Black holes and Rindler superspace: classical singularity and quantum unitarity

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Canonical quantization of spherically symmetric initial data which is appropriate to classical interior black hole solutions in four dimensions is carried out and solved exactly without gauge fixing. The resultant mini-superspace manifold and arena for quantum geometrodynamic is two-dimensional, of signature (+, −), non-singular, and can in fact be identified precisely with the first Rindler wedge. The associated Wheeler-DeWitt equation with evolution in intrinsic superspace time can be formulated as a free massive Klein-Gordon equation; and the Hamilton-Jacobi semiclassical limit of plane wave solutions can be matched precisely to the interiors of Schwarzschild black holes. Furthermore, classical black hole horizons and singularities correspond to the boundaries of the Rindler wedge. Exact wavefunctions of the first-order-in-superspace intrinsic time Dirac equation are also considered. Precise correspondence between Schwarzschild black holes and free particle mechanics in superspace is noted. Another intriguing observation is that, despite the presence of classical horizons and singularities, hermiticity of the Dirac hamiltonian operator for the mini-superspace, and thus unitarity of the quantum theory, is equivalent to an appropriate boundary condition which must be satisfied by the quantum states. This boundary condition holds for quite generic quantum wavepackets of energy eigenstates, but fails for the usual Rindler fermion modes which are precisely eigenstates with zero uncertainty in energy.

PACS numbers: 04.60.-m, 04.60.Ds

I. INTRODUCTION AND OVERVIEW

Quantization of the spherically symmetric sector of four-dimensional pure General Relativity has been investigated in a number of articles[1, 2, 3, 4, 5, 6, 7, 8, 9]. Despite the simplicity of this toy model, it is an extremely interesting testing (toy)ground for many intriguing and challenging quantum gravity issues, both on the technical as well as conceptual fronts. Birkhoff’s theorem[10] ensures that the classical solutions of this sector are none other than Schwarzschild black holes, and as such they come with classical horizons and singularities (and even naked classical singularities for interior Schwarzschild solutions). How do these manifest themselves, and what roles do they play in the quantum context? Do they affect the unitarity of the quantum theory, and if so in what explicit manner? And for reparametrization invariant theory, one ought to ask whether the unitarity or non-unitarity of quantum evolution is with respect to intrinsic time in superspace[11], or with respect to some other appropriate choice of “time”. Indeed it is not unreasonable to demand any respectable quantum theory of gravity to provide guidance on these issues.

Recent investigations motivated by loop quantum gravity have yielded as insights and possible resolutions an upper bound on the curvature, and reformulation of the Wheeler-DeWitt constraint as a difference time evolution equation wherein classical black hole singularities are not obstacles[7, 9]. Here we adopt a more conservative approach based upon continuum physics and exact canonical quantization with field variables. The Wheeler-DeWitt equation[11, 12] for spherically symmetric mini-superspace which is appropriate to the discussion of interior black hole solutions[9] is solved exactly without gauge fixing. Among the neat features which emerged are the following: 1) The resultant arena for quantum geometrodynamic is two-dimensional, of signature (+, −), non-singular, and can in fact be identified precisely with the first Rindler wedge. 2) Classical black hole horizons and singularities correspond, remarkably, to the boundaries of the Rindler wedge. 3) The classical super-Hamiltonian constraint is equivalent to a massive free-particle dispersion relation in flat superspace; and the Wheeler-DeWitt equation can be rewritten as a Klein-Gordon equation with evolution in intrinsic superspace time. 4) The Hamilton-Jacobi semiclassical limit consists of plane wave solutions which can be matched precisely to the interiors of Schwarzschild black holes. 5) Positive-definite “probability” current for wavefunctions on superspace and other considerations motivate the investigation of “fermionic” solutions of the Dirac equation associated with the Wheeler-DeWitt constraint. 6) With

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respect to the natural inner product, hermiticity of the Dirac Hamiltonian, and thus unitarity of the quantum theory, is then translated into a condition which must be satisfied by the corresponding solutions at the boundary of space-time hypersurfaces of the Rindler wedge. The boundary condition is demonstrated to be satisfied by rather generic wavepackets, but not for the usual energy-eigenstate fermionic Rindler modes. This situation has an analogy in free non-relativistic Schrodinger quantum mechanics wherein hermiticity of the momentum operator requires the physical Hilbert space to be made up of states which are suitable wavepackets which vanish at spatial infinity, and rule out plane wave states with infinitely sharp momentum. From this perspective, the quantum evolution for spherically symmetric gravity is thus physically unitary; and the boundary condition guaranteeing unitarity is much milder than, say, “Quantum Censorship” requirement of vanishing wavefunction at the boundary.

