Three-geometry and reformulation of the Wheeler-DeWitt equation

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Abstract. A reformulation of the Wheeler-DeWitt equation which highlights the role of gauge-invariant three-geometry elements is presented. It is noted that the classical super-Hamiltonian of four-dimensional gravity as simplified by Ashtekar through the use of gauge potential and densitized triad variables can furthermore be succinctly expressed as a vanishing Poisson bracket involving three-geometry elements. This is discussed in the general setting of the Barbero extension of the theory with arbitrary non-vanishing value of the Immirzi parameter, and when a cosmological constant is also present. A proposed quantum constraint of density weight two which is polynomial in the basic conjugate variables is also demonstrated to correspond to a precise simple ordering of the operators, and may thus help to resolve the factor ordering ambiguity in the extrapolation from classical to quantum gravity. Alternative expression of a density weight one quantum constraint which may be more useful in the spin network context is also discussed, but this constraint is non-polynomial and is not motivated by factor ordering. The article also highlights the fact that while the volume operator has become a preeminent object in the current manifestation of loop quantum gravity, the volume element and the Chern-Simons functional can be of equal significance, and need not be mutually exclusive. Both these fundamental objects appear explicitly in the reformulation of the Wheeler-DeWitt constraint.

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1. Introductory remarks

The program of non-perturbative canonical quantization of gravity attempts to overcome the perturbative non-renormalizability of Einstein’s theory by constructing exact background-independent quantum geometry. Much excitement and insight have stemmed from Ashtekar’s reformulation of the Hamiltonian theory and the simplification of the constraints through the use of gauge connection and densitized triad variables[1]. Conceptually, the distinction between geometrodynamics and gauge dynamics is bridged by the identification of the densitized triad, \( \tilde{E}^{ia} \) - from which the metric is a derived composite - as the momentum conjugate to the gauge potential \( A^{ia} \). Most intriguing too is the conjunction of the fact the Lorentz group possesses self and anti-self-dual decompositions in four and only in four dimensions with the observation that the Ashtekar-Sen connection is precisely the pullback to the Cauchy surface of the self-(or anti-self)-dual projection of spin connection[2]. The infusion of loop variables[3] and subsequently spin network states[4] have also proved fruitful, and have yielded discrete spectra for well-defined area and volume operators[5]. Indeed by virtue of being area and volume eigenstates, states based upon spin networks -the latter originally introduced by Penrose to explore quantum geometry[6]- are now prominent in the current manifestation of loop quantum gravity. To the extent that exact states and rigorous results are needed, simplifications of the classical and corresponding quantum constraints are crucial steps indeed. These include Ashtekar’s original simplification as well as Thiemann’s observation that \( \epsilon_{abc} \epsilon_{ijk} \tilde{E}^{ia} \tilde{E}^{jb} \) in the super-Hamiltonian constraint is proportional to the Poisson bracket between the connection and the volume operator[7][1]

In this article a reformulation of the super-Hamiltonian constraint and its associated Wheeler-DeWitt Equation is presented. It is noted that the classical super-Hamiltonian of four-dimensional gravity as simplified by Ashtekar through the use of gauge potential and densitized triad variables can furthermore be succinctly expressed as a vanishing Poisson bracket between fundamental invariants. This is discussed in the general setting of the Barbero extension of the theory[8], with arbitrary non-vanishing value of the Immirzi parameter \( \gamma \)[9], and when a cosmological constant \( \lambda \) is also present. The observation naturally suggests a reformulation of quantum gravity wherein the Wheeler-DeWitt equation is reduced to the requirement of the vanishing of the corresponding commutator. Alternative ways of expressing the quantum constraint will also be discussed.

