Fidelity criterion for quantum-domain transmission and storage of coherent states beyond unit-gain constraint

Ryo Namiki, Masato Koashi, and Nobuyuki Imoto

CREST Research Team for Photonic Quantum Information,
Division of Materials Physics, Department of Materials Engineering Science,
Graduate school of Engineering Science, Osaka University, Toyonaka, Osaka 560-8531, Japan

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For the transmission and storage of coherent states, the tight upperbound of the average fidelity achievable by the measure-and-prepare scheme was shown to be $F_c = \frac{1}{2}$ by Hammerer et al. [Phys. Rev. Lett. 94, 150503 (2005)], and surpassing $F_c$ has been serving as an experimental success criterion for quantum teleportation/memory in continuous-variable quantum systems. We generalize this fidelity criterion to be suitable for non-unit-gain condition by considering attenuation/amplification of the coherent-state amplitude. The new criterion can be used for a non-ideal quantum memory and long distance quantum communication as well as quantum devices with amplification process. We also show that the optimal $N$-to-$\infty$ cloning machine of coherent states can be constructed with Gaussian operations.

In quantum information science [1], the manipulation of quantum system is considered to be a channel that transforms a set of quantum states to another set of quantum states. A fundamental distinction is posed on the channel whether it can be simulated by a measure-and-prepare (M&P) scheme or not. The M&P scheme implies that the output of the channel is produced merely based on the classical data processing from the measurement outcomes and that the channel action breaks quantum entanglement shared between the input of the channel and other systems [2]. Therefore, a natural benchmark for quantum-domain (QD) operation of a given experimental quantum manipulation is that the channel is outperforming any M&P scheme. Hammerer et al. [3] have established the criterion for QD operation of continuous-variable (CV) channels by proving a limit of the average fidelity achievable by the M&P schemes, $F_c$, assuming an input set of coherent states: Surpassing $F_c$ ensures the QD operation for transmission and storage of coherent states. This criterion gave a proof for the long-standing conjecture of CV quantum teleportation about $F_c$ [4, 5] and provided a firm foundation for the central claims of experimental CV quantum teleportation [6, 7] and quantum memory [8]. However, the application of this criterion is limited to the unit-gain (UG) channels where the coherent-state amplitudes of the input state and the output state are expected to be the same [4, 6].

Quantum memory for light (QM) is a challenging protocol [10, 11] which requires UG operation and involves not only storage of the states but also transfer of the states between different physical media, such as an optical system and an atomic ensemble. In any implementation [8, 11, 12, 13], it is likely that the effect of linear loss or damping of coherent-state amplitude becomes more significant as the storage period becomes longer. Therefore, some mechanism of gain adjustments seems to be necessary for the complete demonstration of QM. With gain control of the teleportation-based state transfer [11], an above-$F_c$ operation of QM has been reported [5]. Another possible solution for the gain control is to employ an amplifier [13], where it is shown that the quantum-limit phase-insensitive amplifier can be implemented by linear optics and homodyne detection [15]. However, for most of the present experimental approaches, to design QM being compatible with the UG condition is impractical. In addition, most of communication protocols are designed to be tolerant against losses in the optical fibers, and moderate losses in QMs used in such protocols should be acceptable as well. Hence, in a developing step, it is not essential to focus on the UG channel, but non-UG devices should be investigated as well. There are several efforts to ensure “quantum-regime” operation of the storage of light, particularly in the electromagnetically induced transparency (EIT) approaches [12, 14, 15]. For instance, storage of the squeezed state was experimentally demonstrated [17]. Moreover, there is a report of an EIT experiment with an estimation based on a different “teleportation criteria” [18]. However, these efforts need not ensure the QD operation. Therefore, it is widely useful to present an experimental criterion that ensures the QD operation of the non-UG quantum devices. These motivate us to generalize the fidelity criterion to be suitable for the non-UG condition, especially in the presence of loss.

