Classical Interaction Cannot Replace a Quantum Message

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Abstract

We demonstrate a two-player communication problem that can be solved in the one-way quantum model by a quantum protocol of cost $O(\log n)$ but requires exponentially more communication in the classical interactive (two-way) model.

1 Introduction

The ultimate goal of quantum computing is to identify computational tasks where by using the laws of quantum mechanics one can find a solution more efficiently than by using a classical computer.

In this paper we study quantum computation from the perspective of Communication Complexity, first defined by Yao [Y79]. Here two parties, Alice and Bob, try to solve a computational problem that depends on $x$ and $y$. Initially Alice knows only $x$ and Bob knows only $y$; in order to solve the problem they communicate, obeying to the restrictions of a specific communication model. In order to compare the power of two communication models one has to demonstrate a communication task that can be solved more efficiently in one model than in the other.

We will be mostly concerned about the following communication models.

- One-way communication is the model where Alice sends a single message to Bob and he has to give an answer, based on the content of that message and his part of input.

- Interactive (two-way) communication is the model where the players can interactively exchange messages till Bob decides to give an answer, based on the previous communication transcript and his part of input.

Both of these models can be either classical or quantum, according to the nature of communication allowed between the players. The classical versions of the models are denoted by $R^1$ and $R$, and the quantum versions are denoted by $Q^1$ and $Q$, respectively.

Communication tasks can be either functional, corresponding to the case when for each input pair $(x,y)$ there exists at most one correct answer, or relational, when multiple correct answers are allowed for the same input. Input pairs without correct answers are never given to the players.

\[\text{Communication tasks with exactly one correct answer corresponding to each possible input pair are called complete functions.}\]
A communication protocol describes behavior of Alice and Bob in response to each possible input. The cost of a protocol is the maximum total amount of information (bits or qubits) communicated by the parties according to the protocol.

We say that a communication task $P$ is solvable with bounded error in a given communication model by a protocol of cost $O(k)$ if for any constant $\varepsilon$ there exists a corresponding protocol solving $P$ with success probability at least $1 - \varepsilon$. Analogously, $P$ is solvable with 0-error if the protocol refuses to answer with probability at most $\varepsilon$ and solves $P$ correctly whenever it produces an answer.

In this paper we will be interested in communication problems where the quantum model gives exponential savings.

For zero-error one-way and interactive protocols, such problems were demonstrated by Buhrman, Cleve, and Wigderson [BCW98]. In the bounded-error setting the first exponential separation has been demonstrated by Raz [R99], who gave an example of a problem solvable in $Q$ by exponentially more efficient protocol than the best possible in $R$. Later Buhrman, Cleve, Watrous, and de Wolf [BCWW01] demonstrated an exponential separation for simultaneous protocols, which is a communication model even more limited than $R^1$.

All these separations have been demonstrated for functional problems. As of one-way protocols with bounded error, the first exponential separation has been shown by Bar-Yossef, Jayram, and Kerenidis [BJK04] for a relational problem. Later Gavinsky, Kempe, Kerenidis, Raz, and de Wolf [GKKRW07] gave a similar separation for a functional problem.

These result show that quantum communication can be very efficient, by establishing various settings where quantum protocols offer exponential savings over classical solutions. But does there exist a problem that can be solved by a quantum one-way or even simultaneous protocol that is considerably more efficient than any classical two-way protocol? The full answer to this question is not known yet.

1.1 Our result

Theorem 1.1. There exists a relation with input length $N$ whose bounded error communication complexity is $O(\log N)$ in $Q^1$ and $\Omega\left(\frac{N^{1/8}}{\sqrt{\log N}}\right)$ in $R$.

The relation that we use for establishing this result is a modification of a communication task independently suggested by R. Cleve [C] and S. Massar [B] as a possible candidate for such separation.

Some of the intermediate steps in our proof might be of independent interest.

2 Our approach

For any $m$ being a power of 2, let $X_m \overset{\text{def}}{=} GF_2^{log m}$ and denote the identity element by $\bar{0}$. We will sometimes refer to subsets of $X_m$ as elements of $\{0, 1\}^m$. Define the following communication problems.

Definition 1. Let $x \subset X_{n^2}$ and $y \subset X_{n^2}$, such that $|x| = n/2$ and $|y| = n$. Let $z \in X_{n^2} \setminus \{\bar{0}\}$. Then $(x, y, z) \in P_{n \times 1}^{(n)}$ if either $|x \cap y| \neq 2$ or $\langle z, a + b \rangle = 0$ where $x \cap y = \{a, b\}$.

\footnote{In all the examples mentioned here, the first shown super-polynomial separations were, in fact, exponential.}
Definition 2. Let $x \subseteq X_{n^2}$, $|x| = n/2$. Let $y = (y_1, \ldots, y_n)$ be a partition of $X_{n^2}$ into $n$ subsets of equal size such that $\left\{ i \left| x \cap y_i = 2 \right. \right\} \geq \frac{n}{14}$. Let $z = ((i_1, a_1), \ldots, (i_t, a_t))$ for some $t \in \mathbb{N}$. Then $(x, y, z) \in P^{(n)}$ if for every $j$, $1 \leq j \leq t$, it holds that $(x, y_j, a_j) \in P^{(n)}_{I \times 1}$ and some $j_0$ satisfies $|x \cap y_{j_0}| = 2$.

We will show that $P^{(n)}$ is easy to solve in $Q^1$ and is hard for $R$. In order to prove the lower bound we will use the following modification of $P^{(n)}_{I \times 1}$.

Definition 3. Let $x \subseteq X_{n^2}$ and $y \subseteq X_{n^2}$, such that $|x| = n/2$ and $|y| = n$. Let $z \subseteq X_{n^2}$. Then $(x, y, z) \in \tilde{P}^{(n)}_{I \times 1}$ if $x \cap y = z$.

We will use the following generalization of the standard bounded error setting. We say that a protocol solves a problem with probability $\delta$ with error bounded by $\varepsilon$ if with probability at least $\delta$ the protocol produces an answer that is correct with probability at least $1 - \varepsilon$.

Solving $P^{(n)}_{I \times 1}$ when $|x \cap y| = 2$ requires providing an evidence of knowledge of these elements, and intuitively should be as hard as finding them, as required by $\tilde{P}^{(n)}_{I \times 1}$. This intuition is, apparently, false for the quantum 1-way model (we show that $P^{(n)}_{I \times 1}$ can be easily solved in $Q^1$ with probability $1/n$ with small error, which is unlikely to be the case for $\tilde{P}^{(n)}_{I \times 1}$). However, it is true for the model of classical 2-way communication; a “quasi-reduction” from $\tilde{P}^{(n)}_{I \times 1}$ to $P^{(n)}_{I \times 1}$ is one of the central ingredients of our lower bound proof.

The high-level scenario of the proof is the following.

- We claim that if there exists a protocol of cost $k$ that solves $P^{(n)}$ with error bounded by $\varepsilon$ then another protocol of similar cost solves $P^{(n)}_{I \times 1}$ with probability $\Omega(1/n)$ with error $O(\varepsilon)$.
- We reduce the task of solving the (apparently harder) problem $\tilde{P}^{(n)}_{I \times 1}$ to that of solving $P^{(n)}_{I \times 1}$ when $|x \cap y| = 2$.
- We show that the cost of solving $\tilde{P}^{(n)}_{I \times 1}$ with probability $\delta$ when $|x \cap y| = 2$ is $\Omega \left( n \cdot \sqrt{\delta} \right)$.
- We conclude that solving $P^{(n)}$ with bounded error requires an interactive classical protocol of complexity $n^{\Omega(1)}$.

3 Notation and more

We assume basic knowledge of (classical) communication complexity ([KN97]).

We will consider only discrete probability distributions. For a set $A$ we write $U_A$ to denote the uniform distribution over the elements of the set. Given a distribution $D$ over $A$ and some $a_0 \in A$ we denote $D(a_0) \overset{\text{def}}{=} \Pr_{a \sim D}[a = a_0]$; for $B \subseteq A$, $D(B) \overset{\text{def}}{=} \sum_{b \in B} D(b)$. Denote $\text{supp}(D) \overset{\text{def}}{=} \{ a \in A \left| D(a) > 0 \right. \}$. 
We use the following notation.

\[
\text{DISJ} \overset{\text{def}}{=} \{(x, y) | x, y \in \{0, 1\}^n, |x| = |y| > 0, \forall 1 \leq i \leq |x| : x_i = 0 \vee y_i = 0\}
\]

\[
\text{DISJ}_n \overset{\text{def}}{=} \{(x, y) | x, y \in \{0, 1\}^n, (x, y) \in \text{DISJ}\}
\]

\[
\overline{\text{DISJ}} \overset{\text{def}}{=} \{(x, y) | x, y \in \{0, 1\}^n, |x| = |y| > 0, (x, y) \notin \text{DISJ}\}
\]

\[
\overline{\text{DISJ}}_n \overset{\text{def}}{=} \{(x, y) | x, y \in \{0, 1\}^n, (x, y) \in \overline{\text{DISJ}}\}
\]

We use the standard notion of a \textit{(combinatorial) rectangle}. The sides of a rectangle will always correspond to subsets of the input sets of Alice and Bob, as defined by the communication problem under consideration. We will use the same notation for an input rectangle and for the \textit{event that input belongs to the rectangle}.