II. SPHERICALLY SYMMETRIC INITIAL DATA AND CANONICAL QUANTIZATION

We start by following Ref. [4], wherein modulo gauge invariance, the Lie-algebra-valued connection 1-form for spherically symmetric configurations is

\[ A = cT_3d\alpha + (aT_1 + bT_2)d\theta + (-bT_1 + aT_2)\sin\theta d\varphi + T_3\cos\theta d\varphi, \]

where \(a, b, c\) are constants on space-like Cauchy surfaces, and \(T_a\) are the generators of \(SO(3)\). Equivalently,

\[
\begin{align*}
A_1 &= ad\theta - b\sin\theta d\varphi, \\
A_2 &= bd\theta + a\sin\theta d\varphi, \\
A_3 &= c\cos\theta d\varphi;
\end{align*}
\]

and the corresponding dreibein components

\[
\begin{align*}
&e_1 = \omega_a d\theta - \omega_b \sin\theta d\varphi, \\
&e_2 = \omega_b d\theta + \omega_a \sin\theta d\varphi, \\
&e_3 = \omega_c d\alpha;
\end{align*}
\]

are related to the densitized triad by

\[
E = p_c T_3 \sin\theta \frac{\partial}{\partial \alpha} + (p_a T_1 + p_b T_2) \sin\theta \frac{\partial}{\partial \theta} + (-p_b T_1 + p_a T_2) \sin\theta \frac{\partial}{\partial \varphi},
\]

with

\[
\omega_a = \sqrt{p_c p_a}, \quad \omega_b = \sqrt{p_c p_b}, \quad \omega_c = \frac{\text{sgn}(p_c) \sqrt{p_a^2 + p_b^2}}{\sqrt{p_c}}.
\]

The dreibein determines the compatible torsionless spin connection as \(\Gamma = \cos\theta T_3 d\varphi\); and the extrinsic curvature 1-form \(K\) is then

\[
\gamma K := A - \Gamma
\]

and the corresponding field strength components are

\[
\begin{align*}
F_1 &= -b c \cos\theta d\alpha \land d\varphi, \\
F_2 &= b c \cos\theta d\alpha \land d\varphi, \\
F_3 &= (a^2 + b^2 - 1) \sin\theta d\theta \land d\varphi.
\end{align*}
\]

In the Ashtekar-Barbero formulation, the super-Hamiltonian is

\[
\int d^3x N e^{-1} \left[ \epsilon_i^j F_{ab}^i E_j^a E_k^b - 2(1+\gamma^2) K_{iab} K_{ikb} E_i^a E_j^b \right],
\]

wherein \(e := \sqrt{\text{det} E} \text{sgn}(\text{det} E)\). Using the explicit forms of \(F, K\) and \(E\) above, the super-Hamiltonian constraint simplifies to

\[
\mathcal{H} \propto -i c p_c (a p_a + b p_b) + \left( p_a^2 + p_b^2 \right) (a^2 + b^2 + \gamma^2) \approx 0;
\]

and the symplectic form of the spherically symmetric sector is

\[
\Omega = \frac{1}{2\gamma G} (2\alpha^2 - \delta p_a + 2\delta b - \delta p_b + \delta c \land \delta p_c). \tag{9}
\]

We next make following change of notations:

\[
p_a \equiv x; \quad p_b \equiv y; \quad p_c \equiv z; \tag{10}
\]

and invoke the canonical quantization rules

\[
\frac{2a}{2\gamma G} \equiv \frac{\hbar}{i} \frac{\partial}{\partial x} \equiv \hat{p}_x; \quad \frac{2b}{2\gamma G} \equiv \frac{\hbar}{i} \frac{\partial}{\partial y} \equiv \hat{p}_y; \quad \frac{c}{2\gamma G} \equiv \frac{\hbar}{i} \frac{\partial}{\partial z} \equiv \hat{p}_z.
\]

Consequently, the super-Hamiltonian constraint becomes

\[
\frac{1}{2} \left( \hat{p}_x \hat{p}_x + y \hat{p}_y \right) + (x^2 + y^2) \left( \frac{\hat{p}_x^2 + \hat{p}_y^2}{4} + \frac{x^2 + y^2}{4G^2} \right) = 0. \tag{11}
\]

Parametrizing the variables by \(x = r \cos \vartheta, y = r \sin \vartheta\), results in \(x^2 + y^2 = r^2, x\hat{p}_x + y\hat{p}_y = r \hat{p}_r\). The quantum constraint acting on wavefunctions \(\Phi\) is thus

\[
\frac{1}{4} \left( \frac{\partial^2}{\partial r^2} + \frac{\partial^2}{\partial \vartheta^2} + \frac{\partial^2}{\partial \varphi^2} \right) + \frac{r^2}{4G^2} \Phi = 0. \tag{12}
\]