It has long been known that 3-dimensional diffeomorphism invariance would require the quantum states to be functionals of 3-geometries[10][11]. It may therefore be surmised that, albeit a nontrivial endeavor, it ought to be possible to express the Wheeler-DeWitt Equation of the full theory in terms of explicit 3-geometry elements. However the constraint is also required to be satisfied at each point on the Cauchy surface. Both these requirements are remarkably realized in the reformulation here in

\( SO(3) \) indices are denoted by lower case Latin letters at the beginning of the alphabet from a to h, while spatial indices on the 3-dimensional Cauchy surface are denoted by Latin letters from i onwards.
that the Wheeler-DeWitt constraint is equivalent to the vanishing of the commutator between \( \hat{v}^2(\vec{x}) = \det(\hat{E}^{\alpha}(\vec{x})) \) at each point and a combination of the integrals involving the extrinsic curvature and the Chern-Simons functional of the gauge connection. The reformulation not only highlights the role of gauge-invariant 3-geometry elements in the Wheeler-DeWitt Equation, but also spells out which specific superspace functionals are involved.

The Chern-Simons functional has served as a fertile link between quantum field theories of three and four dimensions. In General Relativity with Ashtekar variables, it has the additional significance of being the carrier of information of both intrinsic and extrinsic curvatures (see for instance Eq.(6) later). Although it is not as extensively explored in spin networks in present formulations of loop quantum gravity as in quantum field theories, the expectation that the Chern-Simons invariant has a very significant, and even direct, role in 4-dimensional quantum gravity is in fact borne out by the discussions in this article. It should also be emphasized that while the volume operator has become a preeminent object in the current manifestation of loop quantum gravity, the volume element and the Chern-Simons functional can be of equal significance and need not be mutually exclusive. Both these fundamental objects appear explicitly in the reformulation of the Wheeler-DeWitt constraint.

There is another feature of the reformulation which is worth emphasizing. Unlike the Gauss Law and super-momentum constraints which are kinematic and have straightforward group-theoretic interpretations, the factor ordering ambiguity of the non-commuting variables in quantum super-Hamiltonian constraint is a more intricate matter. There is no unique prescription for defining a quantum theory from its classical correspondence. Thus the factor ordering problem has to be decided through other means; for instance, through mathematical consistency (sometimes expediency) and the absence of quantum anomalies. Even so, these may or may not yield a unique ordering. With complex Ashtekar variables, Hermiticity of the super-Hamiltonian too cannot be adopted as a criterion. Often when dealing with the factor ordering of a complicated constraint, an initial motivation is needed; and a specific ordering is assumed first before checking the consistency of the composite operator.

A proposed quantum constraint of density weight two motivated by 3-geometry considerations here and which is also polynomial in the basic conjugate variables will be demonstrated to correspond to a precise simple ordering of the quantum operators, and may thus help to resolve the factor ordering ambiguity in the extrapolation from classical to quantum gravity. However, it has also been pointed out background independent field theories are ultraviolet self-regulating if the constraint weight is equal to one but not for other density weights\[7\]. To wit, we also discuss an alternative density weight one quantum constraint which may be more useful in the spin network context; but this expression is non-polynomial in the basic conjugate variables, and it is not motivated by factor ordering.
2. Reformulation of the classical super-Hamiltonian constraint

Starting with the fundamental conjugate pair and Poisson bracket,
\[ \{ \tilde{E}^{ia}(\vec{x}), k_{bj}(\vec{y}) \}_{\text{P.B.}} = \beta \delta^i_j \delta^a_b \delta^3(x - y), \] (1)

with \( \beta \equiv (8\pi G/c^3) = 8\pi l_p^2 \) (where \( l_p \) is the Planck length), the Barbero extension\[^8\] of
\( k_{ia} = E^j_a K_{ij} \) to a generalized Ashtekar \( SO(3) \) gauge connection\[^8\],
\[ A_{ai}^\gamma \equiv \gamma k_{ai} + \Gamma_{ai}, \] (2)
yields
\[ \{ \tilde{E}^{ia}(\vec{x}), A_{bj}^\gamma(\vec{y}) \}_{\text{P.B.}} = \gamma \beta \delta^i_j \delta^a_b \delta^3(x - y). \] (3)

In the above \( \Gamma_a \) is the torsionless \((dE_a + \epsilon_{abc} \Gamma_b \wedge E_c = 0)\) connection compatible with the dreibein 1-form \( E_a = E_{ai} dx^i \) on the Cauchy manifold \( M \); and \( K_{ij} \) denotes the extrinsic curvature. In terms of 3-dimensional torsionless spin connection \( \omega_{ab} \), \( \Gamma_{ai} = -\frac{1}{2} \epsilon_{abc} \omega_{abc} \).