Next we define context-sensitive “projection operators” \(||\) and \(\cdot\|\), as follows. For a discrete set \(A\), \(x \subseteq A\) and \(I \subseteq A\), let \(x|I \overset{\text{def}}{=} x \cap I\). For \(B \subseteq 2^A\), let \(B||I \overset{\text{def}}{=} \{x|I | x \in B\}\). For a distribution \(D\) over \(A\), let \(D|I\) be the conditional distribution of \(x \sim D\), subject to \(x \in I\). For a distribution \(D\) over \(2^A\), let \(D|I\) be the marginal distribution of \(y = x|I\) when \(x \sim D\).

We will use special notation for “one-sided” projections of input pairs. Let \((x, y) \in A \times B\), where \(A\) and \(B\) are input sets of Alice and Bob, respectively. Then \((x, y)|A_{1} \overset{\text{def}}{=} x\) and \((x, y)|B_{0} \overset{\text{def}}{=} y\). Similarly define the operators \(\|_{A_{1}}\) and \(\|_{B_{0}}\) for distributions and sets.

\[3.1\] Definitions related to \(P_{1 \times 1}^{(n)}\) and \(P^{(n)}\)

Define the following events characterizing input to \(P_{1 \times 1}^{(n)}\) or \(\tilde{P}_{1 \times 1}^{(n)}\).

**Definition 4.** For \(j \in \mathbb{N}\), let \(X_{j}\) be the event that the input pair \((x, y)\) satisfies \(|x \cap y| = j\). For \(i, j \in \mathbb{N}\) let \(X_{1}(i)\) and \(X_{2}(i, j)\) be, respectively, the events that \(x \cap y = \{i\}\) and \(x \cap y = \{i, j\}\).

We will use the same notation to address the subsets of input that give rise to these events, i.e., \(X_{0} \overset{\text{def}}{=} \cup_{n=2i} \{(x, y) \in X_{n^{2}} \times X_{n^{2}} | x \cap y = \emptyset\}\), and so forth.

Let \(\mathcal{U}_{1 \times 1}^{(n)}\) be the uniform distribution of input to \(P_{1 \times 1}^{(n)}\). \(\mathcal{U}_{A_{1}} \overset{\text{def}}{=} \mathcal{U}_{1 \times 1}^{(n)}||_{A_{1}}\) and \(\mathcal{U}_{B_{0}} \overset{\text{def}}{=} \mathcal{U}_{1 \times 1}^{(n)}||_{B_{0}}\).

**Definition 5.** For \(k_{1}, \ldots, k_{t} \in \mathbb{N}\), let \(\mathcal{U}_{1 \times 1}^{(n;k_{1}, \ldots, k_{t})} \overset{\text{def}}{=} \mathcal{U}_{1 \times 1}^{(n)}|_{X_{k_{1}} \cup \ldots \cup X_{k_{t}}}\) and \(\mathcal{U}_{1 \times 1}^{(n;k_{1}+)} \overset{\text{def}}{=} \mathcal{U}_{1 \times 1}^{(n)}|_{\cup_{i \geq k_{t}} X_{i}}\).

**Definition 6.** Given input set \(A\) (not necessarily a rectangle), define \(\mathcal{U}_{A} \overset{\text{def}}{=} \mathcal{U}_{1 \times 1}^{(n)}|_{A}, \mathcal{U}_{A_{1}} \overset{\text{def}}{=} \mathcal{U}_{A}||_{A_{1}}\) and \(\mathcal{U}_{A_{1}}^{B_{0}} \overset{\text{def}}{=} \mathcal{U}_{A}||_{B_{0}}\). Given \(k_{1}, \ldots, k_{t} \in \mathbb{N}\), let \(\mathcal{U}_{A}^{(k_{1}, \ldots, k_{t})} \overset{\text{def}}{=} \mathcal{U}_{A}|_{X_{k_{1}} \cup \ldots \cup X_{k_{t}}}\).

The following observation is important for our proof.

**Claim 3.1.** For \(n\) large enough it holds that \(\mathcal{U}_{1 \times 1}^{(n)}(X_{0}) \geq 1/3, \mathcal{U}_{1 \times 1}^{(n)}(X_{1}) \geq 1/6\) and \(\mathcal{U}_{1 \times 1}^{(n)}(X_{2}) \geq 1/13\). On the other hand, for any \(t \leq n/2\) it holds that \(\mathcal{U}_{1 \times 1}^{(n)}(\cup_{i \geq t} X_{i}) \leq \left(\frac{3}{4}\right)^{t}\).

**Proof of Claim 3.1.** Think about choosing \((x, y) \sim \mathcal{U}_{1 \times 1}^{(n)}\) as selecting a random subset \(y \subseteq X_{n^{2}}, |y| = n\), followed by selecting \(n/2\) different elements for \(x\). Under such interpretation it
is clear that $U_{1 \times 1}^{(n)}(\bigcup_{i \leq t} X_i) \leq (n/2)^t \cdot \left(\frac{n}{n^2 - n/2}\right)^t$. Therefore, $U_{1 \times 1}^{(n)}(X_0) \geq 1 - n/2 \cdot \frac{n}{n^2 - n/2} \geq \frac{1}{3}$ and $U_{1 \times 1}^{(n)}(\bigcup_{i \leq t} X_i) \leq \left(\frac{n}{2}\right)^t \cdot \left(\frac{2}{n^2}\right)^t = \left(\frac{2}{n}\right)^t$, for $n \geq 2$.

Let $E_i$ be the event that $i \in x \cap y$. It clearly follows from the symmetry between all $E_i$-s and from the fact that the events are mutually exclusive when conditioned upon $X_i$ that $U_{1 \times 1}^{(n)}(X_i)$ is equal to $n/2$ times the probability that the first element selected for $x$ belongs to $y$ and all the following are not in $y$. The former occurs with probability at least $1/n$ and the latter with probability not smaller than $U_{1 \times 1}^{(n)}(X_0)$, therefore $U_{1 \times 1}^{(n)}(X_i) \geq \frac{n}{2} \cdot \frac{1}{n} \cdot \frac{1}{3} \geq \frac{1}{6}$.

Similarly, $U_{1 \times 1}^{(n)}(X_2) \geq \left(\frac{n/2}{2}\right) \cdot \frac{1}{n} \cdot \frac{n-1}{n^2 - 1} \cdot U_{1 \times 1}^{(n)}(X_0) > \frac{1}{13}$ for sufficiently large $n$. ■

4 Efficient protocol for $P^{(n)}$ in $Q^{I}$

We give a 1-way quantum protocol $S^{(n)}$ that receives input to $P^{(n)}$, communicates $O(\log n)$ qubits and either produces a pair $(i, a)$ satisfying $(x, y, a) \in P_{I \times I}^{(n)}$ or returns no answer. Moreover, for $n$ large enough $S^{(n)}$ produces output satisfying $|x \cap y_i| = 2$ with probability at least $\frac{1}{3}$. Therefore, for any given $\varepsilon$ one can run $t \in O(\log \left(\frac{1}{\varepsilon}\right))$ “instances” of $S^{(n)}$ in parallel and output the list of the obtained answers, that would give a protocol for solving $P^{(n)}$ with error at most $\varepsilon$. The communication cost of the new protocol remains in $O(\log n)$ as long as $\varepsilon$ is a constant.

Let us see how $S^{(n)}$ works.

1. Alice sends to Bob the state $|\alpha\rangle \overset{\text{def}}{=} \frac{1}{\sqrt{n/2}} \sum_{j \in x} |j\rangle$.

2. Bob measures $|\alpha\rangle$ with the $n$ projectors $E_i \overset{\text{def}}{=} \sum_{j \in y_i} |j\rangle\langle j|$, let $i_0$ be the index of the outcome of the measurement and $|\alpha_{i_0}\rangle$ be the projected state. Bob applies the Hadamard transform over $G^{2 \log n}$ to $|\alpha_{i_0}\rangle$ and measures the result in the computational basis.

Denote by $a_{i_0}$ be the outcome of the measurement.

3. If $a_{i_0} = 0$ then Bob refuses to answer, otherwise he outputs $(i_0, a_{i_0})$.

Obviously, the protocol transmits $O(\log n)$ qubits.

After Bob’s first measurement the register remains in the state $|\alpha_{i_0}\rangle = \frac{1}{\sqrt{|x \cap y_{i_0}|}} \sum_{j \in x \cap y_{i_0}} |j\rangle$.

