Novel Topological Invariant in the $U(1)$ Gauge Field Theory

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Abstract

Based on the decomposition of $U(1)$ gauge potential theory and the $\phi$-mapping topological current theory, the three-dimensional knot invariant and a four-dimensional new topological invariant are discussed in the $U(1)$ gauge field.

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I. INTRODUCTION

Knotlike configurations appear in a variety of physical, chemical and biological scenarios, including the structure of elementary particles [1, 2, 3], the early universe cosmology [4, 5, 6], the Bose-Einstein condensation [7, 8], the polymer folding [9] and the DNA replication, transcription and recombination [10]. Generally, knotlike configurations are researched by using the Jones polynomial and path integral methods in physics [11, 12, 13]. Meanwhile, it is known that for a knot family there are very important characteristic numbers to describe its topology [14]. So in the research of the knotlike configurations in physics, one should also pay much attention to the geometric and topological characteristics.

As is well known, magnetic helicity has attracted a great deal of attention since it provides a lower bound for the magnetic energy [15, 16, 17] and is a very robust invariant. Magnetic helicity can be considered as a special case of the cross helicity, which is defined for two distinct magnetic fields \( B \) and \( B' \)

\[
H(B, B') = \int_{M^3} A \wedge dA'.
\]  

So the question arises whether there exists a generalization to electromagnetic field, especially to the physically interesting four-dimensional case. Principally, the electromagnetic field can be investigated in the frame of a \( U(1) \) gauge field theory. In this paper, we intend to study the geometric and topological characteristic of knotlike vortex lines in the electromagnetic field by using the decomposition of \( U(1) \) gauge potential theory and the \( \phi \)-mapping theory. The gauge potential decomposition theory [18] establishes a direct relationship between differential geometry and topology, and the \( \phi \)-mapping topological current theory provides an important method in investigating the inner topological structure of field configurations [19, 20, 21]. The purpose of the present paper is twofold. First we show that there exist vortex lines in the \( U(1) \) gauge field and obtain a knot invariant for knotlike vortex lines. The second purpose is to extend to four-dimensional case and give a new topological invariant.

This paper is arranged as follows. In Sec. III using the \( U(1) \) gauge potential decomposition theory and the \( \phi \)-mapping topological current theory, we discuss the inner topological structure of the knotlike vortex lines in the \( U(1) \) gauge field and get a knot invariant. In Sec. III a new topological invariant is obtained in a four-dimensional case. The conclusion of this paper is given in Sec. IV.
II. KNOT INVARIANT IN THE U(1) GAUGE FIELD

We know that the complex scalar field \( \phi \) can be regarded as the complex representation of a two-dimensional vector field \( \vec{\phi} = (\phi^1, \phi^2) \) over the base space, i.e. \( \phi = \phi^1 + i\phi^2 \), where \( \phi^a \) \((a = 1, 2)\) are real functions. The covariant derivative of \( \phi \) is defined as

\[
D_\mu \phi = \partial_\mu \phi - iA_\mu \phi, \quad (\mu = 1, 2, 3),
\]

where \( A_\mu \) is the \( U(1) \) gauge potential. The \( U(1) \) gauge field tensor is given by

\[
F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.
\]

In the subsection we will show that in the \( U(1) \) gauge field theory there exist the vortex line structures. Defining the unit vector \( n^a \) as

\[
n^a = \frac{\phi^a}{||\phi||}, \quad ||\phi||^2 = \phi\phi^*,
\]

one can prove that \[20\], \( A_\mu \) can be decomposed in terms of \( n^a \): \( A_\mu = \epsilon^{ab} n^a \partial_\mu n^b - \partial_\mu \theta \), where \( \theta \) is a phase factor. Since the term \((\partial_\mu \theta)\) does not contribute to the gauge field tensor, \( A_\mu \) and \( F_{\mu\nu} \) can be rewritten as

\[
A_\mu = \epsilon^{ab} n^a \partial_\mu n^b, \quad F_{\mu\nu} = 2\epsilon^{ab} \partial_\mu n^a \partial_\nu n^b.
\]

In the following, we will use these above decomposition expressions Eq. (4) to research the topological feature in the \( U(1) \) gauge field theory. As an analog of Eq. (1), define a action \( I \) in terms of two distinct electromagnetic field tensors \( F^i \) and \( F^j \) by

\[
I(F^i, F^j) = \frac{1}{4\pi} \int_M A^i \wedge F^j = \frac{1}{8\pi} \int_M \epsilon^{\mu\nu\lambda} A^i_\mu F^j_{\nu\lambda} d^3x.
\]