Another example of lossy channel is found in CV quantum key distribution (QKD) using coherent states aimed at a long distance transmission [19, 20]. In QKD, the M&P scheme is an intercept-resend attack, and no secret key can be generated if the input-output relation is explained by an M&P scheme. Hence confirming the QD operation is an important prerequisite for any QKD [21].

The problem of the fidelity optimization over the M&P schemes is equivalent to a type of state estimation. This problem frequently appears in relation with the optimal cloning [22]. Recently it was shown that the state estimation is equivalent to the classical-limit of quantum cloning where the number of clones tends to infinity for any given ensemble of pure states [23]. Accordingly, the fidelity bound $F_c$ is obtained from the optimal fidelity of 1-to-$\infty$ cloning machine of coherent states where $N$-to-M cloning means $M$ imperfect copies are produced from the $N$ copies of the input. An interesting fact is that the optimal cloner is Gaussian in 1-to-$\infty$ case whereas the optimal 1-to-2 cloner is non-Gaussian [24]. For $N \geq 2$, whether the non-Gaussian cloning machine outperforms the Gaussian cloning machine in the $N$-to-$\infty$ case is an open question.

In this Letter, we provide a general classical boundary of the fidelity by considering attenuation/amplification of the coherent-state amplitude including the cases where $N$ copies of the coherent state are given. As criterion for QD operation, the application is not only for experiments of transfer, storage, and transmission with dissipation, but also for quantum devices with amplification process. From the $N$-copy case we prove that the optimal $N$-to-$\infty$ cloning machine of coherent states can be constructed with Gaussian operations.

We define the average fidelity for the transformation task from a set of input states $\{|\psi_x\rangle\}$ to a set of ideal output states $\{|\psi'_x\rangle\}$,

$$F = \sum_x p_x \langle \psi'_x \rangle E(|\psi_x\rangle \langle \psi_x|) |\psi'_x\rangle$$ (1)

where $p_x$ is the prior distribution of the input states and $E$ is
the channel action described by a completely positive trace-preserving (CPTP) map \([1]\). Whereas \(E\) is a physical map, the task \(\{\psi_x\} \rightarrow \{\psi'_x\}\) can be physically impossible (unphysical) map, such as perfect cloning for a set of non-orthogonal states \(\{|\psi_x\}\rightarrow \{|\psi_x\rangle^\otimes 2\}\). The channel is simulated by the M&P scheme when we can write

\[
E(\{|\psi_x\rangle\} = \sum_k (\psi_x|\hat{M}_k|\psi_x)\hat{\rho}_k \tag{2}
\]

where \(\{\hat{M}_k\}\) is a positive-operator valued measure (POVM) and \(\hat{\rho}_k\) is a density operator. The classical boundary of the average fidelity for the task \(\{\psi_x\} \rightarrow \{\psi'_x\}\) is defined by the optimization over the M&P schemes:

\[
F_c \equiv \sup_{\{\hat{M}_k\},\{\hat{\rho}_k\}} \sum_k p_x \text{Tr}\left(\hat{M}_k|\psi_x\rangle\langle\psi_x|\hat{\rho}_k|\psi_x\rangle\right). \tag{3}
\]

We consider a transformation task of the coherent states \(\langle|\alpha\rangle\rangle\) from \(N\) identical copies to a single copy with a modulated amplitude by the factor of \(\eta > 0\): \(\{|\alpha\rangle\rangle^\otimes N\} \rightarrow \{|\sqrt{\eta}\alpha\rangle\rangle\}\), and estimate the classical fidelity assuming the prior distribution of a symmetric Gaussian function

\[
p(\alpha) = \frac{\lambda}{\pi} \exp(-\lambda|\alpha|^2), \tag{4}
\]

where \(\{\hat{M}'_k\}\) is POVM for the single mode. This formula implies that

\[
F_{N,\eta,\lambda} = \frac{\sum_k \int d^2\alpha p(\alpha)|\langle\phi_\alpha|\rangle^2|\langle\sqrt{\eta}\alpha\rangle^2|^2}{\sum_k \int d^2\alpha p(\alpha)|\langle\phi_\alpha|\rangle^2|\langle\sqrt{\eta}\alpha\rangle^2|^2}, \tag{5}
\]

and thus the optimization problem for the \(N\)-copy case is reduced to the \(N = 1\) case when we take into account the factor \(\eta\).