Gauss Law constraint for \( SO(3) \) gauge invariance is equivalent to
\[ 0 \approx \epsilon^{ab} c k_{ib} \tilde{E}^{ic} = \frac{1}{\gamma} D^A \tilde{E}^{ia}. \] (4)

\( D^A \) means the covariant derivative with respect to the connection \( A \); when there is no danger of confusion, we shall suppress the \( \gamma \) index in the connection and denote it simply by \( A_{ai} \), with implicit dependence on the Immirzi parameter \( \gamma \).

Four-dimensional General Relativity as a theory of the conjugate pair of densitized triad and gauge variables, \((\tilde{E}^{ia}, A_{ai})\), seems to be anchored on a few fundamental physical objects: the volume element \((\tilde{v})\), the Chern-Simons functional of the gauge potential \((C[A])\), and invariants constructed from the extrinsic curvature \((K \text{ and } D)\). All are gauge invariant, but apart from \( \tilde{v} \) which is a tensor density, they are in addition also invariant under three-dimensional diffeomorphisms i.e. they are elements of 3-geometry.

The definitions for these objects will be discussed below:
\[ \tilde{v}(\vec{x}) \equiv \sqrt[3]{\epsilon_{abc} \epsilon_{ijk} \tilde{E}^{ia}(\vec{x}) \tilde{E}^{jb}(\vec{x}) \tilde{E}^{kc}(\vec{x})} = | \det E_{ai} |. \] (5)

Its integral over the Cauchy surface, \( M \), is the volume, \( V = \int_M \tilde{v}(\vec{x}) d^3x \). The Chern-Simons functional is
\[ C[A^\gamma] \equiv \frac{1}{2} \int_M \left( A^a \wedge dA_a + \frac{1}{3} \epsilon^{abc} A_a \wedge A_b \wedge A_c \right) \] 
\[ = C[\Gamma] + \gamma \int_M R^a_a \wedge k_a + \frac{\gamma^2}{2} \int_M k^a \wedge (D^F k)_a \] 
\[ + \frac{\gamma^3}{3!} \int_M \epsilon^{abc} k_a \wedge k_b \wedge k_c. \] (6)

\[ § \] It has been pointed out that this generalized connection is however not the pullback onto spatial slices of a four-dimensional spin connection unless \( \gamma = \pm i \).
In the last equality we have expanded the Chern-Simons functional for $A^\gamma = \gamma k + \Gamma$ in terms of $C[\Gamma]$ which is the Chern-Simons functional for the connection $\Gamma$, and higher order terms; and $R_\Gamma$ is the curvature 2-form of the connection $\Gamma$.

Moreover, $\mathcal{K}$ can be written totally in terms of the volume $V$ and $A$ from the observation that the dreibein $E_{ai}$ (inverse of the triad $E^{ia}$) can also be expressed as

$$E_{ai} = \frac{2}{\gamma^2} \{V, A_{ai}\}_{P.B.}. \quad (8)$$

With the above definitions and the fundamental relation of Eq. (3), it follows that the Poisson brackets below are true:

$$\{\tilde{E}^{ia}(\bar{x}), C[A^\gamma]\}_{P.B.} = (\beta\gamma)\tilde{B}^{ia}(\bar{x})$$

$$\{\tilde{E}^{ia}(\bar{x}), \mathcal{K}\}_{P.B.} = \beta\tilde{E}^{ia}(\bar{x})$$

