b_5 (x^-_i + y^-_i)(x^-_i - y^-_i) + b_6 (x^-_0 + y^-_0)(x^-_i - y^-_i)$$

$$- b_7 (x^-_i + y^-_i)(x^-_0 - y^-_0) - b_8 (x^-_0 + y^-_0)(x^-_0 - y^-_0),$$

and

$$S_R = a_{11} (x^+_i - y^+_i)^2 + a_{22} (x^+_0 - y^+_0)^2$$

$$+ a_{12} (x^+_0 - y^+_0)(x^+_i - y^+_i).$$

The functions $b_i(t)$ and $a_{ij}(t)$ depend on the environment and can be constructed from the solutions to the equations

$$b_2(t) \equiv \frac{1}{2} \dot{w}_1(t), \quad b_1(t) \equiv \frac{1}{2} \dot{w}_2(t), \quad b_0(t) \equiv \frac{1}{2} \dot{w}_1(t), \quad b_5(t) \equiv \frac{1}{2} \dot{w}_2(t),$$

$$b_4(t) \equiv \frac{1}{2} \dot{w}_1(0), \quad b_3(t) \equiv \frac{1}{2} \dot{w}_2(0), \quad b_8(t) \equiv \frac{1}{2} \dot{w}_1(0), \quad b_7(t) \equiv \frac{1}{2} \dot{w}_2(0),$$

where $w_i(t)$ are functions which satisfy the following equation

$$\ddot{\Sigma}(s) + \Omega^2 \Sigma(s) = 0,$$

with the boundary conditions:

$$w_1(0) = 1 = w_2(t), \quad w_1(t) = 0 = w_2(0),$$

$$a_{ij}(t) = \frac{1}{2} \int_0^t ds_1 \int_0^t ds_2 u_i(s_1) \nu(s_1 - s_2) u_j(s_2).$$
With this explicit expression of the evolution kernel, given any initial reduced density matrix $\rho_r$ at $t_0$ we can calculate $\rho_r$ at any later time $t$ without having to solve the complicated second order partial differential equation with time-dependent coefficient functions.

With the expression of $J_r$, we can also derive the master equation for the reduced density matrix. The derivation will be given in the appendix.

IV. QUANTUM DECOHERENCE AND DISENTANGLEMENT

In this section we give an example for the application of this master equation to the decoherence and disentanglement of two coupled harmonic oscillators in a common heat bath. For some simplified cases we obtain analytic results which show interesting features such as finite-time disentanglement \[31, 32\].

A. Dynamics of Quantum Coherence

We will assume that the system and the environment are initially uncorrelated. The total density matrix at time $t = 0$ then factorizes into a product of density matrices for the system and the environment. As usual, we further assume that the environment is initially in thermal equilibrium at a given temperature $T$.

We assume initially the 2HO (labeled as 1 and 2) are separated with distance $2L_0$ and the initial wave function of the 1-2 system is given by

$$
\Psi(x_1, x_2, t = 0) = s_1 \Psi_1(x_1) \Psi_1(x_2) + s_2 \Psi_1(x_1) \Psi_2(x_2) + s_3 \Psi_2(x_1) \Psi_1(x_2) + s_4 \Psi_2(x_1) \Psi_2(x_2),
$$

where we have defined the displaced Gaussian states as

$$
\Psi_{1,2}(x) = N \exp \left[ -\frac{(x \mp L_0)^2}{2\delta^2} \right] \exp (\pm iP_0 x),
$$

and $s_i$ are any complex numbers subject to normalization conditions. (We use 1,2 to label different initial positions of the center of the Gaussian wave function of harmonic oscillators while $x, y$ label different time paths.)

With an initial reduced density matrix

$$
\rho_r(x_{10}, x_{20}; y_{10}, y_{20}; t = 0) = \langle x_{10}, x_{20} | \Psi(0) \rangle \langle \Psi(0) | y_{10}, y_{20} \rangle
$$

$$
\equiv \sum_{i,j} s_i \bar{s}_j \rho_{ij}(x_{10}, x_{20}; y_{10}, y_{20}; t = 0),
$$

(62)
the reduced density matrix at \( t \) is given by
\[
\rho_r(x_1, x_2; y_1, y_2; t) = \int dx_0 dy_0 J_r(x_1, x_2, y_1, y_2; x_0, x_20, y_0, y_20, 0) \\
\times \rho_r(x_0, x_20; y_0, y_20; t = 0).
\] (63)

Because the QBM model is linear and the initial state is Gaussian, we can solve the master equation exactly for the dynamics of the 2HO system interacting with an environment with a general spectral density at any temperature. Therefore, we can obtain the total density matrix if the explicit solutions for each component are known,
\[
\rho_{ij}(x_1, x_2; y_1, y_2; t) = \int dx_0 dy_0 J_r(x_1, x_2, y_1, y_2; x_0, x_20, y_0, y_20, 0) \\
\times \rho_{ij}(x_0, x_20; y_0, y_20; t = 0).
\] (64)

Note that since \( J_r \) and \( \rho_{ij} \) are in the form of an exponential with an exponent which is a quadratic function in \((x_10, x_20; y_10, y_20)\), we can use a standard trick for the evaluation of the integral,
\[
\rho_{ij}(t) = \int dx_0 dy_0 J_t \times \rho_{ij}(t = 0) \\
= \int dx_0 dy_0 \exp \left[ -\vec{x}^T \cdot G_{ij} \cdot \vec{x} + \frac{1}{2} \vec{F}_{ij}^T \cdot \vec{x} + \frac{1}{2} \vec{x}^T \cdot \vec{F}_{ij} + c_{ij} \right] \\
= \frac{(\sqrt{\pi})^4}{\sqrt{\det G_{ij}}} \exp (c_{ij} + \frac{1}{4} \vec{F}_{ij}^T \cdot G_{ij}^{-1} \cdot \vec{F}_{ij}),
\] (65)
where \( \vec{x}^T = (x_0, x_20, y_0, y_20) \).

Once we have \( \rho_{ij}(x_1, x_2; y_1, y_2; t) \) we can perform the following substitution \( x_1 \mapsto X_1 - \frac{z_1}{2}; x_2 \mapsto X_2 - \frac{z_2}{2}; y_1 \mapsto X_1 + \frac{z_1}{2}; y_2 \mapsto X_2 + \frac{z_2}{2} \) and then do the Fourier transform to get the Wigner function at a later time \( t \):
\[
W_{ij}(X_1, X_2, P_1, P_2, t) = \int \int \frac{dz_1 dz_2}{(2\pi\hbar)^2} \exp (iP_1 z_1 + iP_2 z_2) \\
\times \rho_{ij}(X_1 - \frac{z_1}{2}, X_2 - \frac{z_2}{2}; X_1 + \frac{z_1}{2}, X_2 + \frac{z_2}{2}; t).
\] (66)

Since after the substitution the exponent of \( \rho_{ij} \) is quadratic in \( z_1, z_2 \), the above integration can be evaluated explicitly. These solutions (65) and (66) will be useful in decoherence and disentanglement analysis below. The detailed results and the explicit expressions of \( \rho_{ij} \) can be found in Appendix B.