In the following we show the upperbound on \(F_{1,\eta,\lambda}\), which immediately provides the upperbound on \(F_{N,\eta,\lambda}\) by relation \(7\). After that we prove the bound is achievable. The derivation of the upperbound is a straightforward extension of \(8\).

The original result is reproduced in the case \(\eta = 1\) and \(N = 1\).

Let us start with rewriting the formula \(9\) for \(N = 1\) as

\[
F_{1,\eta,\lambda} = \sup_{\{|\phi_\alpha\rangle\},\{|\chi_\alpha\rangle\}} \sum_y \int d^2\alpha p(\alpha)|\langle\phi_\alpha|\rangle^2|\langle\sqrt{\eta}\alpha\rangle^2|^2, \tag{6}
\]

where we decomposed \(\hat{M}'_k\) and \(\hat{\rho}_k\) by rank-1 projections as \(\hat{M}'_k = \sum_j |r_{k_j}\rangle\langle r_{k_j}|(\alpha)\rangle\langle \alpha|\rangle\langle \chi_{k_j}|\chi_{k_j}\rangle\rangle\) with \(\sum_j p_{k_j} = 1\) and \(\langle \chi_{k_j}|\chi_{k_j}\rangle\rangle = 1\), and defined \(\hat{\phi}_\alpha \equiv (\sqrt{\eta}\alpha)^{-1}\langle \chi_{k_j}|\chi_{k_j}\rangle\rangle\) and \(\hat{\chi}_\alpha \equiv |\chi_{k_j}\rangle\rangle\). The condition of POVM \(\sum_k \hat{M}'_k = I\) yields which represents the uniform distribution of coherent states in the limit \(\lambda \to 0\). For \(N = 1\), the task can be achieved by a lossy channel when \(0 < \eta \leq 1\), while it becomes an unphysical noiseless amplification of coherent-state amplitude and never be achieved faithfully by any CPTP when \(\eta > 1\). For \(\eta = 1\) the fidelity optimization corresponds to the situation of the \(N\)-to-\(\infty\) cloning of coherent states.

From Eq. \(7\) the fidelity to be estimated is formally written as the optimization of

\[
F = \sum_k \int d^2\alpha p(\alpha)\text{Tr}\left(\hat{M}_k|\sqrt{\eta}\alpha\rangle\langle\sqrt{\eta}\alpha|\hat{\rho}_k|\sqrt{\eta}\alpha\rangle\langle\sqrt{\eta}\alpha|\right) \tag{8}
\]

where \(\{\hat{M}_k\}\) is considered to be a joint measurement that operates on the \(N\) modes. Since the transformation from \(|\sqrt{N}\alpha\rangle\rangle\) to \(|\alpha\rangle\rangle^\otimes N\) is reversible, the optimized value \(F_{N,\eta,\lambda}\) of Eq. \(5\) can be written as

\[
\sum_y |\phi_\alpha\rangle\langle\phi_\alpha| = I. \text{ Then, } F_{1,\eta,\lambda} \text{ is expressed by a simpler form } \tag{9}
\]

\[
F_{1,\eta,\lambda} = \sup_{\{|\phi_\alpha\rangle\},\{|\chi_\alpha\rangle\}} \sum_y ||\hat{A}_{\phi_\alpha}||_\infty \tag{10}
\]

and \(p\) norm of an operator \(\hat{A}\) is defined by

\[
||\hat{A}||_p \equiv [\text{Tr}(|\hat{A}|^p)]^{1/p}. \tag{11}
\]

Note that the optimization over \(|\chi_\alpha\rangle\rangle\) is absorbed in the operator norm.