The Gauss law constraint \((ap_b - bp_a) \propto xp_y - yp_x = 0\) is obviously equivalent to invariance under \(\vartheta\)-rotations about the z-axis, which is solved precisely by independence of \(\Phi\) w.r.t. \(\vartheta\); and the diffeomorphism super-momentum constraint gives rise to no further restrictions. The Wheeler-DeWitt equation is thus

\[
\left[ -\frac{\partial^2}{\partial z^2} - \frac{\partial}{\partial z} \left( \frac{r^2}{4} \frac{\partial}{\partial r} \right) - \frac{1}{4} \frac{\partial}{\partial r} \left( \frac{r^2}{2} \frac{\partial}{\partial \vartheta} \right) + \frac{r^2}{4G^2} \right] \Phi(r, z) = 0. \tag{13}
\]

It is readily verified the equation is not separable in \(r\) and \(z\); nevertheless it may be checked that

\[
\Phi(r, z) = C \exp \left[ ik \frac{z^{1/2}}{4} + \frac{r^2}{4ik} \frac{z}{G^2} \right] \Phi(r, z). \tag{14}
\]
with $C, k_-$ being constants is an explicit solution which has an essential singularity at $z = 0$.

To discuss the complete set of solutions, the following redefinitions are crucial:

$$X_- = \sqrt{z}; \quad X_+ = \frac{r^2}{4\hbar^2 G^2 \sqrt{z}}. \quad (15)$$

which can be inverted to give

$$X_+ X_- = \frac{r^2}{4\hbar^2 G^2}; \quad z = X_2. \quad (16)$$

Consequently,

$$\frac{\partial}{\partial X_+} = \frac{\partial z}{\partial X_+} \frac{\partial}{\partial z} + \frac{\partial r}{\partial X_+} \frac{\partial}{\partial r} = \frac{(4\hbar^2 G^2)^{\sqrt{z}}}{2r} \frac{\partial}{\partial r},$$

$$\frac{\partial}{\partial X_-} = \frac{\partial z}{\partial X_-} \frac{\partial}{\partial z} + \frac{\partial r}{\partial X_-} \frac{\partial}{\partial r} = 2\sqrt{z} \frac{\partial}{\partial z} + \frac{r}{2\sqrt{z}} \frac{\partial}{\partial r}. \quad (17)$$

Owing to the operator identity, $-X_- \frac{\partial}{\partial X_+} \frac{\partial}{\partial X_-} = -\frac{\partial}{\partial X_+} + 2\frac{\partial}{\partial X_-}$, when expressed in the new variables $X_{\pm}$, the Wheeler-DeWitt constraint now reduces to

$$X_+ X_- \left[ -\frac{\partial}{\partial X_+} \frac{\partial}{\partial X_-} + 1 \right] \Phi(X_+, X_-) = 0. \quad (18)$$

Therefore we may consider instead the simple Klein-Gordon wave equation in light cone coordinates $X_{\pm} = \frac{1}{2}(X \pm T)$ with unit “mass” ($m = 1$) as the quantum evolution equation in superspace i.e.

$$\left[ -\frac{\partial}{\partial X_+} \frac{\partial}{\partial X_-} + 1 \right] \Phi(X_+, X_-) = \left[ \frac{\partial^2}{\partial T^2} - \frac{\partial^2}{\partial X_-^2} + 1 \right] \Phi = 0. \quad (19)$$

Plane waves $\Phi(X_+, X_-) = \exp(ik_+X_+ + ik_-X_-)$ (with the mass shell condition $k_+k_- = 1 = 0$) and their superpositions are solutions of this Klein-Gordon equation.

### III. SEMICLASSICAL LIMIT, HAMILTON-JACOBI EQUATIONS, AND SCHWARZSCHILD BLACK HOLES

The semiclassical limit can be addressed in a general context by considering the usual semiclassical states $\hat{A} \exp(\Phi)$; with $\hat{A}$ being a slow varying function, and $\Phi$ satisfying the Hamilton-Jacobi equation, $\frac{\partial \Phi}{\partial X_+} \frac{\partial S}{\partial X_-} + 1 = 0$. The method of separation of variables[12] with $S(X_+, X_-) = S_+(X_+) + S_-(X_-)$ leads to $\frac{\partial S_+}{\partial X_+} \frac{\partial S_-}{\partial X_-} + 1 = 0$. This implies that $\frac{\partial S_+}{\partial X_+} = p_+$ are constants related by $p_+ = -1/p_-$, thus yielding, up to a constant, the Hamilton function $S(X_+, X_-) = p_+X_+ + p_-X_-$. These semiclassical states may therefore be identified with plane waves $A \exp(i(k_+X_+ + k_-X_-))$ where $p_\pm = \hbar k_\pm$.

