Erasable and unerasable correlations

G. M. D’Ariano,1 R. Demkowicz-Dobrzański,2 P. Perinotti,1 and M. F. Sacchi1,3

1QUIT Group, Dipartimento di Fisica “A. Volta”, via Bassi 6, I-27100 Pavia, Italy and CNISM.
2Center for Theoretical Physics of the Polish Academy of Sciences, Aleja Lotników 32/44, 02-668 Warszawa, Poland.
3CNR - Istituto Nazionale per la Fisica della Materia, Unità di Pavia, Italy.

We address the problem of removing correlation from sets of states while preserving as much local quantum information as possible. We prove that states obtained from universal cloning can only be decorrelated at the expense of complete erasure of local information (i.e. information about the copied state). We solve analytically the problem of decorrelation for two qubits and two qumodes (harmonic oscillators in Gaussian states), and provide sets of decorrelable states and the minimum amount of noise to be added for decorrelation.

The laws of quantum mechanics impose a number of restrictions on the processing of quantum information. Examples of such impossible tasks are provided by the famous no-cloning theorem [1] or by the theorem on non-existence of the universal-NOT gate [2]. Despite their discouraging appearance, such limitations can sometimes be proved useful. This is the case with the no-cloning theorem which is at the core of quantum cryptography, as it prevents an eavesdropper from creating perfect copies of a transmitted quantum state.

In this Letter we attempt to broaden our understanding of the limitations imposed on the quantum information processing, by investigating the possibility of decorrelating quantum states – i.e. removing unwanted correlations between quantum subsystems while preserving local information encoded in each of them.

To be more specific, we say that an operation \( D \) faithfully decorrelates an \( N \)-partite state \( \rho_N \) (by \( N \) we denote a set of indices \( \{1, \ldots, N\} \)) if the following equation holds:

\[
D(\rho_N) = \rho_1 \otimes \cdots \otimes \rho_N, \tag{1}
\]

where \( \rho_i \) is the \( i \)-th party reduced density matrix of \( \rho_N \). Now, the problem of decorrelability can be stated as follows: given a set of states \( E = \{\rho_N\} \), we ask whether there exists a single physically realizable operation (completely positive map) \( D \) that satisfies (1) for every state \( \rho_N \) drawn from the set \( E \).

Analogously as in the case of the no-cloning theorem, the answer will strongly depend on the chosen set of states. In particular, if the set \( E \) consists of only one element \( \rho_N \), then the problem of decorrelability is trivial. One can always choose \( D \) to be a map producing \( \otimes_{i=1}^{N} \rho_i \) out of any input. Such a map is completely positive and hence every single-element set is decorrelable.

Moving to the other extreme, and asking whether a set \( E \) consisting of all density matrices is decorrelable, one finds out that due to linearity of quantum mechanics it is not \( \mathbb{F} \). Actually, from the proof of \( \mathbb{F} \) one can easily draw a stronger conclusion, namely:

**Conclusion from \( \mathbb{F} \).** If a set \( E \) contains a state \( \rho_N' \), \( \rho_N'' \) and their convex combination \( \lambda \rho_N' + (1 - \lambda) \rho_N'' \), and the reduced states of \( \rho_N' \) and \( \rho_N'' \) are different at least for two sites, then faithful decorrelability of the set \( E \) is impossible.

Moreover, in \( \mathbb{F} \) nondecorrelability of certain two-elements set was shown using the fact that after decorrelation distinguishability of states cannot increase (see also \( \mathbb{E} \) for some results on disentangling rather than decorrelating states). Apart from the above particular sets, very little is known on the decorrelability of general sets of quantum states. In this Letter we search for explicit solutions to the decorrelation problem in interesting settings.

The key factor for deciding on decorrelability and nondecorrelability is the choice of the set of states. In this Letter such a choice is motivated by the need of considering the problem of decorrelation in information-processing context. We stress that we decorrelate quantum states by keeping the quantum signals. We propose the following paradigm.