When viewed from the center of mass coordinate the physics of decoherence for a 2HO system is essentially similar to that described in [19, 33] using the Hu-Paz-Zhang master equation for 1HO.
because the environment couples to the system only through the center of mass coordinate \( X \) and is independent of the relative coordinate \( x \). The evolution of the relative coordinate part in the reduced density matrix is unitary and hence will not affect the decoherence processes. One can easily recognize these features from (22) and (21). The effects of environment-induced decoherence are encoded in the coefficient functions \( a(t), b(t), c(t), d(t) \) of (22). As one can see from this example four of the matrix elements \( \rho_{11}, \rho_{14}, \rho_{41}, \rho_{44} \) are similar to those in the example considered in [33] sans the relative coordinates.

However, the issue of disentanglement is quite different because usually the entanglement measure is related to the global property of the whole reduced density matrix. In general, entanglement involves both the center of mass and the relative coordinate dynamics. It is difficult to make any prediction on how disentanglement evolves from the information of only the 1HO system. For instance, while the cm coherence always disappear asymptotically, in contrast, entanglement of the two particles may terminate in a finite time. In the following subsection, we will address this issue by a simple illustrative example.

**B. Dynamics of Entanglement: An Example**

As shown in the last subsection, the decoherent effects of the thermal heat bath is manifested by the influential functional appearing in (63). An environment that destroys quantum coherence can also disentangle two quantum Brownian particles. The dynamics of decoherence and entanglement of two harmonic oscillators interacting with a common environment is useful for understanding some basic issues in macroscopic quantum phenomena. We will present a more detailed study of this issue in a later paper. Here we show a simple example which has analytic solutions. Take as initial state the Wigner function:

\[
W(x_1, x_2, P_1, P_2) = W_{cm}(X, P)W_{rel}(x, p) = e^{-\frac{X^2}{2\sigma^2}} e^{-\frac{P^2}{2\beta^2}} e^{-\frac{x^2}{2\alpha^2}} e^{-\frac{p^2}{2\delta^2}}. \tag{67}
\]

where \( P, X, x \) and \( p \) are canonical variables defined in (5) and (6). We have omitted an irrelevant normalization factor. Note that the widths \( a^2, b^2, c^2 \) and \( d^2 \) cannot be chosen arbitrarily since they have to satisfy the uncertainty relations:

\[
a^2b^2 \geq \frac{\hbar^2}{4}, \quad c^2d^2 \geq \frac{\hbar^2}{4}. \tag{68}
\]

For a wide range of parameters \( a, b, c \) and \( d \), the Wigner function \( W(X, P, x, p) \) is entangled, since generally it cannot be written as a product of \( W_1(x_1, P_1) \) and \( W_2(x_2, P_2) \). At any time \( t \), it is known that the separability of the state (67) can be easily detected [34, 35].
Now we consider the dynamics of this state under the influence of a common environment. For
greatest simplicity, we assume two free particles coupled to a Markovian thermal bath (Setting
\( \Omega = 0 \) and \( \kappa = 0 \)) and assume the dissipation in cm coordinates is negligible. Under these
conditions, the Wigner equation \( W_{\text{cm}}(X,P) \) for cm coordinates takes on a simple form:
\[
\frac{\partial W_{\text{cm}}}{\partial t} = - \frac{P}{M_1} \frac{\partial W_{\text{cm}}}{\partial X} + D \frac{\partial^2 W_{\text{cm}}}{\partial P^2},
\]
where \( D = 2M_1\gamma k_B T \). The solution for the dissipative evolution of the center of mass can be easily
obtained, and from it, we can compute the variances of \( X \) and \( P \) at time \( t \) to be:
\[
(\Delta X^2)(t) = \frac{2Dt^3}{3M^2} + \frac{b^2t^2}{4M^2} + a^2,
\]
\[
(\Delta P^2)(t) = 2Dt + b^2.
\]
Since the evolution of the Wigner function \( W_{\text{rel}}(x,p) \) for the relative coordinates \( x,p \) is unitary,
\[
\frac{\partial W_{\text{rel}}}{\partial t} = - \frac{p}{M_2} \frac{\partial W_{\text{rel}}}{\partial x},
\]
the variances at \( t \) are simply given by
\[
(\Delta x^2)(t) = \frac{4d^2}{M_2} t^2 + c^2,
\]
\[
(\Delta p^2)(t) = d^2.
\]
According to [34], we may choose the EPR-like operators as :
\[
u = \tilde{x}_1 - \tilde{x}_2, v = \tilde{P}_1 + \tilde{P}_2,
\]
where \( \tilde{x}_i, \tilde{P}_i \) \( (i = 1, 2) \) are the dimensionless variables satisfying \([\tilde{x}_i, \tilde{P}_j] = i\delta_{ij}\),
\[
\tilde{x}_i = \left( \frac{MD}{h^3} \right)^{\frac{1}{4}} x_i, \quad \tilde{P}_i = \left( \frac{1}{hMD} \right)^{\frac{1}{4}} P_i, \quad (i = 1, 2).
\]
Then the Gaussian state at \( t \) is disentangled if and only if the following inequality is satisfied
\[
(\Delta u^2)(t) + (\Delta v^2)(t) \geq 2.
\]
Inserting (71) and (73) into the above inequality, one gets,
\[
At^2 + Bt + C \geq 2,
\]
where
\[
A = \frac{4d^2}{M^2} \sqrt{\frac{MD}{h^3}},
\]
\[
B = 2 \sqrt{\frac{D}{hM}},
\]
\[
C = \frac{b^2}{\sqrt{hMD}} + c^2 \sqrt{\frac{MD}{h^3}}.
\]
From (78), the disentanglement time $t_{\text{dent}}$ can be determined to be

$$
t_{\text{dent}} = \frac{-B + \sqrt{B^2 - 4AC + 8A}}{2A}.
$$

(82)

Thus after $t \geq t_{\text{dent}}$ the state (67) becomes completely separable.

In situations when the 2HO are coupled or share the same environment, it is expected that for some initial states entanglement will persist longer than the case when there is no direct coupling between the two oscillators and each of them is coupled to a separate environment (See, e.g. [36] for two qubits in a common electromagnetic field). This is what one might anticipate would happen for our model in the more general cases. On the other hand, as shown in this simplified example, finite-time disentanglement may yet occur for some initial states when there is no direct coupling between the two oscillators.

Such finite-time decay behavior has been noted before in several cases where two qubits [31] or two harmonic oscillators [32, 37] are individually coupled to their own heat baths. We show here the onset of the finite-time decay for the case of a common heat bath. However, it should be emphasized again that the finite-time disentanglement process found here depends crucially on the choice of initial states because for some initial states the mutual actions between the two harmonic oscillators may lead to entanglement generation. As shown in the case of two-qubits under phase noises, when the initial states are protected by a decoherence-free subspace quantum entanglement is shown to be robust against the thermal noise [38]. The 2HO model considered here will exhibit similar features, but further details will go beyond the scope of this paper.

\section{V. CONCLUDING REMARKS}

We have studied the quantum Brownian motion model with two coupled harmonic oscillators interacting with a common thermal bath. The exact master equation is derived and decoherence properties are discussed. As an illustrative example, we show how finite-time disentanglement takes place for a Gaussian state. In addition to models for multi-harmonic oscillators, the interaction between a two level system with a harmonic oscillator (such as for a two level atom in a multi-mode cavity) is also of general interest. One can devise methods to derive exact master equations to describe the nonMarkovian dynamics of such systems (e.g., [30]). One could also apply a level reduction scheme such as that used in [30] to one of the two harmonic oscillators, turning the 2HO-bath model into an effective 1HO-spin-boson model for such purposes.