In order to estimate the norm we introduce the operator \(\hat{B}\) that satisfies

\[
(||\hat{A}_\phi||_p)^p = \text{Tr}(\hat{\phi})|\phi\rangle\langle\phi|\rho_p \hat{B}). \tag{12}
\]

The operator \(\hat{B}\) is explicitly written as

\[
\hat{B} \equiv \int d^2\alpha d^2\alpha_2 \cdots d^2\alpha_p (\sqrt{\eta}\alpha) (\sqrt{\eta}\alpha_2) (\sqrt{\eta}\alpha_3) \cdots (\sqrt{\eta}\alpha_p) p(\alpha_1)p(\alpha_2)\cdots p(\alpha_p) \bigotimes_{j=1}^p |\alpha_j\rangle \langle \alpha_j| \]

\[
= \int d^2\alpha \left(\frac{\lambda}{\pi}\right)^p \exp\left(-\alpha^2 M_{\lambda+\eta,\eta}\alpha\right) \bigotimes_{j=1}^p |\alpha_j\rangle \langle \alpha_j| = \int d^2\alpha \left(\frac{\lambda}{\pi}\right)^p \exp\left(-\alpha^2 V^\dagger V M_{\lambda+\eta,\eta} V^\dagger V\alpha\right) \bigotimes_{j=1}^p |\alpha_j\rangle \langle \alpha_j| \]

\[
= \frac{\lambda}{\pi} \int d^2\beta_j \exp\left(-\chi_j|\beta_j|^2\right) |\beta_j\rangle \langle \beta_j| \tag{13}
\]
where in the second line we wrote \( \tilde{\alpha} = (\alpha_1, \alpha_2, \cdots, \alpha_p)^T \) and defined a \( p \times p \) matrix
\[
M_{\alpha, \eta} \equiv \lambda E_p - \eta C
\]
with the identity \( E_p \) and the basic circulant permutation matrix
\[
C \equiv \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & 1 \\
1 & 0 & \cdots & 0 & 0
\end{pmatrix}.
\]
(15)

Then we diagonalized \( M_{\alpha, \eta} \) by a \( p \times p \) unitary matrix \( V \) whose element is \( \langle V \rangle_k,l = e^{-2\pi i k l / p} / \sqrt{p} \). The \( j \)-th eigenvalue of \( M_{\alpha, \eta} \) is given by
\[
\chi_j = \lambda + \eta \left( 1 + e^{\frac{2\pi j}{p}} \right).
\]
(16)

We obtained the final expression by introducing new modes defined by the transformation \( \tilde{\beta} \equiv V \tilde{\alpha} \) and using the following relation
\[
\bigotimes_{j=1}^{p} |\alpha_j \rangle \langle \alpha_j | = \exp(\tilde{\beta}^T \tilde{\alpha} - \tilde{\alpha}^T \tilde{\beta}) |0 \rangle \langle 0 | \exp(\tilde{\alpha}^T \tilde{\alpha} - \tilde{\beta}^T \tilde{\beta})
\]
\[
= \exp(\tilde{\beta}^T \tilde{\beta} - \tilde{\tilde{\beta}}^T \tilde{\beta}) |0 \rangle \langle 0 | \exp(\tilde{\beta}^T \tilde{\beta} - \tilde{\tilde{\beta}}^T \tilde{\beta})
\]
\[
= \bigotimes_{j=1}^{p} |\beta_j \rangle \langle \beta_j |
\]
(17)

where we wrote the set of the annihilation operators of the original \( p \) modes in a vector form \( \tilde{\alpha} = (\tilde{\alpha}_1, \tilde{\alpha}_2, \cdots, \tilde{\alpha}_p)^T \), and defined that of new modes by \( \tilde{\beta} = V \tilde{\alpha} \).