We may furthermore map precisely the equation of motion for plane wave solutions to classical black holes by studying the correspondence of the classical initial data to the Hamilton-Jacobi theory. To achieve this, we note that the familiar Schwarzschild metric is

$$ds^2 = -c_0^2 + a^2 + c_2^2 + c_3^2$$

$$= -(1 - \frac{2GM}{R})dt^2 + (1 - \frac{2GM}{R})^{-1}dR^2 + R^2(d\theta^2 + \sin \theta d\phi^2). \quad (20)$$

Interior to the horizon ($0 < R < 2GM$), we may use constant-$R$ Cauchy surfaces and adopt the vierbein $e^A = (\frac{dR}{\sqrt{(2GM/R)^3}} + \frac{R}{\sqrt{2}}(d\theta - \sin \theta d\phi), \frac{d}{\sqrt{2}}(d\theta + \sin \theta d\phi), \sqrt{(2GM/R)^3} - 1) dt)$. Computing the Ashtekar potential from $A_a = \gamma \omega_{a}^0 - \frac{1}{2}e_{abc}\omega^b$, where $\omega_{AB}$ is the torsionless spin connection results in

$$A_1 = \gamma \sqrt{\frac{2GM}{R}} \frac{1}{d\theta - \sin \theta d\phi},$$

$$A_2 = \gamma \sqrt{\frac{2GM}{R}} \frac{1}{d\theta + \sin \theta d\phi},$$

$$A_3 = -\gamma \frac{GM}{R^2} dt + \cos \theta d\phi. \quad (21)$$

Direct comparisons on constant-$R$ Cauchy hypersurface with Eqs. (2) and (3) yield

$$\omega_a = \omega_b = \frac{R}{\sqrt{2}} \omega_c da = \frac{R}{\sqrt{2}} da = \sqrt{(2GM/R)^3} - 1) dt \quad (22)$$

$$a = b = \gamma \sqrt{\frac{2GM}{R}} - 1); \quad c da = -\gamma \frac{GM}{R^2} dt \quad (23)$$

It follows that $X_2 = z = p_+ = \omega_x^2 + \omega_y^2 = R^2$. To infer the equation of motion from the Hamilton-Jacobi theory, we note that on plane wave states $\Phi \propto e^{(ik_+X_+ + ik_-X_-)}$

$$\tilde{p}_x \Phi = \frac{(2X_+ + \hbar k_+)}{r} \Phi; \quad \tilde{p}_- \Phi = \frac{k_-X_- - k_+X_+}{2X_-^2} \hbar \Phi. \quad (24)$$

Thus semiclassically,

$$\gamma^2 \frac{2GM}{X_-} - 1 = a^2 + b^2 = (\gamma G)^2 (p_x^2 + p_y^2)$$

$$= (\gamma G)^2 \frac{1}{r} \left( p_x r p_y \right)$$

$$\sim (\gamma G)^2 \frac{p_+^2}{\gamma k_+^2 X_+/X_-}, \quad (25)$$

yielding, as expected, the “straight-line trajectory” equation of motion,

$$X_+ = \frac{1}{k_+^2} (2GM - X_-), \quad (26)$$

which is independent of $\gamma$. Furthermore, $\frac{r}{\sqrt{2}} da = \sqrt{(2GM/R)^3} - 1) dt$ yields the relation $\frac{r}{\sqrt{2}} dt = \frac{k_-}{2GM} \hbar$ on applying the equation of motion. Eqs. (23) and (24) imply

$$-\gamma \frac{GM}{X_-^2} \left( \frac{dt}{da} \right) = c = 2\gamma G p_z = \gamma G \frac{k_- X_- - k_+ X_+}{X_-^2} \hbar, \quad (27)$$
which is consistently true taking into account \( \frac{d\mu}{dx} = \frac{4GM}{x^3} \) and the equation of motion \([25]\).

### IV. RINDLER SUPERSPACE

It should be noted that \( X_\pm \geq 0 \). Thus the associated superspace is not the whole of Minkowski spacetime; rather, it is precisely the first Rindler wedge, which can be parametrized by \((\xi \geq 0, -\infty < \tau < \infty)\), with \( X = \xi \cosh \tau \) and \( T = \xi \sinh \tau \), and endowed with the metric
ds^2 = dT^2 - dX^2 = \xi^2 d\tau^2 - d\xi^2. \quad (28)

Correspondence between the boundaries of the Rindler wedge and physical horizons and singularities of black holes can be established. Referring to the equation of motion addressed in the previous section, black hole horizons and singularities occur at \((X_- = R = 2GM, X_+ = \frac{1}{k^2}(2GM - X_-) = 0)\) and \((X_- = 0, X_+ = \frac{1}{k^2}(2GM - X_-) = 2GM \) respectively. Thus as we span over the range of black hole masses \( M \), we observe that the lower boundary \((X_- = 2GM, X_+ = 0)\) and upper boundary \((X_- = 0, \frac{2GM}{k^2})\) of the Rindler wedge correspond precisely to the physical classical black hole horizons and singularities.