As we mentioned in the beginning, a range of issues of interest to MQP can be addressed with
the master equation (or the associated Langevin or Fokker-Planck equations) derived here. In particular, decoherence and disentanglement in 2HO system under more general conditions and $N$-harmonic oscillators systems \[22\] are currently under study. It can also be applied to analyze quantum decoherence, entanglement, fluctuations, dissipation and teleportation of electro-optomechanical systems and superposition of moving mirrors due to quantum and radiative effects.

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APPENDIX A: DERIVATION OF EXACT MASTER EQUATION FROM PATH INTEGRAL

Deriving the master equation from the path integral is lengthy, but one of the advantages of this derivation is that the explicit form of the propagator can be used to find an explicit solution of the equation in many interesting cases. We will mainly follow the steps in [19] and outline the key steps in deriving the master equation from the path integral method.

From (46), it is easy to see that, to get the master equation, one first needs to calculate $J_r(t + dt, 0) - J_r(t, 0)$. The complete derivation can be decomposed into the following four steps.

1. Step one

Our first task is to take the functional representation of $J_r(t + dt, 0)$ and divide each of the path integrals into two parts. We introduce four intermediate points $x_{1m}, x_{2m}, y_{1m}, y_{2m}$ at time $t$ and integrate over them, thus symbolically, we write

$$
A[x_1, x_2, y_1, y_2] = S_S[x_1, x_2] - S_S[y_1, y_2] + \delta A[x_1, x_2, y_1, y_2]
$$

$$
= A[\bar{x}_1, \bar{x}_2, \bar{y}_1, \bar{y}_2] + A[\tilde{x}_1, \tilde{x}_2, \tilde{y}_1, \tilde{y}_2] + A_i[\bar{x}_1, \tilde{x}_2, \bar{y}_1, \tilde{y}_2],
$$

(A2)

where $A_i$ term mixes the $\tilde{x}$ histories with $\bar{x}$ ones. The appearance of the $A_i$ term is due to the non-locality of the influence functional.

2. Step two

Next, we will use straight line histories approximation of $(\tilde{x}_1, \tilde{x}_2, \tilde{y}_1, \tilde{y}_2)$. First, note that

$$
\tilde{x}_1(s) = x_{1m} + (x_{1f} - x_{1m}) \frac{s - t}{dt} \equiv x_{1m} + \beta_{1x} \frac{s - t}{dt},
$$

(A3)
and similarly,
\[\ddot{x}_2(s) = x_{2m} + (x_{2f} - x_{2m}) \frac{s - t}{dt} \equiv x_{2m} + \beta_{2x} \frac{s - t}{dt},\]  
(A4)
\[\ddot{y}_1(s) = y_{1m} + \beta_{1y} \frac{s - t}{dt}, \quad \ddot{y}_2(s) = y_{2m} + \beta_{2y} \frac{s - t}{dt}.\]  
(A5)

To compute the time derivative of \(J_r\), take the limit \(dt \to 0\). Thus we can write
\[
\prod_{k=1}^{2} \int_{t=0}^{t+dt} \mathcal{D}x_k \int_{t=0}^{t+dt} \mathcal{D}y_k \exp \left( \frac{i}{\hbar} A[x_1, x_2, y_1, y_2] \right)
\]
\[= N(t) \prod_{k=1}^{2} \int_{t=0}^{\infty} dx_{km} dy_{km} \exp \left( \frac{i}{\hbar} A[\bar{x}_1, \bar{x}_2, \bar{y}_1, \bar{y}_2] \right) \times \prod_{k=1}^{2} \int_{t=0}^{t+dk} \mathcal{D}x_k \int_{t=0}^{t+dk} \mathcal{D}y_k \exp \left( \frac{i}{\hbar} A[\bar{x}_1, \bar{x}_2, \bar{y}_1, \bar{y}_2] \right) \exp \left( \frac{i}{\hbar} A[x_1, x_2, y_1, y_2] \right). \]  
(A6)

Expanding \(A\) in \(dt\) and keeping the contributions of the first order terms, we get,
\[A[\bar{x}_1, \bar{x}_2, \bar{y}_1, \bar{y}_2] \approx \frac{m}{2dt} (\beta_{1x}^2 + \beta_{2x}^2 - \beta_{1y}^2 - \beta_{2y}^2) - \frac{1}{2} m \Omega^2 dt (x_{1f}^2 + x_{2f}^2 - y_{1f}^2 - y_{2f}^2) + \cdots, \]  
(A7)

and
\[A[x_1, x_2, y_1, y_2] \approx -dt \int_0^t ds J_{\Sigma_1}(s)(\bar{\Sigma}_1(s) + \bar{\Sigma}_2(s)) + dt \int_0^t ds J_{\Delta_1}(s)(\bar{\Delta}_1(s) + \bar{\Delta}_2(s)), \]  
(A8)

where
\[J_{\Sigma_1} + J_{\Sigma_2} \equiv J_{\Sigma}(s) \frac{2}{dt} \int_t^{t+dt} ds' (\bar{\Delta}_1(s') + \bar{\Delta}_2(s')) \eta(s' - s) \approx 2(x_{1f} - y_{1f}) + x_{2f} - y_{2f}) \eta(t - s) + \cdots, \]  
(A9)

and
\[J_{\Delta_1} + J_{\Delta_2} \equiv J_{\Delta}(s) \frac{1}{dt} \int_t^{t+dt} ds' (\bar{\Delta}_1(s') + \bar{\Delta}_2(s')) \nu(s' - s) \approx (x_{1f} - y_{1f}) + x_{2f} - y_{2f}) \nu(t - s) + \cdots. \]  
(A10)

Here we can keep only terms up to the first order in \(\beta_{1x}^2\).

In summary, the propagator \(J_r\) can be formally written as
\[
J_r(x_{1f}, x_{2f}, y_{1f}, y_{2f}, t + dt| x_{10}, x_{20}, y_{10}, y_{20}, 0)
\approx N(t) \int_0^\infty d\beta_{1x} \int_0^\infty d\beta_{2x} \int_0^\infty d\beta_{1y} \int_0^\infty d\beta_{2y} \exp \left( \frac{im}{2\hbar dt} (\beta_{1x}^2 + \beta_{2x}^2 - \beta_{1y}^2 - \beta_{2y}^2) \right) \]  
(A12)
\[
\times \{ 1 - \frac{i}{\hbar} dt [V(x_{1f}, x_{2f}) - V(y_{1f}, y_{2f})] \}
\times J_r(x_{1m}, x_{2m}, y_{1m}, y_{2m}, t + dt| x_{10}, x_{20}, y_{10}, y_{20}, 0; \hat{b}), \]  
(A13)
where
\[
\tilde{J}_r(x_{1m}, x_{2m}, y_{1m}, y_{2m}, t|x_{10}, x_{20}, y_{10}, y_{20}, 0; [\vec{b}]) = \int_{0:x_{10}}^{t:x_{1m}} D\vec{x}_1 \int_{0:y_{10}}^{t:y_{1m}} D\vec{y}_1 \int_{0:y_{20}}^{t:y_{2m}} D\vec{y}_2 \exp \left( \frac{i}{\hbar} A[\vec{x}_1, \vec{x}_2, \vec{y}_1, \vec{y}_2] \right)
\]
(\ref{eq:A14})
\[
\exp \left( \frac{i}{\hbar} (-dt \int_0^t ds J_\Sigma(s)(\Sigma_1(s) + \Sigma_2(s)) + idt \int_0^t ds J_\Delta(s)(\Delta_1(s) + \Delta_2(s))) \right),
\]
(\ref{eq:A16})
and
\[
\vec{b} = \begin{pmatrix} J_\Sigma \\ J_\Delta \end{pmatrix}
\]
(\ref{eq:A17})
where the sources \(\vec{b}\) are functions of the end points. Note that \(\tilde{J}_r(\vec{b})\) can be interpreted as the evolution operator under the action of two external sources.