Denote by $p_i$ the probability that $i_0 = i$, then

$$p_i = \text{tr}(|\alpha\rangle\langle\alpha| \cdot E_i) = \frac{2}{n} \cdot \text{tr} \left( \sum_{j, k \in x} |j\rangle\langle k| + \sum_{j \in x} |j\rangle\langle j| \cdot \sum_{j \in y_i} |j\rangle\langle j| \right) = \frac{2}{n} \cdot \frac{|x \cap y_i|}{n}.$$  

In particular, if $|x \cap y_i| = 2$ then $p_i = \frac{4}{n}$.

The definition of $P^{(n)}$ guarantees that there are not less than $2/t$ different values of $i \in [n]$ satisfying $|x \cap y_i| = 2$, therefore with probability at least $\frac{2}{t}$ it holds that $|x \cap y_{i_0}| = 2$. Note that if this is not the case then $S^{(n)}$ is correct whenever it returns an answer.
Now assume that $|x \cap y_{i_0}| = 2$. Bob applies the Hadamard transform to the state $|a_{i_0}\rangle = \frac{|b_1 + b_2\rangle}{\sqrt{2}}$ where $x \cap y_{i_0} = \{b_1, b_2\}$, denote the outcome of that by $|a'_{i_0}\rangle$. Then

$$|a'_{i_0}\rangle = \frac{1}{n\sqrt{2}} \sum_{j \in X} \left( (-1)^{j \cdot b_1} + (-1)^{j \cdot b_2} \right) |j\rangle = \frac{1}{n\sqrt{2}} \sum_{(j, b_1+b_2) = 0} 2 |j\rangle,$$

and therefore Bob obtains $a_{i_0} \sim U_{\{j \in X | (j, b_1+b_2) = 0\}}$ as the outcome of his second measurement.

If $a_{i_0} = 0$ then Bob refuses to answer, otherwise he returns a pair $(i_0, a_{i_0})$ that satisfies the requirement. Denote the latter event by $E$, it occurs with probability $1 - \frac{2}{n^2}$, conditioned on the event $|x \cap y_{i_0}| = 2$. The unconditional probability of $E$ will be at least $\frac{3}{5} - \frac{1}{n^2} > \frac{1}{10}$ (for $n$ sufficiently large), as required.

## 5 Solving $P^{(n)}$ is expensive in $R$

We will establish a lower bound of $\frac{n^{1/4}}{\sqrt{\log n}}$ for the 2-way classical communication complexity of $P^{(n)}$. We will always assume this model of communication, unless stated otherwise.

In his elegant lower bound proof for DISJ, Razborov [R92] has established the following lemma.

**Lemma 5.1.** [R92] Let $A$ be an input rectangle for $DISJ_n$, assume that $n = 4t - 1$. Let $D$ be the following input distribution – with probability $3/4$ Alice and Bob receive two uniformly distributed disjoint subsets of $[n]$ of size $l$ and with probability $1/4$ they receive two uniformly distributed subsets of $[n]$ of size $l$ that share exactly one element. Then

$$D(A \cap X_1) \geq \frac{1}{135} \cdot D(A \cap X_0) - 2^{-\Omega(n)}.$$ 

We need the following consequence of Lemma 5.1

**Lemma 5.2.** Let $n$ be sufficiently large and $A$ be an input rectangle for $DISJ_n$. Let $D$ be a product distribution w.r.t. two halves of the input, such that Alice receives a uniformly chosen subset of $[n]$ of size $k_1(n)$ and Bob receives a uniformly chosen subset of $[n]$ of size $k_2(n)$, where $\alpha_1 \sqrt{n} \leq k_1(n) \leq k_2(n) \leq \alpha_2 \sqrt{n}$ for some $\alpha_1$, $\alpha_2$. Then for $\delta = \frac{\alpha_1^2}{45 \cdot 4^{\alpha_2^2}}$ it holds that

$$D(A \cap X_1) \geq \delta \cdot D(A \cap X_0) - 2^{-\Omega(\sqrt{n})}.$$ 

**Proof of Lemma 5.2** We will reduce the communication task considered in Lemma 5.1 to that defined in the lemma we are proving. Address the former task by $P'$ and the latter one by $P$ (they both are, in fact, versions of $DISJ$, defined w.r.t. different distributions). We will use $m$ to denote the input length to $P'$. The distribution of input to $P'$ corresponding to $m$ will be denoted by $D'_m$. The length and the distribution of input to $P$ will be denoted by $n$ and $D$, respectively.

Let $m = 4k_1(n) - 1$. Let $T_r$ be a transformation $(x', y') \rightarrow (x, y)$, where $r \in \{0,1\}^s$, $x', y' \in \{0,1\}^m$, and $x, y \in \{0,1\}^m$. Think of $r$ as a uniform random string of sufficient length (we will address this situation by “$r \sim U^s$”) and of $T$ as a randomized transformation of $x'$ and $y'$ only (random bits are implicit taken from $r$). In order to compute $T_r(x', y')$
choose randomly and uniformly a pair \((M, \beta)\) of disjoint subsets of \(|n|\) of sizes \(m\) and \(k_2(n) - l\), respectively (our choice of \(n\) guarantees that the latter value is not negative). Define \((x, y)\) by \(x|_M = x', y|_M = y', x|_{\overline{M}} = \emptyset\) and \(y|_{\overline{M}} = \beta\). Note that \(T\) can be applied locally by Alice and Bob if they share public randomness (that is, \(x\) only depends on \(r\) and \(x'\) and \(y\) only depends on \(r\) and \(y')\).

We can see that \((x, y)\) is input to \(\text{DISJ}_n\) and \(\text{DISJ}_n(x, y) = \text{DISJ}_m(x', y')\), so indeed \(T\) is a reduction from \(\text{DISJ}_m\) to \(\text{DISJ}_n\). If \((x', y')\) comes from \(\mathcal{X}_i \cap \text{supp}(D'_m)\) and \(r \sim \mathcal{U}\) then \(T_r(x', y')\) is uniformly distributed over \(\mathcal{X}_i \cap \text{supp}(D)\), for any \(i \geq 0\). In particular, for \(i \in \{0, 1\}\),

\[
\mathbb{E}_{r \sim \mathcal{U}} \left[ \Pr_{(x', y') \sim D'_m|\mathcal{X}_i} \left[ T_r(x', y') \in A \right] \right] = \Pr_{(x, y) \sim D} \left[ (x, y) \in A|\mathcal{X}_i \right].
\]

For every \(r \in \{0, 1\}^*\) let \(B_r \overset{\text{def}}{=} T_r^{-1}(A)\). It holds that

\[
\Pr_{(x', y') \sim D'_m|\mathcal{X}_i} \left[ T_r(x', y') \in A \right] = D'_m|\mathcal{X}_i(B_r) = \frac{D'_m(B_r \cap \mathcal{X}_i)}{D'_m(\mathcal{X}_i)},
\]

therefore

\[
\mathbb{E}_{r \sim \mathcal{U}} \left[ D'_m(B_r \cap \mathcal{X}_i) \right] = \frac{D'_m(\mathcal{X}_i)}{D(\mathcal{X}_i)} \cdot D(A \cap \mathcal{X}_i).
\]

It is clear that \(T_r\) is rectangle-invariant, so \(B_r\)-s are rectangles and we can apply Lemma 5.1.

\[-2^{-\Omega(\sqrt{n})} = -2^{-\Omega(m)} \leq \mathbb{E}_{r \sim \mathcal{U}} \left[ D'_m(B \cap \mathcal{X}_i) - \frac{D'_m(B \cap \mathcal{X}_0)}{135} \right] \]

\[= \mathbb{E}_{r \sim \mathcal{U}} \left[ D'_m(B \cap \mathcal{X}_i) \right] - \frac{1}{135} \cdot \mathbb{E}_{r \sim \mathcal{U}} \left[ D'_m(B \cap \mathcal{X}_0) \right] \]

\[= \frac{D'_m(\mathcal{X}_i)}{D(\mathcal{X}_i)} \cdot D(A \cap \mathcal{X}_i) - \frac{D'_m(\mathcal{X}_0)}{135} \cdot D(\mathcal{X}_0) \cdot D(A \cap \mathcal{X}_0).
\]

Together with the facts that \(D'_m(\mathcal{X}_0) = \frac{4}{5}\) and \(D'_m(\mathcal{X}_1) = \frac{1}{5}\), it implies that

\[
D(A \cap \mathcal{X}_i) \geq \frac{D(\mathcal{X}_i)}{135} \cdot D(A \cap \mathcal{X}_0) - \frac{D(\mathcal{X}_1)}{135} \cdot D(\mathcal{X}_0) \cdot 2^{-\Omega(\sqrt{n})}
\]

\[
\geq \frac{D(\mathcal{X}_i)}{45} \cdot D(A \cap \mathcal{X}_0) - 2^{-\Omega(\sqrt{n})}.
\]

Note that

\[
D(\mathcal{X}_0) \geq \left( \frac{n - k_1(n) - k_2(n)}{n} \right)^{k_2(n)} \geq \left( 1 - \frac{2\alpha_2}{\sqrt{n}} \right)^{2\alpha_2 \sqrt{n}} \geq \left( \frac{1}{2} \right)^{2\alpha_2^2} = \left( \frac{1}{4} \right)^{\alpha_2^2},
\]

\[
D(\mathcal{X}_1) \geq k_2(n) \cdot \frac{k_1(n)}{n} \cdot D(\mathcal{X}_0) \geq \frac{\alpha_1}{4^{\alpha_2^2}}
\]

(the second inequality can be established analogously to the proof of Claim 3.1). The result follows.  \(\blacksquare\) Lemma 5.3
5.1 Solving $P^{(n)}$ implies solving $P_{1 \times 1}^{(n)}$

Lemma 5.3. Assume that there exists a protocol $S$ of cost $k$ that solves $P^{(n)}$ with error bounded by $\varepsilon$. Then $P_{1 \times 1}^{(n)}$ can be solved w.r.t. $U_{1 \times 1}^{(n;2)}$ with probability $\frac{1}{2n}$ with error bounded by $3\varepsilon$ by a protocol of cost $O(k)$.