$$\{\tilde{v}^2, \mathcal{K}\}_{P.B.} = 3\beta\tilde{v}^2$$

$$\{\tilde{v}^2, C[A^\gamma]\}_{P.B.} = \frac{\beta\gamma}{2} \varepsilon_{abc}\varepsilon_{ijk}\tilde{E}^{ia}\tilde{E}^{jb}\tilde{B}^{kc}_\gamma$$

$$\{\mathcal{K}, C[A^\gamma]\}_{P.B.} = \beta \int_M (\tilde{B}^{ia} k_{ai}) d^3x$$

$$\{\tilde{v}^2, \frac{\gamma}{3}\mathcal{K}\}_{P.B.} = \frac{\beta}{2} \varepsilon_{abc}\varepsilon_{ijk}\tilde{E}^{ia}\tilde{E}^{jb}(\tilde{B}^{kc}_\gamma + \frac{\gamma}{3}\tilde{E}^{kc}). \quad (9)$$

$$\tilde{B}^{ia} = \frac{1}{2}\varepsilon^{ijk}F^{a^\gamma}_{jk}$$ is the magnetic field of $A^\gamma$; with

$$F^{\gamma}_{aij} = R^\gamma_{aij} + \gamma(D^i k_{aj} - D^j k_{ai}) + \gamma^2 \varepsilon^{abc} k_{bi} k_{cj}. \quad (10)$$

Following Ref. [3], the usual ADM super-Hamiltonian constraint $\tilde{H} \approx 0$ can be rewritten with

$$\tilde{v}\tilde{H} \propto (\tilde{v}^2)[Tr(K^2) - (Tr K)^2 - R^\omega + 2\lambda]$$

$$= \varepsilon_{abc}\varepsilon_{ijk}\tilde{E}^{ia}\tilde{E}^{jb}\tilde{B}^{kc}_\gamma - \frac{(1 + \gamma^2)}{2}\varepsilon^{cde}\varepsilon^{klm} k_{dl} k_{em} + \frac{\gamma}{3}\tilde{E}^{kc}]. \quad (11)$$

$R^\omega$ is just the Ricci scalar curvature of the spin connection $\omega$. In the above equality we have used $\varepsilon_{ab} c\tilde{E}^{ia}\tilde{E}^{jb}D^i k_{cj} \propto \varepsilon_{ijk}\tilde{E}^{ia}(D^j k_{k})_a = 0$ by virtue of $\varepsilon_{ijk}D^j \tilde{E}^{kc}_a = 0$ and the Gauss Law constraint which implies $\varepsilon^{ijk}E_{aj} k^a_k = 0$.

We may introduce another gauge-invariant 3-geometry element (essentially the integral of determinant of $k_{ia}$):

$$\mathcal{D} \equiv \frac{1}{3!} \int_M \varepsilon^{abc} k_a \wedge k_b \wedge k_c = \frac{1}{3!} \int_M \varepsilon^{ijk} \varepsilon^{abc} k_{ai} k_{bj} k_{ck} d^3x \quad (12)$$

Its characteristic feature is that it satisfies $\frac{\delta\mathcal{D}}{\delta A_{ai}} = \tilde{B}^{ia}$ if $\partial M = 0$; wherein $\tilde{B}^{ia}$ is the non-Abelian $SO(3)$ magnetic field of $A_{ai}$. If $M$ is with boundary, additional considerations need to be taken into account e.g. the imposition of appropriate boundary conditions, or consideration of whether the addition of a supplementary boundary term to $\mathcal{C}$ can again render the super-Hamiltonian to be a Poisson bracket. On the other hand, one can treat $\partial M = 0$ as a predictive element of the formulation discussed here.
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which has the properties
\[
\{ \hat{E}^{ia}, \mathcal{D} \}_\text{P.B.} = \frac{\beta}{2} \epsilon^{abc} \epsilon^{ijk} k_{bj} k_{kc} \\
\{ \tilde{v}^2, \mathcal{D} \}_\text{P.B.} = \frac{\beta}{4} \epsilon_{abc} \epsilon_{ijk} \hat{E}^{ia} \hat{E}^{jb} \epsilon^{cde} \epsilon^{klm} k_{dl} k_{em}.
\]