### 3. Step three

Computation of the path integral \(\tilde{J}_r(\vec{b})\) can be done as follows. First, one can show that
\[
\tilde{J}_r(x_{1m}, x_{2m}, y_{1m}, y_{2m}, t|x_{10}, x_{20}, y_{10}, y_{20}, 0; [\vec{b}]) = J_r(x_{1m}, x_{2m}, y_{1m}, y_{2m}, t|x_{10}, x_{20}, y_{10}, y_{20}, 0)W(x_{1m}, x_{2m}, y_{1m}, y_{2m}, x_{10}, x_{20}, y_{10}, y_{20}, dt)
\]
(\ref{eq:A19})
(\ref{eq:A15})
(Note that the function \(J_r\) is the evolution operator without source while the function \(W\) is a function of the end points.)

Then we may parametrize the paths, and write
\[
\Sigma_1(s) = \varphi_1(s) + \Sigma_{cl,1}(s), \quad \Sigma_2(s) = \varphi_2(s) + \Sigma_{cl,2}(s)
\]
(\ref{eq:A20})
\[
\Delta_1(s)\psi_1(s) + \Delta_{cl,1}(s), \quad \Delta_2(s) = \psi_2(s) + \Delta_{cl,2}(s)
\]
(\ref{eq:A21})
where the "classical paths" \(\begin{pmatrix} \Sigma \\ \Delta \end{pmatrix}_{cl}\) are the solutions to the equation of motion derived from the real part of \(A[\Sigma_1, \Sigma_2, \Delta_1, \Delta_2]\).

After this path reparametrization and making a saddle point approximation, this path integral \(\tilde{J}_r(x_{1m}, x_{2m}, y_{1m}, y_{2m}, t|x_{10}, x_{20}, y_{10}, y_{20}, 0; [\vec{b}])\) can be written as
\[
\tilde{J}_r(x_{1m}, x_{2m}, y_{1m}, y_{2m}, t|x_{10}, x_{20}, y_{10}, y_{20}, 0; [\vec{b}]) = \tilde{J}_r(0, 0, 0, t|0, 0, 0, 0; [\vec{b}]) \times \exp \left( \frac{i}{\hbar} A[\Sigma_{cl,1}, \Sigma_{cl,2}, \Delta_{cl,1}, \Delta_{cl,2}] \right)
\]
\[
\times \exp \left( \frac{i}{\hbar} (-dt \int_0^t ds J_\Sigma(s)(\Sigma_{cl,1}(s) + \Sigma_{cl,2}(s)) + idt \int_0^t ds J_\Delta(s)(\Delta_{cl,1}(s) + \Delta_{cl,2}(s))) \right),
\]
(\ref{eq:A22})
where
\[
\tilde{J}_r(0, 0, 0, 0, t|0, 0, 0, 0; \theta) = \int_{0;\varphi_1=0}^{t;\varphi_1=0} D\varphi_1 \int_{0;\varphi_2=0}^{t;\varphi_2=0} D\varphi_2 \int_{0;\psi_1=0}^{t;\psi_1=0} D\psi_1 \int_{0;\psi_2=0}^{t;\psi_2=0} D\psi_2 
\exp \left\{ i \int_0^t ds_1 \int_0^t \frac{1}{2} \tilde{\Psi}^T(s_1) \tilde{O}(s_1, s_2) \Psi(s_2) + \int_0^t ds \tilde{\Psi}^T(s) \cdot \tilde{B}(s) \right\}.
\] (A23)

Note that
\[
\Psi = \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = \begin{pmatrix} \varphi_1 \\ \psi_1 \\ \varphi_2 \\ \psi_2 \end{pmatrix}
\] (A24)

and
\[
\tilde{B} = \begin{pmatrix} -dtJ_\Sigma \\ idtJ_\Delta + i\tilde{J}_\Delta \\ -dtJ_\Sigma \\ idtJ_\Delta + i\tilde{J}_\Delta \end{pmatrix},
\] (A25)

where \( \tilde{J}_\Delta \) is a new source which appears because the nonlocality of the influence functional. It couples the classical paths to the \( \Psi \) paths.

\[
\tilde{J}_\Delta(s) = \int_0^t ds'[\Delta_{cl,1}(s') + \Delta_{cl,2}(s')]\nu(s-s').
\] (A26)

The matrix operator \( \tilde{O}(s_1, s_2) \) is defined as follows:

\[
O_{11}(s_1, s_2) = O_{33}(s_1, s_2) = O_{13}(s_1, s_2)O_{31}(s_1, s_2) = 0,
\] (A27)

\[
O_{22}(s_1, s_2) = O_{44}(s_1, s_2) = O_{24}(s_1, s_2)O_{42}(s_1, s_2) = 2i\nu(s_1 - s_2),
\] (A28)

\[
O_{14}(s_1, s_2) = O_{32}(s_1, s_2) = 2\theta(s_2 - s_1)\eta(s_1 - s_2),
\] (A29)

\[
O_{41}(s_1, s_2) = O_{23}(s_1, s_2) = 2\theta(s_1 - s_2)\eta(s_1 - s_2),
\] (A30)

\[
O_{12}(s_1, s_2) = O_{34}(s_1, s_2) = \left( \frac{d^2}{ds_1^2} + \Omega^2 \right) \delta(s_1 - s_2) + 2\theta(s_2 - s_1)\eta(s_1 - s_2),
\] (A31)

\[
O_{21}(s_1, s_2) = O_{43}(s_1, s_2) = \left( \frac{d^2}{ds_1^2} + \Omega^2 \right) \delta(s_1 - s_2) + 2\theta(s_1 - s_2)\eta(s_1 - s_2).
\] (A32)
The Gaussian path integral can be computed in terms of the inverse of the operator $\hat{O}$, which is given by $\hat{G} \equiv \hat{O}^{-1}$. Hence to first order in $dt$, we have

\[
\tilde{J}_r(0,0,0,0,t|0,0,0,0,0;[\vec{b}]) = \tilde{J}_r(0,0,0,0,t|0,0,0,0,0;[\vec{b}]) \exp \{i \int_0^t ds_1 \int_0^t ds_2 [\frac{1}{2} \Psi^T(s_1) \hat{O}(s_1,s_2) \Psi(s_2) + \int_0^t ds \Psi^T(s) \cdot \vec{B}(s)]\} \\
= \int D\varphi_1 \int D\varphi_2 \int D\psi_1 \int D\psi_2 \exp \{i \int_0^t \frac{1}{2} (\Psi^T + \vec{B}^T \cdot \hat{O}^{-1}) \hat{O}(\Psi + \hat{O}^{-1} \cdot \vec{B}) - \frac{1}{2} \vec{B}^T \hat{O}^{-1} \vec{B})\}'n = Z_0(t) \exp \{-\frac{i}{2} \vec{B}^T \hat{O}^{-1} \vec{B}\}'n \approx Z_0(t)(1 - \frac{i}{2} \vec{B}^T \hat{O}^{-1} \vec{B})'n \approx Z_0(t)(1 - dt) \int_0^t ds_1 \int_0^t ds_2 \tilde{G}_{12}(s_1,s_2) [G_{12}(s_1,s_2) + G_{14}(s_1,s_2) + G_{21}(s_2,s_1) + G_{41}(s_1,s_2)] \tilde{J}_\Delta(s_2).'n
\]