Consider an \( N \)-partite correlated “seed” state \( \rho_N \), which should be regarded as the initial state where information is encoded. Let \( U_g \) be a unitary representation of a group \( G \), acting on the Hilbert space of a single party. The representation describes the encoding procedure of a piece of information (the group element \( g \)) on the state of a subsystem. Acting with unitary operations \( U_{g_1} \otimes \cdots \otimes U_{g_N} \) on the seed state \( \rho_N \) should be regarded as encoding \( N \) pieces of information (\( N \) signals \( \mathbb{E} \)) \( \{g_1, \ldots, g_N\} \):

\[
\rho_N(g_1, \ldots, g_N) := U_{g_1} \otimes \cdots \otimes U_{g_N} \rho_N U_{g_1}^\dagger \otimes \cdots \otimes U_{g_N}^\dagger. \tag{2}
\]

The above state is clearly correlated due to the correlation of the seed state \( \rho_N \). The problem of decorrelation is to find a single operation \( D \) that would decorrelate [see Eq. (1)] all states belonging to the set:

\[
E(U_g, \rho_N) = \{\rho_N(g_1, \ldots, g_N), \forall g_1, \ldots, g_N \in G\}. \tag{3}
\]

If we have additional constraints on the signals (e.g. we know that they are identical \( g_1 = \cdots = g_N \)) the above set will get smaller and eventually the problem of decorrelation will become easier. Notice that the reduced density matrices of \( \rho_N(g_1, \ldots, g_N) \) are related to the reduced
density matrices of \( \rho_N \) via: \( \rho_i(g_i) = U_{g_i} \rho_i U_{g_i}^\dagger \), and as a result the decorrelated state still carries the same signals as the initial one. We stress that we do not deal here with
decorrelation of signals, but rather with decorrelation of
states carrying them (hence, there is no contradiction in
performing decorrelation and still claiming, e.g., that the
encoded signals are identical).

To motivate our work further let us recall some facts about cloning and state estimation. We know that
quantum information cannot be copied or broadcast exactly,
due to the no-cloning theorem. Nevertheless, one can
find approximate optimal cloning operations which in-
crease the number of copies of a state at the expense of
the quality. In the presence of noise, however, (i.e.
when transmitting “mixed” states), it can happen that
we are able to increase the number of copies without los-
ing the quality, if we start with sufficiently many identi-
cal originals. Indeed, it is even possible to
perform decorrelation and still claiming, e.g., that the
information cannot be copied or broadcast exactly,
otherwise we would increase the information on the state.
A priori it is not excluded, however, that it is possible to
decorrelate clones at the expense of introducing some addi-
tional noise. One of the results of this Letter is that
clones by universal cloning cannot be decorrelated even
within this relaxed condition. Apart from this negative
result, we provide in this Letter examples of states for
which decorrelation is possible.

Thanks to the structure of the set of states \((3)\) that
we want to decorrelate, a covariant decorrelation must
satisfy the following conditions: (i) \( D \) decorrelates the
seed state; (ii) \( D \) fulfills the covariance condition:

\[
D(U_{g_1} \otimes \cdots \otimes U_{g_N} \rho_N U_{g_1}^\dagger \otimes \cdots \otimes U_{g_N}^\dagger) = U_{g_1} \otimes \cdots \otimes U_{g_N} D(\rho_N) U_{g_1}^\dagger \otimes U_{g_N}^\dagger. \tag{4}
\]

We will generally consider decorrelation with additional
noise on the output local states, namely

\[
D(\rho_N) = \hat{\rho}_1 \otimes \cdots \otimes \hat{\rho}_N, \tag{5}
\]

where generally \( \hat{\rho}_i \neq \rho_i \). As a result, subsystems are
still perfectly decorrelated, but some information about
reduced density matrices is lost. Additionally, in what
follows we will assume that the seed state is permuta-
tionally invariant—in other words we treat all encoded
signals on equal footing. This simplifies the situation
since in this case all single party reduced density matrices
of the seed state are equal (we denote them by \( \rho \))
and the same should hold for the noisy reduced density
matrices \( \hat{\rho} \) after decorrelation. This assumption allows
us, without loss of generality, to impose permutational
invariance on the decorrelating operation \( D \).