Using this last identity and the final Poisson bracket in (9), we can verify that it is possible to express the complete super-Hamiltonian constraint as a vanishing Poisson bracket
\[
0 \approx \tilde{v} \mathcal{H} \propto \{ \tilde{v}^2, \frac{\mathcal{C}[A^\gamma]}{\gamma} + \frac{\lambda}{3} \mathcal{K} - (1 + \gamma^2) \mathcal{D} \}_\text{P.B.}
\]

The self-dual and antiself-dual specializations correspond to \( \gamma = \mp i \) which simplify the expression by eliminating \( \mathcal{D} \) which is cubic in \( k \). Szabados first noted that the super-Hamiltonian constraint is expressible as a Poisson bracket for the further specialization of vanishing cosmological constant [13].

3. Quantization and reformulation of the Wheeler-DeWitt equation

Even though we may invoke Poisson bracket-quantum commutator correspondence \( \{ \cdot, \cdot \}_\text{P.B.} \mapsto (i \hbar)^{-1} [\cdot, \cdot] \), there is no unique prescription for defining a quantum theory from its classical correspondence. The previous observations naturally suggest defining four-dimensional non-perturbative quantum General Relativity as a theory of the conjugate pair \( (\hat{E}^{ia}, A_{ai}) \) with super-Hamiltonian constraint imposed as
\[
[\tilde{v}^2, \frac{\mathcal{C}[A^\gamma]}{\gamma} + \frac{\lambda}{3} \mathcal{K} - (1 + \gamma^2) \mathcal{D}] = 0;
\]

together with the requirement of invariance of the theory under three-dimensional diffeomorphisms and \( SO(3) \) gauge transformations of \( (\hat{E}, A) \).

This reformulation is bolstered by the existence of a precise factor ordering of the non-commuting operators which realizes the quantum Wheeler-DeWitt constraint. To check this we may utilize repeatedly for composite operators the commutator identities
\[
[X Y, Z] = X[Y, Z] + [X, Z] Y \quad \text{and} \quad [X, Y Z] = [X, Y] Z + Y[X, Z].
\]

Thus
\[
[\tilde{v}^2 = \frac{1}{3!} \epsilon_{abc} \epsilon_{ijk} \hat{E}^{ia} \hat{E}^{jb} \hat{E}^{kc}, \mathcal{C}[A]] = (8 \pi i \ell_p^2 \gamma) \frac{1}{3!} \epsilon_{ijk} \epsilon_{abc} (\hat{E}^{ia} \hat{E}^{jb} \hat{B}^{kc} + \hat{E}^{ia} \hat{B}^{jb} \hat{E}^{kc} + \hat{B}^{ia} \hat{E}^{jb} \hat{E}^{kc}),
\]

if we take into account \( [\hat{E}^{ia}, \hat{C}[A]] = (8 \pi i \ell_p^2 \gamma) \hat{B}^{ia} \) which follows from the fundamental commutation relation \( [\hat{E}^{ia}(\vec{x}), \hat{A}_{bj}(\vec{y})] = 8 \pi i \gamma \ell_p^2 \delta^i_j \delta^a_b \delta^3(\vec{x} - \vec{y}) \). In a similar manner,
\[
[\tilde{v}^2, \mathcal{D}] = (8 \pi i \ell_p^2 \gamma) \frac{1}{3!} \epsilon_{abc} \epsilon_{ijk} \epsilon_{cde} \frac{1}{2} \epsilon^{klm} (\hat{E}^{ia} \hat{E}^{jb} k_{dl} k_{em} + \hat{E}^{ia} k_{dl} k_{em} \hat{E}^{jb} + k_{dl} k_{em} \hat{E}^{ia} \hat{E}^{jb}).
\]