Note that the Green's function $(G_{12} + G_{32}) \equiv \tilde{G}_{12}(s_1,s_2)$ satisfies the following equation

\[
\frac{d^2}{ds_1^2} \tilde{G}_{12}(s_1,s_2) + \Omega^2 \tilde{G}_{12}(s_1,s_2) + 4 \int_0^{s_1} d\tau \tilde{G}_{12}(s_1 - \tau) \tilde{G}_{12}(s_1,\tau) = \delta(s_1 - s_2) \tag{A33}\]

with boundary conditions $\tilde{G}_{12}(0,s_2) = \tilde{G}_{12}(s_1,t) = 0$. The equation for $(G_{21} + G_{23}) \equiv \tilde{G}_{21}(s_1,s_2)$ are analogous.

Now we can show that

\[
\tilde{J}_r(x_{m}, x_{m}, y_{m}, y_{m}, t|x_{10}, x_{20}, y_{10}, y_{20}, 0;[\vec{b}])'n = \tilde{J}_r(0,0,0,0,t|0,0,0,0,0;[\vec{b}]) \exp \{i(A[\Sigma_{cl,1}, \Sigma_{cl,2}, \Delta_{cl,1}, \Delta_{cl,2}])\} \times \exp \{i(-dt) \int_0^t ds \tilde{J}_\Sigma(s)(\Sigma_{cl,1}(s) + \Sigma_{cl,2}(s)) + i dt \int_0^t ds \tilde{J}_\Delta(s)(\Delta_{cl,1}(s) + \Delta_{cl,2}(s))\} \approx Z_0(t) \exp \{iA[\Sigma_{cl,1}, \Sigma_{cl,2}, \Delta_{cl,1}, \Delta_{cl,2}]\} \times \{1 - dt \int_0^t ds_1 \int_0^t ds_2 \tilde{G}_{12}(s_1,s_2) [\tilde{G}_{12}(s_1,s_2) + \tilde{G}_{21}(s_1,s_2)] \tilde{J}_\Delta(s_2) \tag{A35}\}

- idt \int_0^t ds \tilde{J}_\Sigma(s)(\Sigma_{cl,1}(s) + \Sigma_{cl,2}(s)) + (i)^2 dt \int_0^t ds \tilde{J}_\Delta(s)(\Delta_{cl,1}(s) + \Delta_{cl,2}(s))\} \tag{A37}\]

\[
= J_r(x_{m}, x_{m}, y_{m}, y_{m}, t|x_{10}, x_{20}, y_{10}, y_{20}, 0) \times W(x_{m}, x_{m}, y_{m}, y_{m}, x_{10}, x_{20}, y_{10}, y_{20}, dt), \tag{A38}\]

\[
= J_r(x_{m}, x_{m}, y_{m}, y_{m}, t|x_{10}, x_{20}, y_{10}, y_{20}, 0) \tag{A39}\]

\[
= J_r(x_{m}, x_{m}, y_{m}, y_{m}, t|x_{10}, x_{20}, y_{10}, y_{20}, 0) \tag{A40}\]
where \( W \) is given by,

\[
W(x_{1m}, x_{2m}, y_{1m}, y_{2m}, x_{10}, x_{20}, y_{10}, y_{20}, dt) = 1 - dt \int_0^t ds \left( \Delta_1 f + \Delta_2 f \right) \eta(t - s) u_1(s) \tilde{\Sigma}_d(0) \\
+ \int_0^t ds \left( \Delta_1 f + \Delta_2 f \right) \eta(t - s) u_2(s) \tilde{\Sigma}_d(t) \\
- dt \int_0^t ds \left( \Delta_1 f + \Delta_2 f \right) \nu(t - s) v_1(s) \tilde{\Delta}_d(0) + \int_0^t ds \left( \Delta_1 f + \Delta_2 f \right) \nu(t - s) v_2(s) \tilde{\Delta}_d(t) \\
- dt \int_0^t ds \int_0^t ds' \int_0^t ds'' \left( \Delta_1 f + \Delta_2 f \right) \eta(t - s_1) \left[ \tilde{G}_{12}(s_1, s_2) + \tilde{G}_{21}(s_2, s_1) \right] \\
\times \nu(s_2 - s_3) v_1(s_3) \tilde{\Delta}_d(0) \\
+ \int_0^t ds \int_0^t ds' \int_0^t ds'' \left( \Delta_1 f + \Delta_2 f \right) \eta(t - s_1) \left[ \tilde{G}_{12}(s_1, s_2) \\
+ \tilde{G}_{21}(s_2, s_1) \right] \nu(s_2 - s_3) v_2(s_3) \tilde{\Delta}_d(t).
\]

(A39)

To simplify the expressions, let us define

\[
d_1(t) = 2 \int_0^t ds \eta(t - s) u_1(s), \quad d_2(t) = 2 \int_0^t ds \eta(t - s) u_2(s), \\
c_1(t) = \int_0^t ds \int_0^t ds' \int_0^t ds'' \left( \Delta_1 f + \Delta_2 f \right) \eta(t - s_1) \left[ \tilde{G}_{12}(s_1, s_2) + \tilde{G}_{21}(s_2, s_1) \right] \nu(s_2 - s_3) v_1(s_3), \\
c_2(t) = \int_0^t ds \int_0^t ds' \int_0^t ds'' \left( \Delta_1 f + \Delta_2 f \right) \eta(t - s_1) \left[ \tilde{G}_{12}(s_1, s_2) + \tilde{G}_{21}(s_2, s_1) \right] \nu(s_2 - s_3) v_2(s_3), \\
e_1(t) = \int_0^t ds \nu(t - s) v_1(s) = \int_0^t ds \nu(t - s) u_1(t - s) = \int_0^t ds \nu(s) u_1(s), \\
e_2(t) = \int_0^t ds \nu(t - s) v_2(s) = \int_0^t ds \nu(t - s) u_2(t - s) = \int_0^t ds \nu(s) u_2(s).
\]

(A43)