Proof of Lemma 5.3. Let $P^{(n)}$ be a version of $P^{(n)}$ without the promise that at least $\frac{1}{14}$th part of the intersections contain 2 elements (in particular, there are input pairs not giving rise to any intersection of size 2 and therefore admitting no right answer). Let $U_{1 \times 1}$ be the uniform input distribution for $P^{(n)}$ (corresponding to choosing $x$ as a uniformly random subset of $X_{n,2}$ of size $n/2$ and $y$ as a uniformly random partition of $X_{n,2}$ into $n$ subsets of equal size).

It follows from Claim 3.1 and Chernoff bound that if we randomly choose $(x, y) \sim U_{1 \times 1}$ then the probability that the pair does not satisfy the condition of $P^{(n)}$ is exponentially small in $n$. Therefore, for $n$ large enough $S$ solves $P^{(n)}$ w.r.t. $U_{1 \times 1}$ with error bounded by $\frac{3\varepsilon}{2}$. Let $S'$ be a deterministic protocol solving $P^{(n)}$ w.r.t. $U_{1 \times 1}$ with error bounded by $\frac{3\varepsilon}{2}$.

Fix $U_{1 \times 1}$ to be the input distribution. Let $C_i$ be the event that the list of pairs produced by $S'$ contains $(i, a)$ for some $a \in X_{n,2}$, such that $|x \cap y_i| = 2$. Define $p_i$ to be the probability that $C_i$ occurs and $q_i$ to be the probability that the protocol is successful, conditioned on $C_i$.

The obvious counting argument implies that there exists $i_0$ such that $p_{i_0} \geq \frac{1}{2n}$ and $q_{i_0} \leq 3\varepsilon$ (assume the opposite, then the error of $S'$ is higher than $(1 - n \cdot \frac{1}{2n}) \cdot 3\varepsilon = \frac{3\varepsilon}{2}$).

Let $(x, y) \sim U_{1 \times 1}^{(n;2)}$. Consider a protocol $S''$ that behaves as follows.

1. Construct $y' = (y'_1, \ldots, y'_n)$ by setting $y_{i_0} = y$ and choosing $(y_1, \ldots, y_{i_0-1}, y_{i_0+1}, \ldots, y_n)$ as a random partition of $X_{n,2} \setminus y$ into $n - 1$ subsets of size $n$ each.
2. Execute $S'$ providing it with input $(x, y')$; let $z = ((i_1, a_1), \ldots, (i_t, a_t))$ be the list produced by $S'$.
3. If $i_0 = i_{j_0}$ for some $1 \leq j_0 \leq t$ output $a_{j_0}$; otherwise refuse to answer.

The protocol $S''$ can be simulated with only constant overhead over $S'$, given access to public randomness.

Observe that the instance of $P^{(n)}$ fed into $S'$ by $S''$ is distributed according to $U_{1 \times 1}$, defined as $U_{1 \times 1}$ with an extra-condition that $|x \cap y_{i_0}| = 2$. But that condition is implied by $C_{i_0}$, therefore $i_0 = i_{j_0}$ for some $j_0$ with probability at least $p_{i_0} \geq \frac{1}{2n}$, in which case $S''$ produces an answer. And the answer is correct with probability at least $1 - q_{i_0} \geq 1 - 3\varepsilon$, as required. \[\square\]

5.2 Solving $\hat{P}_{1 \times 1}^{(n)}$ is as simple as solving $P_{1 \times 1}^{(n)}$

We will show the following.

Theorem 5.4. Assume that there exists a protocol of cost $k \in O(n)$ that solves $P_{1 \times 1}^{(n)}$ w.r.t. $U_{1 \times 1}^{(n;2)}$ with probability $\gamma$ with error bounded by $\frac{1}{10n}$. Then $\hat{P}_{1 \times 1}^{(n)}$ can be solved w.r.t. $U_{1 \times 1}^{(n;2)}$ with probability $\frac{\gamma}{k^2 \log^2(n/\gamma)}$ in 0-error setting by a protocol of cost $O(k + \log^2(n/\gamma))$.

The proof will be done in several stages.
Lemma 5.5. Let $n$ be sufficiently large and $A$ be an input rectangle for $P_{1 \times 1}$, such that $\mathcal{U}_{1 \times 1}^{(n;1)}(A) \in 2^{-o(n)}$. Assume that for some $0 < \varepsilon < 1$ and $I_0 \subseteq X_{n^2}$, $|I_0| \geq \frac{n^2}{2}$, it holds that

$$\sum_{i \in I_0} \mathcal{U}_A^{(1)}(\mathcal{X}_i(i)) \leq \frac{\varepsilon^2}{2600000}.$$ 

Then $\mathcal{U}_A^{(0,1)}(\mathcal{X}_0) < \varepsilon$.

The intuitive meaning of this lemma is that a rectangle that accepts pairs from $\mathcal{X}_i$ intersecting mostly over $X_{n^2} \setminus I_0$ must reject pairs from $\mathcal{X}_0$ with high probability.

Proof of Lemma 5.5. We will analyze subrectangles of $A$. Define a predicate $\chi_A(z_1, z_2)$ over $\{0, 1\}^{I_0} \times \{0, 1\}^{I_0}$ that indicates whether the subrectangle $\{(x, y) \in A | x|_{I_0} = z_1, y|_{I_0} = z_2\}$ recognizes $\text{DISJ}$ over $I_0 \times I_0$ with error at most $\frac{\varepsilon}{2}$. Formally, $\chi_A(z_1, z_2)$ is satisfied if

$$\Pr_{(x, y) \sim \mathcal{U}_A^{(0,1)}} \left[ x \cap y|_{I_0} \neq \emptyset \middle| x|_{I_0} = z_1, y|_{I_0} = z_2 \right] \geq 1 - \frac{\varepsilon}{2}.$$

Proposition. $\chi_A(z_1, z_2)$ holds with probability at least $1 - \frac{\varepsilon}{2}$, when the pair $(z_1, z_2)$ is chosen according to $\mathcal{U}_A^{(0,1)}||_{I_0 \times I_0}$.

Observe that the proposition implies our lemma, as follows.

$$\mathcal{U}_A^{(0,1)}(\mathcal{X}_i) \geq \Pr_{\mathcal{U}_A^{(0,1)}||_{I_0 \times I_0}} \left[ \chi_A(z_1, z_2) \right] \cdot \Pr_{\mathcal{U}_A^{(0,1)}} \left[ x \cap y|_{I_0} \neq \emptyset \middle| x|_{I_0} = z_1, y|_{I_0} = z_2 \right] > 1 - \varepsilon,$$

as required.

Assume that the proposition is false. We will derive a contradiction by showing that the rectangle $A$ contains many instances of $\mathcal{X}_0$ and applying what can be viewed as a stronger form of Lemma 5.2. Not only it follows that the rectangle contains many instances of $\mathcal{X}_i$, but also the distribution of $\{x \cap y\}$ is close to uniform. In particular, the shared element will often belong to $I_0$, which contradicts our assumption.