Thus Eq. (15) corresponds to the precise factor ordering of the super-Hamiltonian constraint which is
\[
\tilde{H} \propto \epsilon_{abc} \epsilon_{ijk} \frac{1}{3} (\hat{E}^{ia} \hat{E}^{jb} \hat{B}^{kc} + \hat{E}^{ia} \hat{B}^{jb} \hat{E}^{kc} + \hat{B}^{ia} \hat{E}^{jb} \hat{E}^{kc}) + \frac{\lambda}{3} \hat{E}^{ia} \hat{E}^{jb} \hat{E}^{kc} - \frac{(1 + \gamma^2)}{2} \frac{1}{3} \epsilon^{cde} \epsilon^{klm} (\hat{E}^{ia} \hat{E}^{jb} k_{dl} k_{em} + \hat{E}^{ia} k_{dl} k_{em} \hat{E}^{jb} + k_{dl} k_{em} \hat{E}^{ia} \hat{E}^{jb})).
From the general quantum Wheeler-DeWitt constraint of Eq.(15) the original Ashtekar self-dual and antiself-dual specializations (with $\gamma = \mp i$) may therefore be expressed succinctly as

$$[\hat{\bar{v}}^2(x), \hat{C}[A]] = \pm \lambda (8\pi l_p^2) \hat{\bar{v}}^2(x).$$

(19)

This corresponds to the symmetric ordering, wherein the operators $\hat{E}$ and $\hat{B}$ appear in every permutation in the constraint with equal weight for each combination (as indicated in Eq.(18) with $\gamma = \mp i$). In the classical limit with commuting operators, (18) reduces to (11). Note also that although it is possible to express $\hat{v}^2$ on the R.H.S. of Eq.(19) as the commutator

$$\left\{ \hat{v}^2, K \right\},$$

it may not be expedient to do so (we shall discuss the related issues shortly) if we do not insist on writing the constraint as a vanishing commutator relation. With Dirac quantization, the reformulated Wheeler-DeWitt Equation is therefore

$$[\hat{\bar{v}}^2, \hat{C}[A]]|_{\Psi_{\text{phys}}} = \pm \lambda (8\pi l_p^2) \hat{\bar{v}}^2|_{\Psi_{\text{phys}}}.$$  

(20)

It has a number of remarkable properties:

1. This equation for the full theory (not just a particular minisuperspace sector) is not merely symbolic but is in fact expressed explicitly in terms of the gauge-invariant 3-geometry element which is none other than the Chern-Simons functional of the Ashtekar connection. This is to be contrasted with the traditional formulation with metric variables, wherein the Equation was symbolically expressed in terms of metric variables 

$$\left\{ \frac{\delta}{\delta G_{ij}}, (R(G) - 2\lambda) \right\}|_{\Psi_{\text{phys}}} = 0.$$  

(21)

2. There are now no factor ordering ambiguities for the composite operators $\hat{v}^2 = \frac{1}{3!} \epsilon_{abcd} \hat{E}^a_{ij} \hat{E}^b jk \hat{E}^c_{k l}$ and $C[A] = \frac{1}{8} \int_M (A^a \wedge dA_a + \frac{1}{3!} \epsilon^{abc} A_a \wedge A_b \wedge A_c)$ which have clear geometric meanings; and each of which is made up of commuting variables (this too is true of the operator $D$ when $\gamma \neq \mp i$ is adopted in the more general context of Eq.(15)). Whatever ordering “ambiguities” that were present in super-Hamiltonian constraint in the transition from classical to quantum theory have been decided by requiring that the Wheeler-DeWitt Equation is expressible in terms of these 3-geometry elements. (3) The operators $\hat{v}^2, C[A]$, and also the combination $\lambda l_p^2$, which appear in the reformulated Wheeler-DeWitt Equation are now all dimensionless.