Finally, we have,

\[
J_r(x_{1f}, x_{2f}, y_{1f}, y_{2f}, t + dt|x_{10}, x_{20}, y_{10}, y_{20}, 0) = N(t) \prod_{k=1}^2 \int_{-\infty}^\infty d\beta_{kx} \int_{-\infty}^\infty d\beta_{ky} \exp \left( \frac{i}{2dt} \left( \beta_{1x}^2 + \beta_{2x}^2 - \beta_{1y}^2 - \beta_{2y}^2 \right) \right) \\
\times \{ 1 - dt[i(V(x_{1f}, x_{2f}) - V(y_{1f}, y_{2f})) + i(\Delta_1 f + \Delta_2 f)(d_1(t) (\Sigma_{i,1} + \Sigma_{i,2}) \\
+ d_2(t) (\Sigma_{1f} + \Sigma_{2f}) + (\Delta_1 f + \Delta_2 f)(\Delta_{i,1} + \Delta_{i,2})(e_2(t) + 2e_1(t)) \\
+ (\Delta_1 f + \Delta_2 f)^2(e_1(t) + 2e_2(t))] \\
\times \{ J_r + \frac{1}{2} \left[ \frac{\partial^2 J_r}{\partial x_{1f}^2} (-\beta_{1x})^2 + \frac{\partial^2 J_r}{\partial x_{2f}^2} (-\beta_{2x})^2 + \frac{\partial^2 J_r}{\partial y_{1f}^2} (-\beta_{1y})^2 + \frac{\partial^2 J_r}{\partial y_{2f}^2} (-\beta_{2y})^2 \right] \} \}.
\]

(A48)
Hence

$$J_r(t + dt) - J_r = -dtJ_r[i\frac{1}{2}\Omega^2(x^2_{1f} + x^2_{2f} - y^2_{1f} - y^2_{2f}) + (\Delta_{1f} + \Delta_{2f}) \times
\[i(d_1(t)(\Sigma_{i,1} + \Sigma_{i,2}) + d_2(t)(\Sigma_{1f} + \Sigma_{2f})) + (\Delta_{i,1} + \Delta_{i,2})(e_2(t) + 2c_1(t))
\] + (\Delta_{1f} + \Delta_{2f})(e_1(t) + 2c_2(t))]$$  \quad (A49)

$$+ \frac{1}{2}dt \frac{\partial^2 J_r}{\partial x^2_f} + \frac{1}{2}dt \frac{\partial^2 J_r}{\partial y^2_f} + \frac{1}{2}i \frac{\partial J_r}{\partial y^1_f} + \frac{1}{2}i \frac{\partial J_r}{\partial y^2_f},$$  \quad (A50)

We can then get the evolution equation for the propagator $J_r$.

\[
i \frac{\partial}{\partial t} J_r(x_{1f}, x_{2f}, y_{1f}, y_{2f}, t|x_{10}, x_{20}, y_{10}, y_{20}, 0) = \frac{\partial}{\partial t}[J_r(x_{1f}, x_{2f}, y_{1f}, y_{2f}, t + dt|x_{10}, x_{20}, y_{10}, y_{20}, 0) - J_r(x_{1f}, x_{2f}, y_{1f}, y_{2f}, t|x_{10}, x_{20}, y_{10}, y_{20}, 0)] = \\
\{ -\frac{1}{2}(\frac{\partial^2}{\partial x^2_{1f}} - \frac{\partial^2}{\partial y^2_{1f}} - \frac{\partial^2}{\partial x^2_{2f}} - \frac{\partial^2}{\partial y^2_{2f}} - \frac{\partial^2}{\partial y^1_{1f}} - \frac{\partial^2}{\partial y^1_{2f}} - \frac{\partial^2}{\partial y^2_{1f}} - \frac{\partial^2}{\partial y^2_{2f}}) + \frac{1}{2}\Omega^2(x^2_{1f} + x^2_{2f} - y^2_{1f} - y^2_{2f})
\] + (\Delta_{1f} + \Delta_{2f})(d_1(t)(\Sigma_{i,1} + \Sigma_{i,2}) + d_2(t)(\Sigma_{1f} + \Sigma_{2f}))
\] - i(\Delta_{1f} + \Delta_{2f})(\Delta_{i,1} + \Delta_{i,2})(e_2(t) + 2c_1(t))
\] - i(\Delta_{1f} + \Delta_{2f})^2(e_1(t) + 2c_2(t))\} J_r(x_{1f}, x_{2f}, y_{1f}, y_{2f}, t|x_{10}, x_{20}, y_{10}, y_{20}, 0). \quad (A51)
\]

4. Step four

Now we have the explicit expression for $J_r$. But we still need to deal with terms of the form like $\Delta_i J$. To do so we can differentiate $J$ with respect to $\Sigma_{1f}$ and get

\[
\partial_{\Sigma_{1f}} J = [ib_1(t)(\Delta_{1f} + \Delta_{2f}) + ib_5(t)(\Delta_{1f} - \Delta_{2f}) - ib_3(t)(\Delta_{1i} + \Delta_{2i}) - ib_7(t)(\Delta_{1i} - \Delta_{2i})]J. \quad (A52)
\]

Similarly if we want $\Delta_{2i} J$, we can differentiate $J$ with respect to $\Sigma_{2f}$ and get

\[
\partial_{\Sigma_{2f}} J = [ib_1(t)(\Delta_{1f} + \Delta_{2f}) - ib_5(t)(\Delta_{1f} - \Delta_{2f}) - ib_3(t)(\Delta_{1i} + \Delta_{2i}) + ib_7(t)(\Delta_{1i} - \Delta_{2i})]J. \quad (A53)
\]

The sum of these two equations gives

\[
(\partial_{\Sigma_{1f}} + \partial_{\Sigma_{2f}}) J = [2ib_1(t)(\Delta_{1f} + \Delta_{2f}) - 2ib_3(t)(\Delta_{1i} + \Delta_{2i})]J. \quad (A54)
\]

This can be written as

\[
(\Delta_{1i} + \Delta_{2i}) J = \frac{1}{2b_3(t)}[i(\partial_{\Sigma_{1f}} + \partial_{\Sigma_{2f}}) + 2b_1(t)(\Delta_{1f} + \Delta_{2f})]J. \quad (A55)
\]
Similarly, we can differentiate with respect to $\Delta_{1f}$ (or $\Delta_{2f}$) to get $\Sigma_{1i}J$ (or $\Sigma_{2i}J$). The sum of these two equations gives

\[
(\partial_{\Delta_{1f}} + \partial_{\Delta_{2f}})J = 2[ib_2(t)(\Sigma_{1i} + \Sigma_{2i}) + ib_1(t)(\Sigma_{1f} + \Sigma_{2f}) - a_{12}(t)(\Delta_{1i} + \Delta_{2i}) - 2a_{11}(t)(\Delta_{1f} + \Delta_{2f})]J
\]  

(A56)

and

\[
(\Sigma_{1i} + \Sigma_{2i})J = \frac{1}{2b_2(t)}[-i(\partial_{\Delta_{1f}} + \partial_{\Delta_{2f}}) + \frac{a_{12}(t)}{b_3(t)}(\partial_{\Sigma_{1f}} + \partial_{\Sigma_{2f}}) - 2b_1(t)(\Sigma_{1f} + \Sigma_{2f}) - i[4a_{11}(t) + 2\frac{a_{12}(t)b_1(t)}{b_3(t)}](\Delta_{1f} + \Delta_{2f})]J.
\]  

(A57)