Let $C$ be the set of pairs $(z_1, z_2) \in \{0, 1\}^{I_0} \times \{0, 1\}^{I_0}$ falsifying $\chi_A(z_1, z_2)$ and $C' \overset{\text{def}}{=} \{(z_1, z_2) \in C | z_1 \cap z_2 = \emptyset\}$. Then $\mathcal{U}_A^{(0,1)}||_{I_0 \times I_0}(C) > \frac{\varepsilon}{2}$ and $\mathcal{U}_A^{(0,1)}||_{I_0 \times I_0}(C') > \frac{\varepsilon}{2} - \frac{\varepsilon^2}{2600000}$, as

$$\Pr_{\mathcal{U}_A^{(0,1)}} \left[ x \cap y|_{I_0} \neq \emptyset \right] \leq \frac{\varepsilon^2}{2600000}.$$

Let

$$G \overset{\text{def}}{=} \{(z_1, z_2) \in \{0, 1\}^{I_0} \times \{0, 1\}^{I_0} | |z_1| < n/6 \text{ or } |z_2| < n/3\}$$

and $C'' \overset{\text{def}}{=} C' \setminus G$, then $\mathcal{U}_A^{(0,1)}||_{I_0 \times I_0}(C'') \geq \frac{\varepsilon}{2} - \frac{\varepsilon^2}{2600000} - \mathcal{U}_A^{(0,1)}||_{I_0 \times I_0}(G)$. Observe that a pair $(z_1, z_2)$ randomly chosen according to $\mathcal{U}_A^{(0,1)}||_{I_0 \times I_0}$ belongs to $G$ with at most twice the probability that a randomly chosen $x \subseteq X_{n^2}$ of size $n/2$ has less than $n/6$ elements inside $I_0$. That is not greater than the probability that some event that occurs with probability $\frac{n^5/2 - n/6}{n^5} > 3/7$ (for large enough $n$) has taken place at most $n/6$ times during $n/2$ independent
trials. By Chernoff bound, that is $2^{-\Omega(n)}$. Based on the lemma assumption that $U^{(n;1)}(A) \in 2^{-\circ(n)}$ we conclude that $U^{(0,1)}_{A}|_{I_0 \times I_0}(G) \leq \frac{U^{(n;1)}_{1 \times 1}(A)|_{I_0 \times I_0}(G)}{U^{(n;1)}_{1 \times 1}(A)} \in 2^{-\Omega(n)}$.

Define $\tilde{C} \equiv \{(x, y) \in A \cap X_0| (x|_{I_0}, y|_{I_0}) \in C^n \}$, it holds that $U^{(0,1)}(\tilde{C}) \geq (\frac{\varepsilon}{2} - \frac{\varepsilon^2}{2000000} - 2^{-\Omega(n)}) \cdot \frac{\varepsilon}{2}$ by the definition of $\chi_A$. Let us bound the size of $\tilde{C}$ as a portion of $X_0 \cup X_1$. For sufficiently large $n$ it holds that

$$U^{(n;0,1)}(\tilde{C}) = U^{(0,1)}_A(\tilde{C}) \cdot U^{(n;0,1)}_1(A) \geq U^{(0,1)}_A(\tilde{C}) \cdot U^{(n;1)}_1(A) \cdot U^{(n;0,1)}_1(X_1)$$

$$\geq U^{(n;1)}_1(A) \cdot (\frac{\varepsilon}{2} - \frac{\varepsilon^2}{2000000} - 2^{-\Omega(n)}) \cdot \frac{\varepsilon}{2} \geq \frac{\varepsilon^2}{25} \cdot U^{(n;1)}_1(A).$$

Take some $(x', y') \in \tilde{C}|_{I_0 \times I_0}$ and consider the rectangle

$$B_{x', y'} \equiv \{(x, y) \in A \mid (x|_{I_0}, y|_{I_0}) = (x', y')\}.$$

We would like to claim that $B_{x', y'}$ is large. However, in fact $B_{x', y'}$ can be small for some pairs $(x', y') \in \tilde{C}|_{I_0 \times I_0}$, but that cannot happen too often w.r.t. randomly chosen $(x', y') \in \tilde{C}|_{I_0 \times I_0}$.

Let us formalize this intuition.

The natural terminology to be used here is that of entropy of distributions. Let $\tilde{C}$ be a subset of $C$ satisfying $\frac{1}{4} \leq U^{(0,1)}_C(\tilde{C}) \leq \frac{1}{2}$ and minimizing the value of

$$E_{(x_1', x_2', y_1', y_2')} \sum_{\tilde{C}} \left[ H_{U^{(0,1)}_C} [x_2, y_2 | x_1 = x_1', y_1 = y_1'] \right]$$

(for $n$ large enough it is always possible to find a subset of right size).

Let us treat $U^{(n)}_{1 \times 1}$ as a distribution of 4-tuples $(x_1, x_2, y_1, y_2)$, where $x_1, y_1 \subseteq I_0$ and $x_2, y_2 \subseteq I_0$, and denote

$$H_{U^{(n;0,1)}_{1 \times 1}} [x_1, y_1] \equiv \frac{H_{U^{(n;0,1)}_{1 \times 1}} [x_1, y_1]}{U^{(n;0,1)}_{1 \times 1} |_{I_0 \times I_0}}$$

and

$$H_{U^{(n;0,1)}_{1 \times 1}} [x_2, y_2 | x_1, y_1] \equiv \frac{H_{U^{(n;0,1)}_{1 \times 1}} [x_2, y_2 | x_1, y_1]}{U^{(n;0,1)}_{1 \times 1} |_{I_0 \times I_0}}.$$
and
\[
\begin{align*}
\mathbf{H}_{\mathcal{U}_{1 \times 1}^{(0,1)}} - \mathbf{H}_{\mathcal{U}_{1 \times 1}^{\rho,1}} &= \mathbf{H}_{\mathcal{U}_{1 \times 1}^{(0,1)}}[x_1, y_1] - \mathbf{H}_{\mathcal{U}_{1 \times 1}^{0,1}}[x_1, y_1] \\
&\quad + \mathbf{H}_{\mathcal{U}_{1 \times 1}^{(0,1)}}[x_2, y_2|x_1, y_1] - \mathbf{H}_{\mathcal{U}_{1 \times 1}^{0,1}}[x_2, y_2|x_1, y_1].
\end{align*}
\]

Since it is obviously true that \(\mathbf{H}_{\mathcal{U}_{1 \times 1}^{(0,1)}}[x_1, y_1] \geq \mathbf{H}_{\mathcal{U}_{1 \times 1}^{\rho,1}}[x_1, y_1]\), we conclude that
\[
\begin{align*}
\mathbf{H}_{\mathcal{U}_{1 \times 1}^{(0,1)}} - \mathbf{H}_{\mathcal{U}_{1 \times 1}^{\rho,1}} \geq \mathbf{H}_{\mathcal{U}_{1 \times 1}^{(0,1)}}[x_2, y_2|x_1, y_1] - \mathbf{H}_{\mathcal{U}_{1 \times 1}^{\rho,1}}[x_2, y_2|x_1, y_1].
\end{align*}
\]

Note that \(\mathbf{H}_{\mathcal{U}_{1 \times 1}^{(0,1)}} - \mathbf{H}_{\mathcal{U}_{1 \times 1}^{\rho,1}}\) is equal to the difference of the logarithms of sizes of corresponding supports, which is equal to the logarithm of the ratio of the sizes. That is,
\[
\mathbf{H}_{\mathcal{U}_{1 \times 1}^{\rho,1}}[x_2, y_2|x_1, y_1] \geq \mathbf{H}_{\mathcal{U}_{1 \times 1}^{(0,1)}}[x_2, y_2|x_1, y_1] - \log \left(\frac{1}{\mathcal{U}_{1 \times 1}^{(0,1)}(\tilde{C})}\right).
\]

Recall that \(\tilde{C} \subseteq \tilde{C}\) is minimizing the value of
\[
E_{(x',y',y_2) \sim \mathcal{U}_{\tilde{C}}^{\rho,1}} \left[ \mathbf{H}_{\mathcal{U}_{1 \times 1}^{\rho,1}}[x_2, y_2|x_1 = x', y_1 = y_1'] \right] \geq \mathbf{H}_{\mathcal{U}_{1 \times 1}^{\rho,1}}[x_2, y_2|x_1, y_1].
\]

Therefore, for any \((x_1', y_1') \in (\tilde{C} \setminus \tilde{C})||_T \times \{0, 1\}^T\) it holds that
\[
\mathbf{H}_{\mathcal{U}_{1 \times 1}^{\rho,1}}[x_2, y_2|x_1 = x', y_1 = y_1'] \geq \mathbf{H}_{\mathcal{U}_{1 \times 1}^{(0,1)}}[x_2, y_2|x_1, y_1] - \log \left(\frac{4}{\mathcal{U}_{1 \times 1}^{(0,1)}(\tilde{C})}\right). \tag{2}
\]