Substituting \( F^i = F^j \) in Eq. (5) returns the expression for Chern-Simon action \[3\]. To simplify the action, we introduce a topological current

\[
J^\mu = \frac{1}{8\pi} \epsilon^{\mu\nu\lambda} F^j_{\nu\lambda}.
\]

Then Eq. (5) can be expressed as

\[
I(F^i, F^j) = \int_M A^i_\mu J^\mu d^3x.
\]

In order to explore the inner topological structure of the action \( I \), we should first investigate that of \( J^\mu \). Then taking account of Eq. (4), using \( \partial_\mu n^a = \partial_\mu \phi^a/||\phi|| + \phi^a \partial_\mu (1/||\phi||) \) and the Green’s function relation in \( \phi \) space \[21\], one can prove that

\[
J^\mu = \delta^2(\vec{\phi}) D^\mu \left( \frac{\vec{\phi}}{x} \right),
\]
where
\[
D^\mu(\frac{\phi}{x}) = \frac{1}{2} \epsilon^{\mu\nu\rho} \epsilon^{ab} \partial_\nu \phi^a \partial_\rho \phi^b
\]  

(9)
is the Jacobian vector. The expression Eq. (8) provides an important conclusion:

\[
J^\mu \begin{cases} 
0 & \text{if and only if } \bar{\phi} \neq 0. \\
\neq 0 & \text{if and only if } \bar{\phi} = 0.
\end{cases}
\]  

(10)

So it is necessary to study the zero points of the \( \bar{\phi} \) field to determine the nonzero solutions of \( J^\mu \). The implicit function theory \([22]\) shows that under the regular condition \( D^\mu(\phi/x) \neq 0 \), the general solutions of

\[
\phi^1(x^1, x^2, x^3) = 0, \quad \phi^2(x^1, x^2, x^3) = 0
\]  

(11)
can be expressed as

\[
x^1 = x^1_k(s), \quad x^2 = x^2_k(s), \quad x^3 = x^3_k(s)
\]  

(12)

which represent \( N \) isolated singular strings \( L_k \) with string parameter \( s \) \((k = 1, 2, ..., N)\). These singular string solutions are just the vortex lines in the \( U(1) \) gauge field.

In the \( \delta \)-function theory \([23]\), one can prove that in three-dimensional space

\[
\delta^2(\bar{\phi}) = \sum_{k=1}^{N} \beta_k \int_{L_k} \frac{\delta^3(\bar{x} - \bar{x}_k(s))}{|D(\bar{x}/u)|_{\Sigma_k}} ds,
\]  

(13)

where

\[
D(\phi/u)_{\Sigma_k} = \frac{1}{2} \epsilon^{\mu\nu\rho} \epsilon_{mn} (\partial \phi^m / \partial u^n)(\partial \phi^n / \partial u^\nu),
\]  

(14)

and \( \Sigma_k \) is the \( k \)th planar element transverse to \( L_k \) with local coordinates \((u^1, u^2)\). The positive integer \( \beta_k \) is the Hopf index of the \( \phi \)-mapping, which means that when \( \bar{x} \) covers the neighborhood of the zero point \( \bar{x}_k(s) \) once, the vector field \( \bar{\phi} \) covers the corresponding region in \( \phi \) space \( \beta_k \) times. Meanwhile taking notice of Eqs. (9) and (14), the direction vector of \( L_k \) is given \([21]\) by

\[
\frac{dx^\mu}{ds} \bigg|_{\bar{x}_k} = \frac{D^\mu(\phi/x)}{D(\phi/u)} \bigg|_{\bar{x}_k}.
\]  

(15)

Then from Eqs. (13) and (15), we obtain the inner structure of \( J^\mu \)

\[
J^\mu = \delta^2(\bar{\phi}) D^\mu(\frac{\phi}{x}) = \sum_{k=1}^{N} W_k \int_{L_k} \frac{dx^\mu}{ds} \delta^3(\bar{x} - \bar{x}_k(s)) ds,
\]  

(16)
in which \( W_k = \beta_k \eta_k \) is the winding number of \( \vec{\varphi} \) around \( L_k \), with \( \eta_k = \text{sgn}D(\phi/u)z_k = \pm 1 \) being the Brouwer degree of \( \phi \)-mapping. It can be seen that when these \( U(1) \) vortex lines are \( N \) closed curves, i.e., a family of \( N \) knots \( \gamma_k(k = 1, ..., N) \), and taking account of Eqs. (7) and (16), the action \( I(F^i, F^j) \) is reexpressed as

\[
I(F^i, F^j) = \int_M A_\mu^i J^\mu dx^3 = \sum_{k=1}^N W_k \oint_{\gamma_k} A_\mu^i dx^\mu. \quad (17)
\]