Substituting in what we already have for $(\Sigma_{1i} + \Sigma_{2i})J$ and $(\Delta_{1i} + \Delta_{2i})J$, and multiplying by $\rho_0$ and integrating over initial coordinates, we obtain

\[
(\Delta_{1f} + \Delta_{2f})d_1(t)(\Sigma_{1i} + \Sigma_{2i})J
\]

\[
= (\Delta_{1f} + \Delta_{2f})d_1(t)[\frac{-i}{2b_2(t)}(\partial_{\Delta_{1f}} + \partial_{\Delta_{2f}}) + \frac{a_{12}(t)}{2b_2(t)b_3(t)}(\partial_{\Sigma_{1f}} + \partial_{\Sigma_{2f}}) - \frac{b_1(t)}{b_2(t)}(\Sigma_{1f} + \Sigma_{2f}) - i[2a_{11}(t) + \frac{a_{12}(t)b_1(t)}{b_2(t)b_3(t)}](\Delta_{1f} + \Delta_{2f})]J,
\]  

(A58)

and

\[
(\Delta_{1f} + \Delta_{2f})(e_2(t) + 2c_1(t))(\Delta_{1i} + \Delta_{2i})J
\]

\[
= (\Delta_{1f} + \Delta_{2f})(e_2(t) + 2c_1(t))[\frac{i}{2b_3(t)}(\partial_{\Sigma_{1f}} + \partial_{\Sigma_{2f}}) + \frac{b_1(t)}{b_3(t)}(\Delta_{1f} + \Delta_{2f})]J.
\]  

(A59)

Hence we can write the evolution equation for the reduced density matrix as

\[
i\frac{\partial}{\partial t}\rho_r = [-\frac{1}{2}(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} - \frac{\partial^2}{\partial y_1^2} - \frac{\partial^2}{\partial y_2^2}) + \frac{1}{2}\Omega^2(x_1^2 + x_2^2 - y_1^2 - y_2^2)]\rho_r
\]

\[
+ \delta\Omega^2(t)(\Delta_{1f} + \Delta_{2f})(\Sigma_{1f} + \Sigma_{2f})\rho_r
\]

\[-iA_1(t)(\Delta_{1f} + \Delta_{2f})(\partial_{\Delta_{1f}} + \partial_{\Delta_{2f}})\rho_r
\]

\[-iA_2(t)(\Delta_{1f} + \Delta_{2f})^2\rho_r
\]

\[+A_3(t)(\Delta_{1f} + \Delta_{2f})(\partial_{\Sigma_{1f}} + \partial_{\Sigma_{2f}})\rho_r
\]  

(A60)

where

\[
\frac{\partial}{\partial x} = \frac{\partial}{\partial x} + \frac{\partial}{\partial y}; \quad \frac{\partial}{\partial \Delta} = \frac{1}{2}(\frac{\partial}{\partial x} - \frac{\partial}{\partial y})
\]  

(A61)

and

\[
\delta\Omega^2(t) \equiv d_2(t) - d_1(t)\frac{b_1(t)}{b_2(t)}, \quad A_1(t) \equiv \frac{d_1(t)}{2b_2(t)}.
\]  

(A62)
A_2(t) \equiv d_1(t)[\frac{2a_{11}(t)}{b_2(t)} + \frac{a_{12}(t)b_1(t)}{b_2(t)b_3(t)}] + (c_1(t) + 2c_2(t)) + (e_2(t) + 2c_1(t)) \frac{b_1(t)}{b_3(t)} \quad \text{(A63)}

A_3(t) \equiv \frac{d_1(t)a_{12}(t)}{2b_2(t)b_3(t)} + \frac{e_2(t) + 2c_1(t)}{2b_3(t)}.

This immediately leads to the general master equation \(^{(20)}\).

**APPENDIX B: EXPLICIT EXPRESSIONS FOR \( \rho_{ij} \)**

We find that the matrix \( G_{ij} \) is the same for all the \( \rho_{ij} \). Thus, we can write \( G_{ij} \equiv G \). The matrix elements for the matrix \( G \) are given by

\[
G_{11} = G_{22} = a_{22} + \frac{ib_4}{2} + \frac{ib_8}{2} + \frac{1}{2\delta^2}, \quad \text{(B1)}
\]

\[
G_{33} = G_{44} = a_{22} - \frac{ib_4}{2} - \frac{ib_8}{2} + \frac{1}{2\delta^2}, \quad \text{(B2)}
\]

\[
G_{12} = G_{21} = \frac{1}{2}(2a_{22} + ib_4 - ib_8), \quad \text{(B3)}
\]

\[
G_{34} = G_{43} = \frac{1}{2}(2a_{22} - ib_4 + ib_8), \quad \text{(B4)}
\]

\[
G_{13} = G_{14} = G_{23} = G_{24} = G_{31} = G_{32} = G_{41} = G_{42} = -a_{22}. \quad \text{(B5)}
\]

Then the determinant of \( G \) can be explicitly computed,

\[
\det G = b_1^2b_8^2 + \frac{1}{16\delta^8} + \frac{a_{22}b_8^2}{2\delta^4} + \frac{b_4^2}{4\delta^4} + \frac{b_8^2}{4\delta^4} + \frac{2a_{22}b_8^2}{\delta^2}. \quad \text{(B6)}
\]

Moreover, the matrix elements of the inverse matrix \( G^{-1} \) are

\[
G_{11}^{-1} = G_{22}^{-1} = \frac{1}{\det G}\left(-\frac{i}{2}b_4b_8 + a_{22}b_8^2 + \frac{ib_4}{2} + \frac{ib_8}{2} + \frac{1}{2\delta^2}, \right) \quad \text{(B7)}
\]

\[
G_{33}^{-1} = G_{44}^{-1} = \frac{1}{\det G}\left(\frac{i}{2}b_4b_8 + a_{22}b_8^2 + \frac{ib_4}{2} - \frac{ib_8}{2} + \frac{1}{2\delta^2} \right), \quad \text{(B8)}
\]

\[
G_{12}^{-1} = G_{21}^{-1} = \frac{1}{\det G}\left(-\frac{i}{2}b_4b_8 - a_{22}b_8^2 - \frac{ib_4}{2} + \frac{ib_8}{2} - \frac{1}{2\delta^2}, \right) \quad \text{(B9)}
\]

\[
G_{34}^{-1} = G_{43}^{-1} = \frac{1}{\det G}\left(-\frac{i}{2}b_4b_8 + a_{22}b_8^2 + \frac{ib_4}{2} + \frac{ib_8}{2} + \frac{1}{2\delta^2} \right), \quad \text{(B10)}
\]

\[
G_{13}^{-1} = G_{14}^{-1} G_{23}^{-1} = G_{24}^{-1} = G_{31}^{-1} = G_{32}^{-1} = G_{41}^{-1} = G_{42}^{-1} = \frac{1}{\det G}(a_{22}b_8^2 + \frac{a_{22}}{4\delta^4}). \quad \text{(B11)}
\]
For the case of $\rho_{11}$:

$$
\rho_{11}(t = 0) = N^4 \exp\left[\frac{-(x_{10} - L_0)^2 + (x_{20} - L_0)^2 + (y_{10} - L_0)^2 + (y_{20} - L_0)^2}{2\delta^2}\right]
\times \exp[iP_0(x_{10} + x_{20} - y_{10} - y_{20})],
$$