We now present the solution for some interesting bi-
partite situations. We analyze qubits, in which informa-
tion is encoded through general unitaries in \( SU(2) \), and
qumodes (harmonic oscillators in Gaussian states), with
information encoded by the representation of the Weyl-
Heisenberg group of displacements. In our analysis we
consider the two situations in which the units represent-
ing signals on the two systems are either independent
or equal. The latter case is relevant for the decorrelabil-
ity of output states of cloning and broadcasting machines.

It turns out that decorrelation is indeed possible in some
cases, at the expense of increasing local noise. The op-
timal decorrelating map adding the minimum amount of
noise is derived in the qubit case, and the optimal depo-
larization factor is evaluated as a function of the input
seed state. For Gaussian states we show that it is al-
ways possible to erase correlations by means of a suitable
Gaussian map.

Consider a couple of qubits \(( A \) and \( B \)). Permutational
invariance of the seed state means that it is block dia-
gonal with respect to singlet and triplet subspaces. For
qubits the state is conveniently described in the Bloch
form. Without loss of generality we may assume that the
reduced density matrices \( \rho = 1/2(\mathbb{1} + \eta \sigma_z) \) of the seed
state \( \rho_{AB} \) are diagonal in the \( \sigma_z \) basis. The information
\((\alpha, \beta)\) is encoded via the action of \( U(\alpha) \otimes U(\beta) \):

\[
\rho_{AB}(\alpha, \beta) = U(\alpha) \otimes U(\beta) \rho_{AB} U(\alpha)^\dagger \otimes U(\beta)^\dagger, \tag{6}
\]

where \( \alpha \) and \( \beta \) are elements of \( SU(2) \). In other words it
is encoded on the direction of the Bloch vectors \( n_A(\alpha) \)
and \( n_B(\beta) \) of the marginal states

\[
\rho_A(\alpha) = \text{Tr}_B[\rho_{AB}(\alpha, \beta)] = \frac{1}{2}(\mathbb{1} + \eta n_A(\alpha) \cdot \sigma),
\rho_B(\beta) = \text{Tr}_A[\rho_{AB}(\alpha, \beta)] = \frac{1}{2}(\mathbb{1} + \eta n_B(\beta) \cdot \sigma), \tag{7}
\]

where \( \sigma = (\sigma_x, \sigma_y, \sigma_z) \) is the vector of Pauli matrices
\( \sigma_\alpha \). Covariance of the decorrelation map means that
the direction of the Bloch vectors \( n_A(\alpha) \) and \( n_B(\beta) \) should
be preserved in the output states, i.e.

\[
\hat{\rho}_A(\alpha) = \frac{1}{2}(\mathbb{1} + \tilde{\eta} n_A(\alpha) \cdot \sigma),
\hat{\rho}_B(\beta) = \frac{1}{2}(\mathbb{1} + \tilde{\eta} n_B(\beta) \cdot \sigma), \tag{8}
\]

namely only the length of the Bloch vector (i.e. the pu-
"arity of the state) is changed \( \eta \rightarrow \tilde{\eta} \). The additional noise
of the output states corresponds to a reduced length of the
Bloch vector \( \tilde{\eta} < \eta \). The directions of the Bloch vectors
\( n_A(\alpha) \) and \( n_B(\beta) \) are completely arbitrary. The
optimal decorrelation map will maximize the length $\tilde{\eta}$ of the Bloch vector, namely it will produce the highest purity of decorrelated states. It can be shown that the general form of a two-qubit channel $\mathcal{D}$ covariant under $U(\alpha) \otimes U(\beta)$ and invariant under permutations of the two qubits can be parameterized with three positive parameters only (effectively two due to normalization)

$$\mathcal{D}(\rho_{AB}) = a \rho_{AB} + b \mathcal{D}_1(\rho_{AB}) + c \mathcal{D}_2(\rho_{AB}),$$