On this Rindler wedge the Klein-Gordon equation becomes

\[
\left[ \frac{1}{\xi^2} \frac{\partial^2}{\partial \tau^2} - \frac{\partial^2}{\partial \xi^2} - \frac{1}{\xi} \frac{\partial}{\partial \xi} + 1 \right] \Phi(\xi, \tau) = 0. \quad (29)
\]

Writing the solutions as \( \Phi_\omega(\xi, \tau) = e^{-i\omega \tau} C_\omega(\xi) \), the equation separates into,

\[
i \frac{\partial}{\partial \tau} \Phi_\omega = \omega \Phi_\omega, \quad (30)
\]

and the modified Bessel equation

\[
\left[ \xi^2 \frac{\partial^2}{\partial \xi^2} + \frac{\partial}{\partial \xi} - (\xi^2 - \omega^2) \right] C_\omega(\xi) = 0. \quad (31)
\]

The orthonormal modes can be expressed explicitly as \( \Phi_\omega(\xi, \tau) = \sqrt{\frac{1-e^{-2\omega \tau}}{2\pi}} B_\omega(\xi, \tau) \), wherein the “Minkowski Bessel modes” \([13]\) are given by \( B_\omega = \frac{1}{\xi} e^{\frac{\pi}{2} K_{i\omega}(\xi)} e^{-i\omega \tau} \), with \( K_{i\omega}(\xi) \) being the MacDonald functions (Bessel function with imaginary argument) of imaginary order. These modes \( \Phi_\omega \) are orthonormal, \( \langle \Phi_\omega(\xi, \tau) | \Phi_\omega(\xi, \tau) \rangle = \delta(\omega - \omega') \), with respect to the Klein-Gordon inner product \([11]\) for the Rindler wedge,

\[
\langle \Phi_\omega | \Phi_\omega' \rangle = \int_0^\infty d\xi \Phi_\omega^* (\xi, \tau) \Phi_\omega' (\xi, \tau) \frac{d\xi}{\xi} \quad (32)
\]

For solutions of the Klein-Gordon equation \( \Phi \), the current density \( j^\mu = i\sqrt{|g|} g^{\mu\nu} (\Phi^* \partial_\nu \Phi - (\partial_\nu \Phi^*) \Phi) \) (with \( \mu = 0 \) denoting \( \tau \) and \( \xi \) respectively, and \( g^{\mu\nu} = \text{diag} (\frac{1}{\xi^2}, -1) \)), \( \sqrt{|g|} = \xi \) obeys the continuity equation \( \partial_\mu j^\mu = 0 \). The Minkowski Bessel modes can in fact be understood as the “rapidity Fourier transform” of plane wave solutions \( e^{i(k_x X + i k_t T)} \) \([12]\) i.e.

\[
B_\omega(\xi, \tau) = \frac{1}{2\pi} \int_0^{\infty} e^{i(k_x X - \omega T)} e^{-i\omega \eta} d\eta = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i(\xi \sinh(\eta - \tau))} e^{-i\omega \eta} d\eta, \quad (33)
\]

where \( k_\pm = k \mp \omega \), and in terms of \( \eta \) (the rapidity variable), \( k = \sinh \eta, \omega = \cosh \eta \). The mass shell condition \( k^2 + k_-^2 = 0 \iff \omega^2 = k^2 = k_+^2 \) for our particular case of unit mass \((m = 1)\). As discussed, these plane wave solutions (which are the inverse rapidity Fourier transform wavepackets of \( B_\omega \), correspond to Schwarzschild black holes. However, as it is well-known in Klein-Gordon theory which is second order in time, positive-definite current density for a generic superpositions cannot be guaranteed \([13]\).

It should be noted that while “bosonic” scalar solutions of Eq.(29) are the first to come to mind, the possibility of “fermionic” solutions of the Wheeler-DeWitt constraint should not be ignored. To wit, we investigate the Dirac equation which is first-order in superspace intrinsic time on the Rindler wedge,

\[
(i\sl{\partial} - m) \Psi = [\gamma^\mu (i\partial_\mu + \Omega_\mu) - m] \Psi = [\gamma^0 i\partial_\tau + \gamma^1 i\gamma^0 = \gamma^2 \partial_\xi - m] \Psi = 0, \quad (34)
\]