(B12)

then the matrix elements for $F$ are,

\begin{align*}
F_{11}^1 &= iP_0 - a_{12}x_1 + \frac{ib_{2}x_1}{2} - \frac{ib_{3}x_1}{2} + \frac{ib_{6}x_1}{2} - \frac{ib_{7}x_1}{2} - a_{12}x_2 + \frac{ib_{2}x_2}{2} - \frac{ib_{3}x_2}{2} - \frac{ib_{6}x_2}{2} + \frac{ib_{7}x_2}{2} \\
&\quad + a_{12}y_1 - \frac{ib_{2}y_1}{2} - \frac{ib_{3}y_1}{2} + \frac{ib_{6}y_1}{2} - \frac{ib_{7}y_1}{2} + a_{12}y_2 - \frac{ib_{2}y_2}{2} - \frac{ib_{3}y_2}{2} + \frac{ib_{6}y_2}{2} + \frac{ib_{7}y_2}{2} + \frac{L_0}{\delta^2}

F_{11}^2 &= iP_0 - a_{12}x_1 + \frac{ib_{2}x_1}{2} - \frac{ib_{3}x_1}{2} - \frac{ib_{6}x_1}{2} + \frac{ib_{7}x_1}{2} - a_{12}x_2 + \frac{ib_{2}x_2}{2} + \frac{ib_{3}x_2}{2} + \frac{ib_{6}x_2}{2} - \frac{ib_{7}x_2}{2} \\
&\quad + a_{12}y_1 - \frac{ib_{2}y_1}{2} - \frac{ib_{3}y_1}{2} + \frac{ib_{6}y_1}{2} + \frac{ib_{7}y_1}{2} + a_{12}y_2 - \frac{ib_{2}y_2}{2} - \frac{ib_{3}y_2}{2} - \frac{ib_{6}y_2}{2} - \frac{ib_{7}y_2}{2} + \frac{L_0}{\delta^2}

F_{11}^3 &= -iP_0 + a_{12}x_1 + \frac{ib_{2}x_1}{2} + \frac{ib_{3}x_1}{2} + \frac{ib_{6}x_1}{2} + \frac{ib_{7}x_1}{2} + a_{12}x_2 + \frac{ib_{2}x_2}{2} + \frac{ib_{3}x_2}{2} + \frac{ib_{6}x_2}{2} + \frac{ib_{7}x_2}{2} \\
&\quad - a_{12}y_1 - \frac{ib_{2}y_1}{2} + \frac{ib_{3}y_1}{2} - \frac{ib_{6}y_1}{2} + \frac{ib_{7}y_1}{2} - a_{12}y_2 - \frac{ib_{2}y_2}{2} + \frac{ib_{3}y_2}{2} - \frac{ib_{6}y_2}{2} - \frac{ib_{7}y_2}{2} + \frac{L_0}{\delta^2}

F_{11}^4 &= -iP_0 + a_{12}x_1 + \frac{ib_{2}x_1}{2} + \frac{ib_{3}x_1}{2} - \frac{ib_{6}x_1}{2} - \frac{ib_{7}x_1}{2} + a_{12}x_2 + \frac{ib_{2}x_2}{2} + \frac{ib_{3}x_2}{2} - \frac{ib_{6}x_2}{2} - \frac{ib_{7}x_2}{2} \\
&\quad - a_{12}y_1 - \frac{ib_{2}y_1}{2} + \frac{ib_{3}y_1}{2} - \frac{ib_{6}y_1}{2} - \frac{ib_{7}y_1}{2} + a_{12}y_2 - \frac{ib_{2}y_2}{2} + \frac{ib_{3}y_2}{2} - \frac{ib_{6}y_2}{2} - \frac{ib_{7}y_2}{2} + \frac{L_0}{\delta^2},
\end{align*}

where $F_{11}^T = (F_{11}^1, F_{11}^2, F_{11}^3, F_{11}^4)$ and

\begin{align*}
c_{11} &= -a_{11}x_2^2 + \frac{i}{2}b_1x_2^2 + \frac{i}{2}b_5x_2^2 - 2a_{11}x_1x_2 + ib_1x_1x_2 - ib_5x_1x_2 \\
&\quad - a_{11}x_2^2 + \frac{i}{2}b_1x_2^2 + \frac{i}{2}b_5x_2^2 + 2a_{11}x_1y_1 + 2a_{11}x_2y_1 \\
&\quad - a_{11}y_1^2 - \frac{i}{2}b_1y_1^2 - \frac{i}{2}b_5y_1^2 + 2a_{11}x_1y_2 + ib_1x_1y_2 + 2a_{11}x_2y_2 + ib_1x_2y_2 \\
&\quad - 2a_{11}y_1y_2 + ib_5y_1y_2 - a_{11}y_2^2 + \frac{i}{2}b_1y_2^2 - \frac{i}{2}b_5y_2^2 - \frac{2L_0^2}{\delta^2}.
\end{align*}

(B13)

For the case of $\rho_{12}$:

$$
\rho_{12}(t = 0) = N^4 \exp\left[\frac{-(x_{10} - L_0)^2 + (x_{20} - L_0)^2 + (y_{10} - L_0)^2 + (y_{20} + L_0)^2}{2\delta^2}\right]
\times \exp[iP_0(x_{10} + x_{20} - y_{10} + y_{20})]
$$

(B14)

\begin{align*}
F_{12}^1 &= F_{11}^1, \quad F_{12}^2 = F_{11}^2, \quad F_{12}^3 = F_{11}^3, \quad F_{12}^4 = F_{11}^4 + 2iP_0 - \frac{2L_0}{\delta^2}, \quad c_{12} = c_{11}.
\end{align*}

(B15)

For the case of $\rho_{13}$:

$$
\rho_{13}(t = 0) = N^4 \exp\left[\frac{-(x_{10} - L_0)^2 + (x_{20} - L_0)^2 + (y_{10} + L_0)^2 + (y_{20} - L_0)^2}{2\delta^2}\right]
\times \exp[iP_0(x_{10} + x_{20} + y_{10} - y_{20})]
$$

(B16)
\[ F^1_{13} = F^1_{11}, \quad F^2_{13} = F^2_{11}, \quad F^3_{13} = F^3_{11} + 2iP_0 - \frac{2L_0}{\delta z}, \quad F^4_{13} = F^4_{11}, \quad c_{13} = c_{11}. \] (B17)

For the case of \( \rho_{14} \):

\[
\rho_{14}(t = 0) = N^4 \exp \left[ -\frac{(x_{10} - L_0)^2 + (x_{20} - L_0)^2 + (y_{10} + L_0)^2 + (y_{20} + L_0)^2}{2\delta^2} \right] \\
\times \exp \left[ iP_0(x_{10} + x_{20} + y_{10} + y_{20}) \right] \] (B18)

\[ F^1_{14} = F^1_{11}, \quad F^2_{14} = F^2_{11}, \quad F^3_{14} = F^3_{11} + 2iP_0 - \frac{2L_0}{\delta z}, \quad F^4_{14} = F^4_{11} + 2iP_0 - \frac{2L_0}{\delta z}, \quad c_{14} = c_{11} \] (B19)

Similarly, one can work out the cases for \( \rho_{2i}, \rho_{3i} \) and \( \rho_{4i} \) \( (i = 1, 2, 3, 4) \).

[29] This equation was originally derived for the Markovian limit. We have added the name Markov to it since we want to call the equation (30) which covers the more general cases the Fokker-Planck (FP) equation. This more general equation is sometime called the Wigner equation.