where $\mathcal{D}_1$ and $\mathcal{D}_2$ are given by

$$\mathcal{D}_1(\rho_{AB}) = \frac{1}{3}(\rho_A \otimes \mathbb{1} + \mathbb{1} \otimes \rho_B - \rho_{AB})$$

and the trace preserving condition gives $a + b + c = 1$. This is a very restricted set of operations, due to the fact that the covariance condition is very strong. As a consequence the condition for decorrelating the seed state

$$\mathcal{D}(\rho_{AB}) = \tilde{\rho}_{\text{sym}}^2 = [1/2(1 + \tilde{\eta} \sigma_z)]_{\text{sym}}^2$$

cannot be satisfied for a generic seed state $\rho_{AB}$ (apart from the trivial decorrelation to a maximally mixed state).

FIG. 1: Length $\tilde{\eta}$ of the Bloch vectors of the decorrelated states of two qubits starting from the joint state in Eq. (13). The plot depicts the maximal achievable $\tilde{\eta}$ in gray scale versus the parameters $\eta$ and $\lambda$ of the input state.

The seed states for which nontrivial decorrelation is possible [which means that we can find such $a, b, c$ and $\tilde{\eta} > 0$ satisfying Eq. (12)] have the form [7]

$$\rho_{AB} = \frac{1}{4}[\mathbb{1} \otimes \mathbb{1} + \eta(\sigma_z \otimes \mathbb{1} + \mathbb{1} \otimes \sigma_z) + \lambda \sigma_z \otimes \sigma_z].$$

We emphasize that for a generic seed state $\rho_{AB}$ one can reduce correlations, but only sets arising from the seed state of the form (13) can be decorrelated completely in a nontrivial way (apart from the cases when $\eta = 0$ or $\lambda = 0$). The noise of the decorrelated states depends on parameters $\eta$ and $\lambda$ as depicted in Figure 1.

In order to study the decorrelability of the output states of cloning machines, we consider now the case where the same unitary is encoded on the two qubits (identical signals). Differently from the case of independent signals, $\mathcal{D}$ has to be covariant with respect to $U(\alpha) \otimes U(\beta)$, which is a weaker condition than covariance with respect to $U(\alpha) \otimes U(\beta)$. Using the methods from [7] one can parameterize these class of operations with six parameters $s_j, t_j$ satisfying two constraints, so effectively one enjoys a four parameter freedom on covariant operations. Thanks to this larger freedom it can be shown by straightforward calculation [7] that the decorrelation condition $\mathcal{D}(\rho_{AB}) = \tilde{\rho}_{\text{sym}}^2$ is non trivially satisfied (i.e. for $\tilde{\eta} > 0$) for a generic state $\rho_{AB}$ which is diagonal in the singlet triplet basis. Such a state can be written in the form:

$$\rho_{AB} = p|\Psi^-\rangle\langle \Psi^-| + (1-p)\rho_{\text{sym}},$$

where $\rho_{\text{sym}}$ is a state supported on the triplet (symmetric) subspace, and $|\Psi^-\rangle$ is the singlet state. Analogously to Eq. (13), $\rho_{\text{sym}}$ can be written with the help of Pauli matrices:

$$\rho_{\text{sym}} = \frac{1}{4}[\mathbb{1} \otimes \mathbb{1} + \eta(\sigma_z \otimes \mathbb{1} + \mathbb{1} \otimes \sigma_z) + (1 + \lambda)/2 (\sigma_x \otimes \sigma_x + \sigma_y \otimes \sigma_y) - \lambda \sigma_z \otimes \sigma_z].$$