This is a very important expression. Consider the \( U(1) \) gauge transformation of \( A_\mu \):

\[
A'^\mu_\mu = A_\mu + \partial_\mu \theta, \quad (18)
\]

where \( \theta \) is a phase factor denoting the \( U(1) \) transformation. It can be shown that the \( (\partial_\mu \theta) \) term in Eq. (18) contributes nothing to the integral \( I \), that is, the expression Eq. (17) is invariant under the \( U(1) \) gauge transformation. Meanwhile we know that \( I \) is independent of the metric \( g_{\mu\nu} \). Therefore one can conclude that the action \( I(F^i, F^j) \) in Eq. (5) is a topological invariant for the knotlike vortex lines in the \( U(1) \) gauge field theory.

III. THE NEW FOUR-DIMENSIONAL TOPOLOGICAL INVARIANT

The action \( I(F^i, F^j) \) is of three-dimensional nature due to the integral over \( M^3 \), so the question arises whether there exists a generalization to higher dimensions, especially to the physically interesting four-dimensional case [24]. In this section, we construct a new action in four dimensions, which is based on the so-called Novikov invariant [25].

In the following the field tensor \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \) is also interpreted as the \( U(1) \) gauge field over a simply connected domain \( M^4 \), in which \( A_\mu \) is the \( U(1) \) gauge potential. The new action is defined in terms of three distinct electromagnetic field tensor \( F^i, F^j \) and \( F^k \) [24]

\[
N(F^i, F^j, F^k) = \frac{1}{4\pi} \int_{M^4} A^i \wedge A^j \wedge F^k = \frac{1}{8\pi} \int_{M^4} \epsilon^{\mu\nu\lambda\rho} A^i_\mu A^j_\nu F^k_{\lambda\rho} d^4x, \quad (19)
\]

where \( \mu, \nu, \lambda, \rho = 1, 2, 3, 4 \). And then we will research the inner topological structure of the new action. Introduce a two-dimensional topological tensor current, which is denoted as [26]

\[
J^{\mu\nu} = \frac{1}{8\pi} \epsilon^{\mu\nu\lambda\rho} F^k_{\lambda\rho}. \quad (20)
\]

Substituting Eq. (20) into Eq. (19), the new action can be simplified as

\[
N(F^i, F^j, F^k) = \int_{M^4} A^i_\mu A^j_\nu J^{\mu\nu} d^4x. \quad (21)
\]
To explore the inner topological structure of the action $N(F^i F^j F^k)$, we research that of $J^{\mu \nu}$. Using the $\phi$-mapping topological theory [21], we can obtain

$$J^{\mu \nu} = \delta^2(\vec{\phi}) D^{\mu \nu}(\frac{\phi}{x}),$$  \hspace{1cm} (22)


$$D^{\mu \nu}(\frac{\phi}{x}) = \frac{1}{2} \epsilon^{\mu \nu \lambda \rho} \epsilon^a_b \partial_{\lambda} \phi^a \partial_{\rho} \phi^b.$$  \hspace{1cm} (23)

Considering the Eq. (22), we can come to an important conclusion

$$J^{\mu \nu} \begin{cases} = 0 & \text{if and only if } \vec{\phi} \neq 0. \\ \neq 0 & \text{if and only if } \vec{\phi} = 0. \end{cases}$$  \hspace{1cm} (24)

Now we should first study the zero points of $\vec{\phi}$ to determine the nonzero solutions of $J^{\mu \nu}$. The implicit function theory [22] shows that under the regular condition

$$D^{\mu \nu}(\phi/x) \neq 0,$$  \hspace{1cm} (25)

the general solutions of

$$\phi^1(x^1, x^2, x^3, x^4) = 0, \quad \phi^2(x^1, x^2, x^3, x^4) = 0$$  \hspace{1cm} (26)

can be expressed as

$$x^1 = x^1_i(\sigma), \quad x^2 = x^2_i(\sigma), \quad x^3 = x^3_i(\sigma), \quad x^4 = x^4_i(\sigma),$$  \hspace{1cm} (27)