Denote by \(G'\) the set of pairs \((x_1', y_1') \in \{0, 1\}^T \times \{0, 1\}^T\) for which it holds that
\[
\mathbf{H}_{\mathcal{U}_{1 \times 1}^{(0,1)}}[x_2, y_2|x_1 = x', y_1 = y_1'] \geq \mathbf{H}_{\mathcal{U}_{1 \times 1}^{(0,1)}}[x_2, y_2|x_1, y_1] + \log \left(\frac{4}{\mathcal{U}_{1 \times 1}^{(0,1)}(\tilde{C})}\right),
\]

and let \(G' \equiv \{(x_1, x_2, y_1, y_2) | (x_1, y_1) \in G'\}\). Then
\[
\begin{align*}
\mathbf{H}_{\mathcal{U}_{1 \times 1}^{(0,1)}}[x_1, x_2, y_1, y_2] &= \mathbf{H}_{\mathcal{U}_{1 \times 1}^{\rho,1}}[x_1, y_1] + \mathbf{H}_{\mathcal{U}_{1 \times 1}^{(0,1)}}[x_2, y_2|x_1, y_1] \\
&\geq \mathbf{H}_{\mathcal{U}_{G'}^{\rho,1}}[x_1, x_2, y_1, y_2] = \mathbf{H}_{\mathcal{U}_{G'}^{\rho,1}}[x_1, y_1] + \mathbf{H}_{\mathcal{U}_{G'}^{\rho,1}}[x_2, y_2|x_1, y_1] \\
&\geq \mathbf{H}_{\mathcal{U}_{G'}^{\rho,1}}[x_1, y_1] + \mathbf{H}_{\mathcal{U}_{1 \times 1}^{(0,1)}}[x_2, y_2|x_1, y_1] + \log \left(\frac{4}{\mathcal{U}_{1 \times 1}^{(0,1)}(\tilde{C})}\right),
\end{align*}
\]
Therefore,

\[ \mathcal{U}_{x_{0}}^{(n,1)}(\bar{G}^0) = 1 \left/ \exp \left( \frac{\mathbf{H}_{x_{0}}^{(n,1)}[x_{1}, y_{1}] - \mathbf{H}_{x_{0}}^{(n,1)}[x_{1}, y_{1}]}{4 \mathcal{U}_{x_{0}}^{(n,1)}(\bar{C})} \right) \right. \]

Define

\[ \tilde{C} \overset{\text{def}}{=} \left\{ (x', y') \in \tilde{C} \left| \exists (x'_1, y'_1) \in (\tilde{C} \setminus \tilde{C}) \right| \mathcal{T}_0 \times \mathcal{T}_0 \right\} \]

By definition, \( \mathcal{U}_{x_{0}}^{(n,1)}(\tilde{C}) \leq \frac{1}{2} \cdot \mathcal{U}_{x_{0}}^{(n,1)}(\bar{C}) \), and because \( \tilde{C} \supseteq (\tilde{C} \setminus \tilde{C}) \),

\[ \mathcal{U}_{x_{0}}^{(n,1)}(\tilde{C}) \geq \mathcal{U}_{x_{0}}^{(n,1)}(\tilde{C}) - \mathcal{U}_{x_{0}}^{(n,1)}(\tilde{C}) \geq \frac{1}{4} \cdot \mathcal{U}_{x_{0}}^{(n,1)}(\bar{C}) \quad \text{(3)} \]

Now fix any \((x', y') \in \tilde{C} \setminus \mathcal{T}_0 \times \mathcal{T}_0\) and consider \(B_{x', y'}\). We want to apply Lemma 5.2 to \(B_{x', y'}\). Consider the function \(d(\cdot, \cdot)\) over \(\{0, 1\}^I \times \{0, 1\}^I\), defined as \(d(a, b) = \text{DISJ}(x_{a}, y_{b})\), where \(x_{a} \subseteq X_{n^2}\) defined by \(x_{a}|_{I_0} = a\) and \(x_{a}|_{I_0} = x'\), and similarly \(y_{b} \subseteq X_{n^2}\) satisfies \(y_{b}|_{I_0} = b\) and \(y_{b}|_{I_0} = y'\). Our construction guarantees that \(x' \cap y' = \emptyset\), and therefore \(d(a, b)\) is in fact equal to \(\text{DISJ}_{I_0}(a, b)\).

Let \(\mathcal{U}_{d}\) be the distribution resulting from \(\mathcal{U}_{x_{0}}^{(n,1)}\) conditioned upon \((x|_{I_0}, y|_{I_0}) = (x', y')\). Note that \(B_{x', y'} = B_{x', y'}|_{I_0} \times I_0\) is a rectangle for \(d(a, b)\) satisfying

\[ \mathcal{U}_{d}|_{I_0 \times I_0} \geq \mathcal{U}_{d}|_{I_0 \times I_0} \left( \left( B_{x', y'} \cap \tilde{C}' \right) \right|_{I_0 \times I_0} \]

\[ = \mathbf{Pr}_{\mathcal{U}_{x_{0}}^{(n,1)}}[(x, y) \in \tilde{C}' \left| (x|_{I_0}, y|_{I_0}) = (x', y') \right.]. \]

It follows from the definition of \(G'\) together with (2) that

\[ \mathbf{H}_{x_{0}}^{(n,1)}[x_{2}, y_{2} | x_{1} = x', y_{1} = y'] \geq \mathbf{H}_{x_{0}}^{(n,1)}[x_{2}, y_{2} | x_{1} = x', y_{1} = y'] - 2 \log \left( \frac{4 \mathcal{U}_{x_{0}}^{(n,1)}(\bar{C})}{16} \right), \]

in other words

\[ \mathbf{Pr}_{\mathcal{U}_{x_{0}}^{(n,1)}}[(x, y) \in \tilde{C}' \left| (x|_{I_0}, y|_{I_0}) = (x', y') \right.] \geq \frac{\left( \mathcal{U}_{x_{0}}^{(n,1)}(\bar{C}) \right)^2}{16}. \]

We conclude that

\[ \mathcal{U}_{d}|_{I_0 \times I_0} \geq \frac{\left( \mathcal{U}_{x_{0}}^{(n,1)}(\bar{C}) \right)^2}{16}. \]

Let us apply Lemma 5.2. The function \(d(a, b) = \text{DISJ}_{I_0}(a, b)\) with input distribution \(\mathcal{U}_{d}|_{I_0 \times I_0}\) and rectangle \(B_{x', y'}\) satisfy the requirements of the lemma with parameters \(\alpha_1 = \frac{n}{6 \sqrt{|I_0|}}\) and \(\alpha_2 = \frac{n}{\sqrt{|I_0|}}\), where \(|I_0| = n^2\) and \(|I_0| = \frac{n^2}{2}\) are respectively upper and lower bounds on the size of \(I_0\). Lemma 5.2 guarantees that for \(\delta = \frac{1}{25920}\),

\[ \mathcal{U}_{d}|_{I_0 \times I_0}(B_{x', y'} \cap X_{I}) \geq \delta \cdot \mathcal{U}_{d}|_{I_0 \times I_0}(B_{x', y'} \cap X_{I}) - 2^{-\Omega(n)}. \]
We have established that
\[ U_d \| I_0 \times I_0 \left( B'_{x',y'} \cap X_0 \right) \in \Omega \left( \left( U_{1 \times 1}^{(n,0,1)}(\tilde{C}) \right)^2 \right) \in \Omega \left( \left( U_{1 \times 1}^{(n,1)}(A) \right)^2 \right) \in 2^{-o(n)}, \]
where the second inclusion follows from (3). Therefore for \( n \) sufficiently large it holds that
\[ U_d \| I_0 \times I_0 \left( B'_{x',y'} \cap X_0 \right) \geq \frac{U_d \| I_0 \times I_0 \left( B'_{x',y'} \cap X_0 \right)}{25921}. \]

Recall that we view \( B_{x',y'} \) as a function of \( x' \) and \( y' \) and the above is true for any \((x', y') \in \tilde{C}' | T_0 \times T_0 \). So,
\[
\sum_{i \in I_0} U_{1 \times 1}^{(n;0,1)}(A \cap X_j(i)) \geq \frac{U_{1 \times 1}^{(n;0,1)}(\tilde{C}' \cap X_0)}{25921} = \frac{U_{1 \times 1}^{(n;0,1)}(\tilde{C}')}{25921} \geq \frac{U_{1 \times 1}^{(n;0,1)}(\tilde{C})}{103684} \geq \frac{\varepsilon^2 \cdot U_{1 \times 1}^{(n;1)}(A)}{2592100},
\]
where the second and the last inequalities follow from (3) and (1), respectively. Therefore,
\[
\sum_{i \in I_0} U_{1 \times 1}^{(1)}(X_j(i)) \geq \frac{1}{U_{1 \times 1}^{(n;1)}(A)} \cdot \sum_{i \in I_0} U_{1 \times 1}^{(n;0,1)}(A \cap X_j(i)) > \frac{\varepsilon^2}{2600000},
\]
which contradicts a lemma assumption. The result follows. \( \blacksquare \)

**Lemma 5.6**

**Corollary 5.6.** Let \( n \) be sufficiently large and \( A \) be an input rectangle for \( P_{1 \times 1}^{(n)} \), such that \( U_{1 \times 1}^{(n,2)}(A) \in 2^{-o(n)} \). Assume that for some \( 0 < \varepsilon < 1 \) and every \( i \in X_{n^2} \) there exists \( I_0(i) \subseteq X_{n^2} \setminus \{i\} \), \( |I_0(i)| \geq n^2 / 2 \), such that for every \( j \in X_{n^2} \), \( i \in I_0^{(j)} \) if and only if \( j \in I_0^{(i)} \) and
\[
\frac{1}{2} \sum_{i \in X_{n^2} \atop j \in I_0^{(i)}} U_{2}^{(i)}(X_2(i, j)) \leq \frac{\varepsilon^3}{10^{17}}.
\]

Then \( U_{A}^{(0,1,2)}(X_0 \cup X_1) < \varepsilon \).