So far the reformulation is not confined to specific representations in the quantum theory, and we have used $\hat{v}^2$, which is polynomial in $\hat{E}$, instead of $\hat{v}$ to express the constraint in polynomial form. This results in a super-Hamiltonian of density weight

\[\hat{K}, \psi\rceil_{\Psi_{\text{phys}}} = \{\hat{\bar{v}}^2, K\} = \beta \hat{\bar{v}}^2\]

so apart from a multiplicative constant, $K$ is in fact conjugate to the variable $\ln \hat{\bar{v}}$. In the quantum context $\hat{K}$ is thus proportional to the generator of translations in $\ln \hat{\bar{v}} = \ln |\det E_{ai}|[14]$, which is a fundamental function of the superspace “intrinsic time variable” $\ln |\det E_{ai}|$ discovered by DeWitt in his seminal study of canonical quantum gravity and the signature of the supermetric many years ago[11].

1 The physical meaning of the operator $\hat{K}$ is not readily apparent, but we may nevertheless deduce its connection to “intrinsic time” in quantum gravity from the Poisson bracket \(\{\hat{v}^2(x), K\}\) $= 3\beta \hat{v}^2(x)$; so apart from a multiplicative constant, $K$ is in fact conjugate to the variable $\ln \hat{v}$. In the quantum context $\hat{K}$ is thus proportional to the generator of translations in $\ln \hat{\bar{v}} = \ln |\det E_{ai}|[14]$. This variable is furthermore a monotonic function of the superspace “intrinsic time variable” $\ln |\det E_{ai}|$ discovered by DeWitt in his seminal study of canonical quantum gravity and the signature of the supermetric many years ago[11].

1 Ref. [14] contains related discussions on the reformulation.
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Thus it may be desirable to use \( \tilde{v} \) instead and conjecture an alternative Wheeler-DeWitt equation of the form

\[
[\hat{\tilde{v}}, \hat{C}[A]]|\Psi_{\text{phys}}\rangle = \pm \lambda (4\pi l_p^2) \tilde{v} |\Psi_{\text{phys}}\rangle.
\]

(22)

Although \( \tilde{v} \) and the associated density weight one constraint is non-polynomial in the basic variables, this version of the constraint does lead to a necessary condition involving the volume operator \( \hat{V} = \int \hat{\tilde{v}} d^3x \) that is

\[
[\hat{V}, \hat{C}[A]]|\Psi_{\text{phys}}\rangle = \pm \lambda (4\pi l_p^2) \hat{V} |\Psi_{\text{phys}}\rangle.
\]

(23)

Explicit realizations and representations of eigenstates of the volume operator can be precisely associated with spin network states\[5\]. Furthermore the volume operator is classically real and in the quantum context should be Hermitian, implying its eigenstates form a complete basis; and all physical states can be expanded in terms of these spin network volume eigenstates. Instead of treating \( \tilde{v} \) as

\[
\sqrt{\frac{1}{3!} \varepsilon_{abc} \varepsilon_{ijk} \tilde{E}^{ia} \tilde{E}^{jb} \tilde{E}^{kc}},
\]

an alternative expression in terms of the dreibein \( E_{ai} \) (inverse to the triad) which is also of interest is

\[
\tilde{v} = \frac{1}{3!} \varepsilon_{ijk} \varepsilon^{abc} E_{ai} E_{bj} E_{ck}
\]

with each dreibein operator (as suggested in Ref.\[7\]) expressed as \( E_{ai} = \frac{1}{4\pi l_p^2} [V, A_{ai}] \). Since the constraint Eq.(22) is now non-polynomial in the basic conjugate variables, the quantum constraint does not correspond to a simple factor ordering of the basic variables as for the case of the polynomial constraint in Eq.(15). Thus Eq.(22) discussed here is not motivated by simple factor ordering. Although the classical Poisson bracket

\[
\{\tilde{v}, C[A]\}_{P.B.} = \pm \lambda (\frac{4\pi G}{c^3}) \tilde{v}
\]

is true, and may appear as a motivation for Eq.(22), there is no rigorous justification for promoting Poisson brackets between composite operators to quantum commutator relations.