which represent $N$ isolated singular sheets with surface parameter $\sigma$. Without loss of generality, we consider these singular sheet solutions bordering by two vortex lines with string parameter $s$ ($l = 1, 2, ..., N$). Generalizing Eq. (13) to four-dimensional case with two vortex lines $L_k$ and $L_{\gamma}$, $\delta(\vec{\phi})$ can be expressed as

$$\delta^2(\vec{\phi}) = \sum_{k=1}^{N} \sum_{l=1}^{N} \beta_k \beta_l \int_{L_k} \int_{L_l} \frac{\delta^2(\vec{x} - \vec{x}_k(s)) \delta^2(\vec{x} - \vec{x}_l(s)) \partial^2 s}{|D(\vec{u})|_{\Sigma_k} |D(\vec{v})|_{\Sigma_l}},$$  \hspace{1cm} (28)

where $\Sigma_k$ is the $k$th planar element transverse to $L_k$ with local coordinates $(u^1, u^2)$ and $\Sigma_l$ is transverse to $L_l$ with $(v^1, v^2)$. Taking notice of Eqs. (14) and (23), we can obtain

$$\frac{dx^\mu \wedge dx^\nu}{d^2 s} \bigg|_{\vec{x}_k, \vec{x}_l} = \frac{D^{\mu \nu}(\phi/x)}{D(\phi/u)|_{\Sigma_k} D(\phi/v)|_{\Sigma_l}}.$$  \hspace{1cm} (29)
Then from Eqs. (28) and (29), we get the inner structure of \( J^{\mu\nu} \)
\[
J^{\mu\nu} = \delta^2(\vec{\phi}) D^{\mu\nu}(\frac{\partial}{\partial x}) \\
= \sum_{k=1}^{N} \sum_{l=1}^{N} W_k W_l \int_{L_k} \int_{L_l} \frac{dx^\mu \wedge dx^{\nu}}{d^2 s} \\
\delta^2(\vec{x} - \vec{x}_k(s)) \delta^2(\vec{x} - \vec{x}_l(s)) d^2 s,
\]
in which \( W_k = \beta_k \eta_k \) is the winding number of \( \vec{\phi} \) around \( L_k \) with \( \eta_k = \text{sgn} D(\frac{\partial}{\partial x}) \Sigma_k = \pm 1 \) being the Brower degree of the \( \phi \)-mapping, while \( W_l = \beta_l \eta_l \) is the winding number corresponding to \( L_l \). When these vortex lines are closed curves, i.e. a family of knots \( \gamma_k, \gamma_l \) \( (k, l = 1 \cdots N) \), the new action Eq. (21) can be expressed as
\[
N(F^i, F^j, F^k) = \sum_{k=1}^{N} \sum_{l=1}^{N} W_k W_l \oint_{\gamma_k} A^i_\mu dx^\mu \wedge \oint_{\gamma_l} A^j_\nu dx^{\nu}.
\]
Considering the \( U(1) \) transformation of \( A \)
\[
A'^\mu = A^i_\mu + \partial_\mu \alpha, \quad A'^j_\nu = A^i_\nu + \partial_\nu \theta,
\]
one can seen that the \( \partial_\mu \alpha \) and \( \partial_\nu \theta \) contribute nothing to the action \( N(F^i, F^j, F^k) \). So the expression Eq. (31) is invariant under the gauge transformation. Meanwhile we know that \( N(F^i, F^j, F^k) \) is independent of the metric. Therefore one can conclude that \( N(F^i, F^j, F^k) \) is a topological invariant. Since the invariant \( N(F^i, F^j, F^k) \) is concerned with two families of knotlike vortex lines \( \gamma_k \) and \( \gamma_l \) \( (k, l = 1 \cdots N) \), we conjecture that it is invariant for the linkage of vortex lines. As follows, we will research the relationship between the new invariant and the linking numbers of the knots family. We should first express \( A^i_\mu \) in terms of the vector field which carries the geometric information of the linkage, namely, we need to decompose \( A^i_\mu \) and \( A^j_\nu \) in terms of another two-dimensional unit vector \( \vec{e} \) which is different from the two-dimensional vector \( \vec{n} \) in Sec. II. Define the Gauss mapping \( \vec{m} \)
\[
\vec{m}(\vec{y}, \vec{x}) = \frac{\vec{y} - \vec{x}}{||\vec{y} - \vec{x}||},
\]
where \( \vec{x} \) and \( \vec{y} \) are two points, respectively, on the knots \( \gamma_k \) and \( \gamma_l \). Denote the two-dimensional unit vector \( \vec{e} = \vec{e}(\vec{x}, \vec{y}) \) as
\[
e^a e^a = 1 \quad (a = 1, 2; \vec{e} \perp \vec{m}).
\]
Then according to Ref. [20], $A_{\mu}$ can be decomposed in terms of this two-dimensional vector $e^a$: $A_{\mu}^i = \epsilon^{ab} e^a \partial_\mu e^b - \partial_\mu \varphi$; $A_{\mu}^j = \epsilon^{ab} e^a \partial_\mu e^b - \partial_\mu \theta$, where $\varphi$ and $\theta$ are phase factors. Since the terms $\partial_\mu \varphi$ and $\partial_\mu \theta$ contribute nothing to the integral $N(F^i, F^j, F^k)$, $A_{\mu}$ can in fact be expressed as