with \( \Omega \) being the spin connection compatible with Rindler space vielbein \( E^{0.1}_x \), \( dx^\mu = (\xi d\tau, d\xi) \). Moreover, the Lorentz-invariant current density \( \sl{\bar{J}}^\mu = \text{det}(E) \sl{\bar{\gamma}}^\gamma \gamma^\mu \sl{\bar{\psi}} \) now obeys \( \partial_\mu \sl{\bar{J}}^\mu = 0 \) and \( \sl{\bar{J}}^0 = \text{det}(E) E^0_\tau \sl{\bar{\gamma}}^\tau \sl{\bar{\gamma}}^4 \sl{\bar{\psi}} = \sl{\bar{\psi}} \Psi \) is also positive-definite. Orthonormal modes with respect to the inner product \( \langle \sl{\bar{\psi}} | \Psi \rangle = \int_0^{\infty} \sl{\bar{\psi}}^\dagger \Psi d\xi \) can be constructed \([14, 17]\); and they can be expressed as \( \sl{\bar{\psi}}(\tau, \xi) = N_\nu e^{-i\nu \tau} \sl{\bar{K}}_{\nu}(\xi) \chi_\nu \), with the Dirac equation yielding (for \( m = 1 \)) the restriction \((1 + i\gamma^1) \chi = 0 \) on the constant spinor \( \chi \). It is possible to choose \( \chi = \chi^+ + \chi^- \), with orthonormal \( \pm \)-eigenspinors, \( \chi^\pm \), of \( \gamma^0 \gamma^1 \); and the normalization constant is then fixed to be \( N_\nu = \sqrt{\frac{\cosh(\pi \nu)}{\pi}} \). With \( \gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \) and \( \gamma^1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \), the general solution of \((1 + i\gamma^1) \chi = 0 \) has \( \chi^\pm = \begin{pmatrix} a_\pm \\ \mp ia_\pm \end{pmatrix} \) and \( a_- = -ia_+ \). An explicit pair is \( \chi^+ = \frac{i\chi}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix} \).
V. QUANTUM UNITARITY DESPITE THE PRESENCE OF APPARENT CLASSICAL SINGULARITIES

With respect to the inner product discussed above, the Dirac Hamiltonian operator on the Rindler wedge, $H_D = -i\gamma^\nu \partial_\nu - \frac{m^2}{2} + m\gamma^0$, is hermitian i.e. $\langle H_D \Psi | \Psi \rangle = \langle \Psi | H_D \Psi \rangle$ iff for all $-\infty < \tau < \infty, \lim_{\xi \to 0} \gamma^{\alpha} \Psi(\tau, \xi)^\dagger \gamma^\beta \Psi(\tau, \xi) = 0$. In an investigation of the Unruh effect, Ori[17] conjectured that the fermionic solutions should therefore obey $\lim_{\xi \to 0} \Psi(\xi) = 0$. In our present context of quantum gravity, it is intriguing to observe that this last equation is a boundary condition which guarantees hermiticity of the hamiltonian operator and thus unitarity of the quantum evolution with respect to superspace intrinsic time despite the threat of apparent classical black hole singularities.

We next discuss the nature of this boundary condition in a general context. The asymptotic behavior of $K_n(\xi)$ is quite simple:

$$
\lim_{\xi \to 0} \xi^{\frac{1}{2}} K_{\nu \pm \frac{1}{2}}(\xi) = \sqrt{2} \Gamma\left(\frac{1}{2} \pm i\nu\right) \xi^{\mp i\nu}. \tag{35}
$$

Moreover, $|\Gamma\left(\frac{1}{2} \pm i\nu\right)|^2 = \frac{\pi}{\cosh(\pi\nu)} = \frac{1}{\pi(N\nu)^2}$, and $\Theta(\nu) := \arg\{\Gamma\left(\frac{1}{2} \pm i\nu\right)\} = (\pm \nu)F\left(\frac{1}{2}\right) + \sum_{n=0}^{\infty} \left[\left(\frac{\pm \nu}{2 + \pi\nu}\right) - \tan^{-1}\left(\frac{\pm \nu}{2 + \pi\nu}\right)\right]$, where the digamma function $F\left(\frac{1}{2}\right) = -\gamma_{EM} - 2\ln 2$ and $\gamma_{EM}$ is the Euler-Mascheroni constant.

Since $\{\Psi_{\nu}(\tau, \xi)\}$ forms an orthonormal set, the boundary condition for unitarity implies the restriction (on $f(\nu)$) on a generic state $\Psi(\tau, \xi)$ which is

$$
0 = \lim_{\xi \to 0} \xi^{\frac{1}{2}} \Psi(\tau, \xi) = \sqrt{2} \Gamma\left(\frac{1}{2} \pm i\nu\right) \xi^{\mp i\nu},
$$

$$
\lim_{\xi \to 0} \sqrt{2} \Gamma\left(\frac{1}{2} \pm i\nu\right) \xi^{\mp i\nu} \int_{-\infty}^{\infty} d\nu f(\nu) \Psi_{\nu}(\tau, \xi) = 0.
$$