Expanding \( |\beta_j \rangle \) by Fock-basis and performing the integration of \( \beta \) noting that \( \Re (1 + \chi_j) > 0 \), we obtain the diagonal expression
\[
\hat{B} = \bigotimes_{j=1}^{p} \frac{\lambda}{1 + \chi_j} \sum_{n_j=0}^{\infty} \left( \frac{1}{1 + \chi_j} \right)^{n_j} |n_j \rangle \langle n_j |
\]
\[
= \frac{\lambda^p}{(1 + \lambda + \eta)^p - \eta^p} \bigotimes_{j=1}^{p} \sum_{n_j=0}^{\infty} \left( \frac{1}{1 + \chi_j} \right)^{n_j} |n_j \rangle \langle n_j |
\]
(18)

where \( \eta \) is an arbitrary density operator with \( \rho_{\phi} \equiv \langle \phi | \rho | \phi \rangle \). We obtained the fidelity bound on the \( N \)-copy case by the replacement \( (\lambda, \eta) \rightarrow (\frac{\lambda}{N + \lambda}, \frac{\eta}{N + \lambda}) \):
\[
F_{N, \eta, \lambda} \leq \frac{1 + \lambda}{1 + \lambda + \eta} \leq \frac{N + \lambda}{N + \lambda + \eta} = F(N, \eta, \lambda).
\]
(24)

Next we show that we can achieve the bound by an M&P scheme, which is the projection onto coherent states and preparation of states according to the outcome. The measurement operator can be represented by \( \hat{A}_\alpha \equiv |\alpha \rangle \langle \alpha | / \sqrt{\pi} \). It gives the POVM element \( \hat{M}_{\alpha} \equiv \hat{A}_\alpha \hat{A}_\alpha^\dagger \), \( |\alpha \rangle \langle \alpha | / \pi \) and the corresponding state preparation \( \hat{\rho}_\alpha \equiv \hat{A}_\alpha \hat{\rho} \hat{A}_\alpha^\dagger / \text{Tr}(\hat{A}_\alpha \hat{\rho} \hat{A}_\alpha^\dagger) \) where \( |\alpha \rangle \rangle \) represents Fock states of the new \( j \)-th mode. From Eq. \((18)\), \((19)\) and \((21)\), we obtain
\[
\text{Tr} \left( |\phi \rangle \langle \phi | \hat{B} \right) \leq \text{Tr} \left( |\phi \rangle \langle \phi | \hat{\rho}_\lambda \right)
\]
\[
\leq \frac{(1 + \lambda)^p}{(1 + \lambda + \eta)^p - \eta^p} \
\times \text{Tr} \left( |\phi \rangle \langle \phi | \hat{\rho}_\lambda \right)
\]
(25)

Therefore, we can conclude that \( F_{N, \eta, \lambda} = F(N, \eta, \lambda) \).
The classical bound on the fidelity for the task \( \{ |\alpha\rangle \} \rightarrow \{ |\sqrt{\eta}|\alpha\rangle \} \), which includes transfer, storage, transmission, amplification, and so on, is simply described by a single formula:

\[
F_{1,0,\lambda} = F(1, \eta, \lambda) = \frac{1 + \lambda}{1 + \eta + \lambda}.
\] (26)

An interesting application is the experimental success criterion for non-UG quantum devices. Suppose that the input and the output of quadrature operators are related by \( X_{\text{out}} = gX_{\text{in}} + X_{\text{noise}} \) by a positive gain factor \( g \) and noise operator \( X_{\text{noise}} \). If the average fidelity \( \bar{F} \) of the task \( \{ |\alpha\rangle \} \rightarrow \{ |\sqrt{\eta}\alpha\rangle \} \) surpasses the classical boundary \( F(1, g^2, \lambda) \), it ensures that any M&P scheme cannot simulate the experiment. Note that the classical bound on the fidelity for a more general class of tasks \( \{ U|\alpha\rangle \} \rightarrow \{ V|\sqrt{\eta}\alpha\rangle \} \) is also \( F_{1,0,\lambda} \), where \( U \) and \( V \) are any unitary operators. This is because the replacement \( \{ M_k, \rho_k \} \rightarrow \{ U^\dagger M_k U, V^\dagger \rho_k V \} \) provides the same optimization problem. Thus, we can select \( U \) and \( V \) as well as \( \eta \) to ensure the QD operation. A trivial example is phase-space displacement/rotation, which is implicitly adjusted in the experiments.