$$A_{\mu}^i = \epsilon^{ab} e^a \partial_\mu e^b, \quad A_{\mu}^j = \epsilon^{ab} e^a \partial_\mu e^b.$$  \hspace{1cm} (35)

Substituting Eq. (35) into Eq. (31) and considering $\vec{x}$ and $\vec{y}$ are two points, respectively, on the knots $\gamma_k$ and $\gamma_l$, we have

$$N(F^i, F^j, F^k) = \frac{1}{4\pi} \sum_{k=1}^{N} \sum_{l=1}^{N} W_k W_l \oint_{\gamma_k} \oint_{\gamma_l} \epsilon^{ab} \partial_\mu e^a (\vec{x}, \vec{y}) \partial_\nu e^b (\vec{x}, \vec{y}) dx^\mu \wedge dy^\nu.$$  \hspace{1cm} (36)

Using the relation $\epsilon^{ab} \partial_\mu e^a \partial_\nu e^b = \frac{1}{2} \vec{m} \cdot (\partial_\mu \vec{m} \times \partial_\nu \vec{m})$, Eq. (36) is just written as

$$N(F^i, F^j, F^k) = \frac{1}{2} \sum_{k=1}^{N} \sum_{l=1}^{N} W_k W_l \oint_{\gamma_k} \oint_{\gamma_l} \vec{m} \cdot (\partial_\mu \vec{m} \times \partial_\nu \vec{m}) dx^\mu \wedge dy^\nu.$$  \hspace{1cm} (37)

Substituting Eq. (33) into Eq. (37), one can prove that

$$\frac{1}{4\pi} \sum_{k=1}^{N} \oint_{\gamma_k} \oint_{\gamma_l} \vec{m} \cdot (\partial_\mu \vec{m} \times \partial_\nu \vec{m}) dx^\mu \wedge dy^\nu = \frac{1}{4\pi} \epsilon_{\mu\nu\lambda} \oint_{\gamma_k} dx^\mu \oint_{\gamma_l} dy^\nu \frac{(x^\lambda - y^\lambda)}{|x - y|^3} = L(\gamma_k, \gamma_l)(k \neq l),$$

where $L(\gamma_k, \gamma_l)(k \neq l)$ is the Gauss linking number between $\gamma_k$ and $\gamma_l$ [27]. Therefore, we arrive at the important result

$$I = 2\pi \sum_{k=1}^{N} \sum_{l=1}^{N} W_k W_l L(\gamma_k, \gamma_l).$$  \hspace{1cm} (39)

This precise expression just reveals the relationship between the new invariant and the linking number of the knots family [28, 29]. Since the linking numbers are invariant characteristic numbers of the knotlike closed curves in topology, $N(F^i, F^j, F^k)$ is an important invariant required to describe the linkage of knotlike vortex lines in the $U(1)$ gauge field.

IV. CONCLUSION

In this paper, using the decomposition of $U(1)$ gauge potential theory and the $\phi$-mapping topological current theory, we obtain the topological structure of the knot invariant in the three-dimensional $U(1)$ gauge field and that of the link invariant in four-dimensional case.
We show that the action \( I(F^i, F^j) \) is a knot invariant for the knotlike vortex lines inhering in the \( U(1) \) gauge field. Furthermore, by generalizing to four-dimensional case, we construct a new action \( N(F^i, F^j, F^k) \) and discuss the topological properties of the new action. It is pointed out that the new action \( N(F^i, F^j, F^k) \) is a topological invariant for the knotlike vortex lines existing in the zeros of \( \vec{\phi} \) field. Finally, by introducing the Gauss mapping, we find out the relationship between the new invariant \( N(F^i, F^j, F^k) \) and the Gauss linking number. Therefore, we reveal that the new action is a link invariant for knotlike vortex lines in four dimensions.

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