This results in two conditions:

$$
\lim_{\nu \to 0} \int_{-\infty}^{\infty} d\nu f(\nu) e^{i\nu \tau} e^{i\nu \omega} = 0; \tag{37}
$$

wherein $X_{\pm} = \frac{\xi}{\sqrt{2}} e^{\pm \tau}$ has been used, and for $-\infty < \tau < \infty$, the limit $\xi \to 0$ implies both $X_{\pm} \to 0$. Thus the boundary condition is equivalent to requiring the Fourier transform of $f(\nu)e^{\pm i\nu \tau}$ to vanish at $\pm \infty$. Since $e^{i\nu \tau}$ is oscillatory (rather than sharply peaked), the boundary condition can be satisfied by a rather generic wavepacket with $f(\nu)$ whose Fourier transform vanishes at $\pm \infty$ (explicitly, for instance, by $f(\nu)$ being Gaussian). However, it should be pointed out that infinitely sharp energy eigenstates i.e. $f(\nu) \propto \delta(\nu - \nu_0)$ which correspond to the usual fermionic Rindler modes $\Psi_{\nu_0}$ discussed previously fail to satisfy the boundary condition. An analogous situation happens in free non-relativistic quantum mechanics wherein hermiticity of the momentum operator requires a physical Hilbert space of suitable wavepackets which vanish at spatial infinity, and rule out plane wave states with infinitely sharp momentum. From this perspective the boundary condition guaranteeing quantum unitarity in our present context of spherically symmetric gravity holds for rather generic wavepackets.

It can also be verified explicitly that a semiclassical black hole state is a wavepacket (in $\nu$) which satisfies the boundary condition explicitly. To wit we note that the energy eigenstates $\Psi_{\nu}(\tau, \xi)$ satisfy the associated Klein-Gordon equation,

$$
0 = (i\partial_t + m)(i\partial_t - m)\Psi_{\nu} = \left[-\frac{1}{\xi^2} (\partial_\tau^2 + \gamma^{\alpha} \partial_\tau^\alpha + \frac{1}{4}) + \partial_\xi^2 + \frac{1}{\xi} \partial_\xi - m^2\right] \Psi_{\nu} = \left[-\frac{1}{\xi^2} e^{-\frac{\nu}{\xi}} (\partial_\tau^2 e^{-\frac{\nu}{\xi}}) + \partial_\xi^2 + \frac{1}{\xi} \partial_\xi - m^2\right] \Psi_{\nu}. \tag{38}
$$

This means that, in general, if $\Psi$ is a solution of the Dirac equation, the fermionic Lorentz-boosted solution $\Phi_{\nu}(\tau, \xi) = e^{\frac{i\nu}{\xi}} \Psi_{\nu}(\tau, \xi)$ will solve our previous Klein-Gordon equation (39). The underlying reason is that Rindler space is flat, thus the spin connection can be gauged away (here by a $\tau$-dependent transformation) and the corresponding transformed fermionic solution of Dirac equation will therefore solve the Klein-Gordon equation. According to our previous analysis following Eq. (24) which is still valid in the fermionic case, semiclassical black hole states are Minkowski plane wave modes, and these can be written (in terms of the rapidity $\eta$, with $k = \sinh \eta, \omega = \cosh \eta$) as

$$
P_{\eta}^{\nu}(X, T) = \frac{1}{2\sqrt{2\pi}} (e^{i\nu \partial_\tau} + i \gamma^\gamma \partial_X + m) e^{i(kX - \omega T)} \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \frac{1}{2\sqrt{2\pi}} (e^{\frac{\eta}{2} \chi^-} + ie^{\frac{\eta}{2} \chi^+}) e^{i(kX - \omega T)} = \frac{1}{2\sqrt{2\pi}} (e^{\frac{\eta}{2} \chi^-} + ie^{\frac{\eta}{2} \chi^+}) e^{i(kX - \omega T)}. \tag{39}
$$

Thus the semiclassical plane wave black hole solution which solves the Dirac equation (24) in the Rindler wedge is just given by the inverse Fourier transform $e^{-\frac{\nu}{\xi}} P_{\nu}^{\nu}(X, T) = 2\pi \int_{-\infty}^{\infty} \frac{e^{\nu \tau}}{2\pi} \Psi_{\nu}(X, T) e^{i\nu \tau} d\nu$ wherein $\hat{X} = \xi \cosh \tau, \hat{T} = \xi \sinh \tau$. This is a wavepacket with $f_0(\nu) = \frac{\nu}{2\pi} e^{i(\nu \tau + \frac{\nu}{2})}$ which quite clearly satisfies the boundary condition.

VI. FURTHER REMARKS

The limitations of classical General Relativity are saliently exposed by the occurrence of singularities in
the theory. While singular potentials do not necessarily pose problems to quantum theory (e.g. in elementary quantum mechanics, situations with delta and Coulomb potentials are tractable and even exactly solvable), classical curvature singularities need not manifest themselves in the quantum context as singular potentials. The quantum theory of spherically symmetric 4-dimensional General Relativity is a simple model whose classical sector is made up of only black hole solutions to Einstein’s theory. In this sense it is precisely a “quantum theory of Schwarzschild black holes”, just as non-relativistic free particle Schrodinger quantum mechanics is a quantum theory of free Newtonian particle mechanics. Amusingly, this analogy is true even in the details, for Schwarzschild solutions are mapped to straight-line trajectories of free motion in flat superspace!