**Proof of Corollary 5.6.** We will show that \( U_{A}^{(0,1,2)}(X_1) \leq \frac{\varepsilon^3}{10^{17}} \) and \( U_{A}^{(0,1,2)}(X_0) \leq \frac{\varepsilon^3}{10^{17}} \).

Define auxiliary constants \( c_1 \overset{\text{def}}{=} 7 \cdot 10^6 \) and \( c_2 \overset{\text{def}}{=} 4 \cdot 10^6 \). Define \( A_i \overset{\text{def}}{=} \{(x, y) \in A | i \in x \cap y \} \) for each \( i \in X_{n^2} \). Let \( D \) be the probability distribution over \( X_{n^2} \) defined by \( D(i) = \frac{1}{2} U_{A}^{(2)}(A_i) \), then choosing \((x, y) \sim U_{A}^{(2)}\) can be viewed as first choosing \( i \sim D \) followed by \((x, y) \sim U_{A}^{(2)}\).

The main condition of the corollary can be written as
\[
\mathbb{E} \left[ \sum_{j \in I_0^{(i)}} U_{A_i}^{(2)}(X_2(i,j)) \right] \leq \frac{\varepsilon^3}{10^{17}}.
\]
Let

\[ I_1 \overset{\text{def}}{=} \left\{ i \in X_n^2 \mid U^{(n;2)}_{1 \times 1}(A_i) < \frac{\varepsilon}{c_2 \cdot n^2} \cdot U^{(n;2)}_{1 \times 1}(A) \right\}, \]

\[ I_2 \overset{\text{def}}{=} \left\{ i \in X_n^2 \mid \sum_{j \in I(i)} U^{(2)}_{A_i}(X_2(i, j)) > \frac{\varepsilon^2}{10^{17}} \right\}. \]

For any \( i_0 \in X_n^2 \setminus I_1 \setminus I_2 \) it holds that \( U^{(n;2)}_{1 \times 1}(A_{i_0}) \geq \frac{\varepsilon}{c_2 \cdot n^2} U^{(n;2)}_{1 \times 1}(A) \in 2^{-\Theta(n)} \). We can treat \( A_{i_0} \) as an input rectangle for \( P_{1 \times 1}^{(n-1)} \) defined over \( X_n^2 \setminus \{i_0\} \). The properties of \( I_0(i_0) \) allow us to apply Lemma 5.3, concluding that \( U^{(1, 2)}_{A_{i_0}}(X_i) < \sqrt{2.6 \cdot 10^{-11} \cdot \varepsilon^2 \cdot c_1}. \)

On the other hand, it holds that

\[ \sum_{i \in I_1} U^{(n;2)}_{1 \times 1}(A_i) < \frac{\varepsilon}{c_2} \cdot U^{(n;2)}_{1 \times 1}(A) \Rightarrow \sum_{i \in I_1} U^{(2)}_{A}(A_i) < \frac{\varepsilon}{c_2}, \]

and

\[ D(I_2) < \frac{\varepsilon}{c_1} \Rightarrow \sum_{i \in I_2} U^{(2)}_{A}(A_i) < \frac{2 \varepsilon}{c_1}, \]

that is,

\[ \sum_{i \in I_1 \cup I_2} U^{(2)}_{A}(A_i) < \frac{\varepsilon}{c_2} + \frac{2 \varepsilon}{c_1}. \]

We treat each \( A_{i_0} \) as an input rectangle for \( P_{1 \times 1}^{(n-1)} \) and apply Lemma 5.2, concluding that for \( \delta = \frac{1}{\sqrt{20}} \) the following holds.

\[ \sum_{i \in I_1 \cup I_2} U^{(n)}_{1 \times 1}(A_i \cap X_i) \leq \frac{1}{\delta} \cdot \sum_{i \in I_1 \cup I_2} U^{(n)}_{1 \times 1}(A_i \cap X_2) + n^2 \cdot 2^{-\Omega(n)}, \]

and because \( U^{(n)}_{1 \times 1}((X_1 \cup X_2) \cap A) \geq U^{(n)}_{1 \times 1}(X_2) \cdot U^{(n;2)}_{1 \times 1}(A) \in 2^{-\Theta(n)}, \)

\[ \sum_{i \in I_1 \cup I_2} U^{(1, 2)}_{A}(A_i \cap X_i) \leq \frac{1}{\delta} \cdot \sum_{i \in I_1 \cup I_2} U^{(1, 2)}_{A}(A_i \cap X_2) + 2^{2 \log n - \Omega(n) + o(n)} \]

\[ \leq \frac{1}{\delta} \cdot \sum_{i \in I_1 \cup I_2} U^{(2)}_{A}(A_i) + 2^{-\Omega(n)} \leq \frac{\varepsilon}{\delta \cdot c_2} + \frac{2 \varepsilon}{\delta \cdot c_1} + 2^{-\Omega(n)}. \]

We conclude:

\[ U^{(0, 1, 2)}_{A}(X_1) \leq U^{(1, 2)}_{A}(X_1) = \sum_{i \in X_n^2} U^{(1, 2)}_{A}(A_i \cap X_i) \]

\[ = \sum_{i \in I_1 \cup I_2} U^{(1, 2)}_{A}(A_i \cap X_i) + \sum_{i \notin I_1 \cup I_2} U^{(1, 2)}_{A}(A_i) \cdot U^{(1, 2)}_{A_{i_0}}(X_i) \]

\[ < \frac{\varepsilon}{\delta \cdot c_2} + \frac{2 \varepsilon}{\delta \cdot c_1} + 2^{-\Omega(n)} + \sqrt{2.6 \cdot 10^{-11} \cdot \varepsilon^2 \cdot c_1} < \frac{\varepsilon}{1441}. \]

\(^3\)Strictly speaking, this violates our requirement that \( n \) is a power of 2 and slightly affects the Hamming waits of \( x \) and \( y \) as functions of \( n \), though the former is irrelevant for the present context and the influence of the latter is negligible for sufficiently large \( n \).
for sufficiently large \( n \).

Let us use Lemma 5.2 again, for the same value of \( \delta \) it holds that
\[
U_{1 \times 1}^{(n)}(A \cap X_0) \leq \frac{1}{\delta} \cdot U_{1 \times 1}^{(n)}(A \cap X_1) + 2^{-\Omega(n)}.
\]

It holds that \( U_{1 \times 1}^{(n)}((X_0 \cup X_1 \cup X_2) \cap A) \geq U_{1 \times 1}^{(n)}(X_2) \cdot U_{1 \times 1}^{(n,2)}(A) \in 2^{-\sigma(n)} \), and therefore
\[
U_{A}^{(0,1,2)}(X_0) \leq \frac{1}{\delta} \cdot U_{A}^{(0,1,2)}(X_1) + 2^{-\Omega(n)} < \frac{720 \cdot \varepsilon}{1441} + 2^{-\Omega(n)} < \varepsilon
\]
for sufficiently large \( n \). The result follows.

**Corollary 5.6**

The next lemma will be the last preparation step before we prove the main result of this section.

**Lemma 5.7.** Let \( n \) be sufficiently large and \( A \) be an input rectangle for \( P_{1 \times 1}^{(n)} \), such that \( U_A(X_0 \cup X_1) \leq \frac{1}{6} \) and for some \( 0 < \delta < 1 \)
\[
\Pr_{y \sim U_{A}^{Bo}} \left[ \exists \{a, b\} \subset y : \Pr_{x \sim U_{A}^{A}}[\{a, b\} \subset x] \geq \delta \right] \leq \frac{1}{3}
\]

Then \( U_{1 \times 1}^{(n)}(A) \in 2^{-\Omega\left(\frac{1}{\sqrt{\delta}}\right)} \).

**Proof of Lemma 5.7.** Let \( B \) be the set of \( y \in X_n^2 \), such that
\[
\Pr_{x \sim U_{A}^{A}}[|x \cap y| < 2] \leq \frac{1}{3} \quad (4)
\]
and
\[
\forall \{a, b\} \subset y : \Pr_{x \sim U_{A}^{A}}[\{a, b\} \subset x] < \delta. \quad (5)
\]

If we choose \( y' \sim U_{A}^{Bo} \) then (4) holds with probability at least \( \frac{1}{2} \) and (5) holds with probability at least \( \frac{2}{3} \), therefore \( |B \cap A||B_0| \geq \frac{1}{6} |A||B_0| \). Denote \( A' \overset{\text{def}}{=} A||A_1 \times B \), then \( U_{1 \times 1}^{(n)}(A') \geq \frac{1}{6} U_{1 \times 1}^{(n)}(A) \).