It may also be worth pointing out that the Chern-Simons operator should have the characteristic property that for a 2-surface \( S \) spanned by arbitrary \( \varepsilon_{abc} E^b \wedge E^c \), its commutator with the basic area operator \( A_a \equiv \frac{i}{2} \int_S \varepsilon_{abc} E^b \wedge E^c \) should yield (in units of \( 8\pi l_p^2 \)) the total non-Abelian magnetic flux traversing \( S \) i.e. \( [A_a, C] = (8\pi l_p^2 \gamma) \int_S F_a \).

4. Further comments

It is also interesting to investigate the situation for more conventional phase space variables. If \((\tilde{E}, k)\) is adopted as the fundamental pair, we may revert to the last equality in the expansion of Eq.(6) for \( C[A = \gamma k + \Gamma] \). However, a simplification occurs here in that \( C[\Gamma] \) commutes with \( \tilde{v} \) and \([\tilde{v}^2, \int_M k_a \wedge (D^F k)^a]\) also vanishes if the Gauss Law constraint holds. Furthermore the \( \gamma^2 \) (cubic in \( k \)) term of \( C[\gamma k + \Gamma]/\gamma \) (please see Eq.(6)) cancels the \( \gamma^2 D \) term in Eq.(15). Thus the ADM super-Hamiltonian constraint with \((\tilde{E}, k)\) as fundamental variables may be simplified to

\[
[\tilde{v}^2, (\int_M R_\gamma^2 \wedge k_a) + \frac{\lambda}{3} \mathcal{K} - \mathcal{D}] = 0;
\]

(24)
and we may use the last equality of Eq. (7) to express $\mathcal{K}$ in terms of $(\tilde{E}, k)$. However, $R^i_i$ remains a non-polynomial function of $\tilde{E}$; but the formalism may become simplify greatly in the minisuperspace context when spatially flat slices can be chosen.

Since $[\tilde{v}^2, \mathcal{K}] \propto \tilde{v}^2$, there is a choice in using only $\tilde{v}^2, \mathcal{C}$ (together with $\mathcal{D}$ for $(\gamma \neq \mp i)$ in the reformulation if we do not insist on expressing the constraint as a commutator. As indicated in the later part of the Section III, this may be desirable in some contexts. However if we wish to utilize $\mathcal{K}$, as alluded to earlier in Section I earlier, it is possible to rewrite $\mathcal{K}$ in terms of $A$ and $V$ through Eq.(8), yielding $\mathcal{K} \propto \int_M \tilde{\epsilon}^{ijk} [A^a_i, V](D^A_j[A_k, V])_a d^3x$.

A choice of $\gamma$ other than the self-dual or antiself-dual values $\mp i$ appears to introduce formidable complications in the commutator, regardless of whether $(\tilde{E}, A)$ or $(\tilde{E}, k)$ is used as the basic conjugate pair. In the former instance, we should either express $k_{ai} = (A_{ai} - \Gamma_{ai})/\gamma$ with non-polynomial $\tilde{E}$ dependence in $\Gamma$ in $\mathcal{D}$; or overcome the non-polynomial nature by writing $k_{ai} \propto [A^a_i, \mathcal{K}]$ with $\mathcal{K}$ expressed as in the previous paragraph, or resort to an alternative expression for this composite operator wherein the loop representation of the operator $\mathcal{K} \propto [V, \int \tilde{B}^a[A_{a}, V] d^3x]$ has also been addressed in detail in Refs.[7, 15].

As it is well known, both the intrinsic and extrinsic curvatures contribute to the Wheeler-DeWitt constraint. In the ADM formulation the intrinsic curvature is related to the metric, while its conjugate momentum is associated with the extrinsic curvature. For the reformuldated equation, $\tilde{v}$ can be related only to the intrinsic curvature, but $\mathcal{C}[A = \gamma k + \Gamma]$ is a carrier of information of both types of curvature.

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