The arena for quantum gravity is not space-time but superspace.[11,12] In this article we adopt a conservative approach, by starting with traditional field variables and adhering to continuum physics; and discover that the corresponding minisuperspace is actually free of singularity and has a surprisingly simple structure. Quantum evolution takes place in the arena of a 2-dimensional Rindler wedge of signature (+,−), equipped with intrinsic superspace time which is the natural time variable of the Wheeler-DeWitt equation. The semiclassical limit derived from the quantum theory (in some sense a “quantum Birkhoff theorem”) is in complete agreement with Birkhoff’s classical result of Schwarzschild solutions for Einstein’s field equations. For these 4-manifold black holes which correspond to straight-line trajectories in superspace, intrinsic time \( \tau \) and \( R \) radial coordinate time of interior Schwarzschild solutions are monotonic functions of each other. Although the spin connection and momentum of Eqs. (21) and (24) for classical Schwarzschild black holes diverge at \( R = X_+ = 0 \), the equation of motion (26) in Minkowski, and also Rindler, coordinates is nevertheless well-defined. The singularity is at the extreme at which correspondence between semiclassical limit of quantum theory and classical Schwarzschild black holes breaks down. Although not pursued here, it is possible to study the extension of the Rindler superspace to the whole of Minkowski space-time and investigate “singularity traversal” and the continuation of the trajectories and their correspondence to, in general, complex 4-manifolds.

We should also comment upon some related deeper issues. Solutions of the Dirac equation associated with the Wheeler-DeWitt constraint were also investigated. In flat Rindler space-time, a solution of the former is also a solution of the latter. This is not necessarily true if superspace is not flat in the full theory [1]. But the existence of fermionic solutions, and “factorizations” of the Wheeler-DeWitt constraint, or its generalizations with and without supersymmetry, should not be ignored. The classical super-Hamiltonian constraint may be playing the analogous role of a “dispersion relation” which can be realized by more than one type of “particle” in the quantum context. Intriguingly too in the spherically symmetric sector our Klein-Gordon and Dirac equations are really the resultant quantum constraints of a massive free particle dispersion relation which is equivalent to the classical super-Hamiltonian constraint of General Relativity. For simplicity, we have adopted a finite-dimensional representation of the Lorentz group of superspace when the Dirac equation and its solutions were discussed explicitly. Since superspace comes with hyperbolic signature, the Lorentz group is non-compact and thus non-unitary for finite-dimensional representations. This has the drawback of the lack of unitary equivalence between states related by local Lorentz transformations in superspace. In quantum field theory, the related issue is resolved by assuming that these wavefunctions are operators acting upon physical states which belong to unitary infinite-dimensional representations. In the present context such a route would in fact result in “third quantization”. Barring this, the solutions here should already be considered as physical states of the theory; however, one is not forbidden to consider these solutions of the Dirac equation with infinite-dimensional unitary representations. This may in fact be closer to the full theory for which superspace is infinite-dimensional.[11]

There is a general belief that black holes are rudimentary objects in General Relativity. This is bolstered by the precise correspondence that can be established here between descriptions of Schwarzschild black holes and those of elementary free particle mechanics in superspace.

Acknowledgments

The research for this work has been supported in part by funds from the National Science Council of Taiwan under Grant No. NSC95-2112-M-006-011-MY3, and the National Center for Theoretical Sciences, Taiwan.

[7] V. Hussain and O. Winkler, Class. Quantum Grav. 22,
[18] It is possible to define another set of modes $\Phi_\omega \equiv \sqrt{2\omega}\Phi_\omega$ and a new inner product
\[
\langle \Phi'|\Phi \rangle \equiv \int_0^{\infty} \Phi'|\Phi d\xi.
\]
With respect to this, the modes $\Phi_\omega$ are orthonormal i.e. $\langle \Phi_\omega|\Phi_{\omega'} \rangle = \delta(\omega - \omega')$. But, unlike the earlier case, this new “probability” density is now positive-definite for generic states $\Phi = \int_\omega f(\omega)\Phi_\omega$. But it does not satisfy a continuity equation.
[19] The boundary of space-like hypersurfaces for $-\infty < \tau < \infty$ are at the origin with $\xi = 0$, and at $\xi = \infty$ where the fall-off behavior of $K_{\omega,\pm\frac{1}{2}}$ ensures that no further restriction arises. The hypersurfaces $\xi = 0, \tau = \pm\infty$ are light-like in superspace rather than space-like.