FIG. 2: Length $\tilde{\eta}$ of the Bloch vectors of the decorrelated states of two qubits starting from a seed state supported on the symmetric subspace parameterized as in Eq. (13). The plot depicts the maximal achievable $\tilde{\eta}$ versus the parameters $\eta$ and $\lambda$ of the input state.
The only states that cannot be nontrivially decorrelated are those with either \( p = 1 \) or \( \eta = 0 \) or \( \lambda = -1/3 \). Since these decorrelable seed states form a three-parameter family, we cannot represent a density plot for the achievable \( \lambda \) with respect to all parameters. As an example in Figure 2 we present the plot for \( p = 0 \) i.e. for seed states supported on the symmetric space. Interestingly, the line \( \lambda = -1/3 \) contains states which can be obtained via universal cloning machines producing two copies out of one copy of a qubit state. Then, clones obtained by 1-to-2 universal cloning cannot be nontrivially decorrelated. Such a result can be shown for general \( N \)-to-\( M \) universal cloning \([8]\).

We consider now the case of decorrelation for qumodes. For a couple of qumodes in a joint state \( \rho_{AB} \), the information \((\alpha, \beta)\) (with \( \alpha \) and \( \beta \) complex) is encoded as follows

\[
\rho_{AB} = \frac{1}{\pi^2} \int d^4q \, e^{-2q^T M q} D(q),
\]

where \( q = (q_1, q_2, q_3, q_4) \), \( D(q) = D(q_1 + i q_2) \otimes D(q_3 + i q_4) \), and \( M \) is the \( 4 \times 4 \) (real, symmetric, and positive) correlation matrix of the state, that satisfies the Heisenberg uncertainty relation \( \left[ M + 1/\lambda \Omega, \Omega \right] \geq 0 \), with \( \Omega = \sum_{\kappa=1}^2 \omega \) and \( \omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \).

A Gaussian decorrelation channel covariant under \( D(\alpha) \otimes D(\beta) \) is given by

\[
\mathcal{D}(\rho) = \frac{\sqrt{\det G}}{(2\pi)^2} \int d^4x \, e^{-x^T G x} D(x) \rho D^T(x),
\]

with suitable positive definite \( G \), and the resulting state \( \mathcal{D}(\rho_{AB}) \) is still Gaussian, with a new block-diagonal covariance matrix \( \tilde{M} \), thus corresponding to a decorrelated state.

A special example of Gaussian state of two qumodes is the so-called twin beam, which can be generated in a quantum optical lab by parametric downconversion of vacuum. In this case \( M \) is given by

\[
M = \frac{1 + \lambda^2}{1 - \lambda^2} \mathbb{1} - \frac{2\lambda}{1 - \lambda^2} \begin{pmatrix} \sigma_z & 0 \\ 0 & \sigma_z \end{pmatrix},
\]

with \( 0 \leq \lambda < 1 \). The map \( \mathcal{D} \) with

\[
G = \frac{2\lambda}{1 - \lambda^2} \begin{pmatrix} \mathbb{1} + (\varepsilon \sigma_z) & \varepsilon \\ \varepsilon & \mathbb{1} + (\varepsilon \sigma_z) \end{pmatrix},
\]

and arbitrary \( \varepsilon > 0 \), provides two decorrelated states with \( \tilde{M} = (1 \pm \lambda \varepsilon) \mathbb{1} \), which correspond to two thermal states with mean photon number \( \bar{n} = \frac{1}{1 - \lambda \varepsilon} + \frac{\varepsilon}{2} \) each. Since the channel in Eq. (18) is covariant also for \( D(\alpha)^{\otimes 2} \), the above derivation then holds for the case of encryption with the same unitary on both qumodes as well.

The striking difference between the qubit and the qumode cases is that for qubits only few states can be decorrelated, whereas for qumodes any joint Gaussian state can be decorrelated. This is due to the fact that the covariance group for qubits comprises all local unitary transformations, whereas for qumodes includes only local displacements, which is a very small subset of all possible local unitary transformations in infinite dimension.

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[9] The word signal suggests sometimes a sequence of pieces of information being transmitted. In our case this will simply correspond to repeated preparation of the state \( \rho_N(g_1, \ldots, g_N) \) with varying set \( \{g_1, \ldots, g_N\} \) in each shot.