In experiments, fidelity to a coherent state \( |\alpha\rangle \) is directly determined by measuring the probability of photon detection after the displacement operation \( D(-\alpha) \) that displaces \( |\alpha\rangle \) to \( |0\rangle \). In CV systems, homodyne measurement is commonly used and it might be useful to provide similar criterion in terms of the quadrature variances. Let us define the quadrature mean-square deviation from \( \sqrt{\eta} \alpha \) by \( V_{\pm} = \langle \Delta^2_x \pm \rangle = (\langle \hat{x}^2 \pm \rangle - \sqrt{\eta} \alpha^2 \pm \rangle)^2 \) associated with the input \( |\alpha\rangle \) and output \( \rho_{\pm} = E(|\alpha\rangle\langle\alpha|) \), where \( \cdot \mp \) \( Tr(\cdot \rho_{\pm}) \), \( \hat{x}_{\pm} = \hat{a} + \hat{a}^\dagger \), \( \Delta_{\pm} \equiv (\hat{x}_{\pm} - \hat{a} - \hat{a}^\dagger) / i \), and \( \alpha_{\pm} \equiv \langle \alpha | \hat{x}_{\pm} | \alpha \rangle \). Since the mean of \( V_{\pm} \) is \( v = (V_+ + V_-)/2 = \text{Tr} \left( 2 \hat{a}^\dagger \hat{a} + 1 \right) \hat{D}(-\sqrt{\eta} \alpha) \hat{\rho}_{\pm} \hat{D}^\dagger (-\sqrt{\eta} \alpha) \right) \) and \( \text{Tr}(\hat{a}^\dagger \hat{a} \rho_{\pm}) \geq 1 - \text{Tr}(|0\rangle\langle 0| \rho_{\pm}) \) for any state \( \rho_{\pm} \), we can bound \( v \geq 3 - 2(\sqrt{\eta} \alpha \rho_{\pm} |\sqrt{\eta} \alpha\rangle \) Averaging both sides over \( \alpha \) with \( P(\alpha) \), we find \( \bar{\delta} \equiv \bar{v} - 1 \geq 2(1 - \bar{F}) \). Then we can see that \( F > F_{1,0,\lambda} \) is satisfied if

\[
\bar{\delta} < 2(1 - F_{1,0,\lambda}) = \frac{2 \eta}{1 + \lambda + \eta}.
\] (27)

This is a sufficient condition for the QD operation of the physical map \( E \).

From the equivalence between the state estimation and the asymptotic cloning \( F(N,1,0) \) is the optimal fidelity of the \( N \)-to-\( \infty \) cloning of coherent states. Since we have derived \( F(N,1,0) \) without assuming Gaussian operation, neither Gaussian nor non-Gaussian cloning machine can surpass \( F(N,1,0) \). On the other hand, \( F(N,1,0) \) is equal to the optimal fidelity of the Gaussian machines given in \( [16, 17] \). Therefore, the optimal \( N \)-to-\( \infty \) cloning machine of coherent states can be constructed with Gaussian operations. Note that \( F_c \) for any task described by \( \{ U|\sqrt{\eta}|\alpha\rangle \} \rightarrow \{ V|\sqrt{\eta}|\alpha\rangle \} \) can be achieved by Gaussian operations.

In conclusion, we provide a general boundary of the average fidelity achieved by the MKP schemes assuming a transformation task that modulates coherent-state amplitude. The formula can be applicable to the experimental success criterion for continuous-variable quantum devices with dissipation/amplification, such as storage of light for a quantum memory. We also found that the optimal \( N \)-to-\( \infty \) cloning machine of coherent states can be constructed with Gaussian operations.

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