For \( a, b \in X_n^2 \), \( a \neq b \), let \( p_a \overset{\text{def}}{=} \Pr_{x \sim U_{A}^{A}}[a \in x] \) and \( p_b^{(a)} \overset{\text{def}}{=} \Pr_{x \sim U_{A}^{A}}[b \in x|a \in x] \). Condition (5) holds only if
\[
\forall a \in y : \left( p_a \geq \sqrt{\delta} \Rightarrow \forall b \in y, b \neq a : p_b^{(a)} < \sqrt{\delta} \right). \quad (6)
\]

Let \( a_0 \in y \) be the lexicographically first value satisfying \( p_{a_0} = \max_{i \in y} \{ p_i \} \). Think about the process of choosing \( y \sim U_{B_0} \) as first choosing \( a_0 \) and then the rest of the elements. We will see that conditions (4) and (6) are not likely to hold simultaneously.

First let us consider the situation when
\[
\forall a \in y : p_a < \sqrt{\delta}. \quad (7)
\]
Since $\Pr[|x \cap y| \geq 1|x \in A\|A]\geq \frac{2}{3}$ only if $\sum_{a \in y} p_a \geq \frac{2}{3}$, the probability that (1) and (7) hold is upper bounded by the probability that

$$\sum_{a \in y} p'_a \geq \frac{2}{3},$$

where $p'_a \overset{\text{def}}{=} \begin{cases} p_a & \text{if } p_a < \sqrt{\delta} \\ 0 & \text{otherwise} \end{cases}$.

Let $Z_1, \ldots, Z_n$ be the elements of $y$ and denote $W_i \overset{\text{def}}{=} p'_{Z_i}$. We want to use Chernoff bound in order to limit from above the value of $\sum_{i=1}^n W_i$. Strictly speaking, the variables $W_i$ are not independent (because all $Z_i$-s are different), but their dependence is relatively small, which makes it possible to apply Chernoff bound using the “worst case” estimation of the variable’s mean values. Note that for $n$ large enough and any $1 \leq i_0 \leq n$ it holds that $W_{i_0} \leq \sqrt{\delta}$ and $\mathbb{E}[W_{i_0}] \leq \frac{n/2}{n^2 - n} < \frac{3}{5n}$, where the mean value is computed w.r.t. repeated “experiments”, for the fixed $i_0$. Based on Chernoff bound, we conclude that

$$\Pr_{y \sim \mathcal{U}_{Bo}} \left[ \sum_{a \in y} p'_a \geq \frac{2}{3} \right] \in 2^{-\Omega\left(\frac{1}{\sqrt{\delta}}\right)}.$$ (9)

Now consider the situation when

$$p_{a_0} \geq \sqrt{\delta} \text{ and } \forall b \in y, b \neq a_0 : \quad p_{b(a_0)} < \sqrt{\delta}. \quad \text{(10)}$$

Since $\Pr[|x \cap y| \geq 2|x \in A\|A], a_0 \in y] \geq \frac{2}{3}$ only if $\sum_{b \in y, b \neq a_0} p_{b(a_0)} \geq \frac{2}{3}$, the probability that (1) and (10) hold is upper bounded by the probability that

$$\sum_{b \in y, b \neq a_0} p_{b(a_0)}' \geq \frac{2}{3},$$

where $p_{b(a_0)}' \overset{\text{def}}{=} \begin{cases} p_{b(a_0)} & \text{if } p_{b(a_0)} < \sqrt{\delta} \\ 0 & \text{otherwise} \end{cases}$.

It is easy to see that bound (9) on the probability that (8) holds is also an upper bound on the probability that (11) holds, and therefore

$$U_{1 \times 1}(A) \leq 6 \cdot U_{1 \times 1}(A') \leq 6 \cdot \Pr_{y \sim \mathcal{U}_{Bo}} [y \in B] \in 2^{-\Omega\left(\frac{1}{\sqrt{\delta}}\right)},$$

as required. \hfill $\blacksquare$

**Lemma 5.7**

We are ready for the

**Proof of Theorem 5.4.** Let $S$ be a deterministic protocol of cost $k$ solving $P^{(n)}_{1 \times 1}$ w.r.t. $U^{(n/2)}_{1 \times 1}$ with probability $\gamma$ with error bounded by $\frac{1}{m}$.

We will call a rectangle $A$ $\delta$-labeled if

$$\Pr_{y \sim \mathcal{U}_{Bo}} \left[ \exists \{a, b\} \subset y : \quad \Pr_{x \sim U_{A}}[\{a, b\} \subset x] \geq \delta \right] \geq \frac{1}{3},$$

as required.
Lemma 5.7 guarantees that if $\mathcal{U}_A(\mathcal{X}_0 \cup \mathcal{X}_1) \leq \frac{1}{6}$ and $\mathcal{U}^{(n)}_{1 \times 1}(A) \geq 2^{-2k}$ then there exists a function $\delta(k) \in \Omega\left(\frac{1}{k}\right)$, such that $A$ is $\delta(k)$-labeled.

Consider the rectangles defined by $S$. We will call a rectangle $A$ dark if it is not possible to define an answer that would solve $P^{(n)}_{1 \times 1}$ with probability at least $1 - \frac{2}{3^{2k}}$ w.r.t. $\mathcal{U}^{(2)}_A$. It follows from the properties of $S$ that $(x, y) \sim \mathcal{U}^{(n)}_{1 \times 1}(A)$ does not belong to a dark rectangle with probability at least $\frac{1}{2}$ (at least half of all pairs $(x, y) \in \mathcal{X}_2$ for which $S$ produces an answer belong to non-dark rectangles, since otherwise the error of $S$ would be greater than the allowed $\frac{1}{10^{19}}$). On the other hand, with probability at least $1 - \frac{1}{4}$ it happens that $(x, y) \sim \mathcal{U}^{(n)}_{1 \times 1}(A)$ falls into a rectangle $A$ satisfying $\mathcal{U}^{(n;2)}_{1 \times 1}(A) \geq 1 - \frac{2}{2^{k+1}}$. Note that for any such $A$ it holds that $\mathcal{U}^{(n)}_{1 \times 1}(A) \geq \mathcal{U}^{(n;2)}_{1 \times 1}(\mathcal{X}_2) \cdot \frac{7}{2^{k+1}} \geq 2^{-2k}$ for $n$ large enough (we can assume that $k \in \omega(1)$).

Call a rectangle $A$ good if it is not dark and $\mathcal{U}^{(n)}_{1 \times 1}(A) \geq 2^{-2k}$. It holds that $(x, y) \sim \mathcal{U}^{(n;2)}_{1 \times 1}(A)$ falls into a good rectangle with probability at least $\frac{1}{2} - \frac{1}{4} = \frac{1}{4}$. Consequently, $(x, y) \sim \mathcal{U}^{(n;2)}_{1 \times 1}(A)$ falls into a good rectangle with probability at least $\mathcal{U}^{(n)}_{1 \times 1}(\mathcal{X}_2) \cdot \frac{7}{2^{k+1}} \geq \frac{7}{8}$.

Let $A$ be good, we will see that $A$ is $\delta(k)$-labeled. We know that there exists some $z_A \in X_{n^2} \setminus \{0\}$, such that

$$1 - \frac{2}{10^{19}} \leq \Pr_{U^{(2)}_A}[\langle x, y, z_A \rangle \in P^{(n)}_{1 \times 1}] = \Pr_{U^{(2)}_A}[\langle z_A, a + b \rangle = 0],$$

where $x \cap y = \{a, b\}$. If we define $I_{0}^{(a)} \stackrel{\text{def}}{=} \{b \in X_{n^2} | \langle z_A, a + b \rangle = 1\}$ that will satisfy the requirement of Corollary 5.6 for $\varepsilon = \frac{1}{10}$, therefore it holds that $A \sim \mathcal{U}^{(n)}_{1 \times 1}(\mathcal{X}_0 \cup \mathcal{X}_1) < \frac{1}{6}$. We know that $\mathcal{U}^{(n)}_{1 \times 1}(A) \geq 2^{-2k}$, so we can apply the contrapositive of Lemma 5.7 (as sketched in the beginning of the proof), which guarantees that $A$ is $\delta(k)$-labeled.

Let us construct a protocol satisfying the promise of our theorem. Intuitively, we will use an efficient randomized mapping of any $(x, y) \in \mathcal{X}_2$ to $(x', y') \sim \mathcal{U}^{(n;2)}_{1 \times 1}(A)$, where $x'$ and $y'$ are disjoint, followed by $j_0$ lexicographically first element from $x$, denoted by $(x_1, \ldots, x_{j_0})$.

Let $D$ be the distribution over $[n]$ satisfying $D(j) \stackrel{\text{def}}{=} \mathcal{U}^{(n;2)}_{1 \times 1}(\mathcal{X}_j)$. Consider the following protocol $S'$.

1. Alice chooses $j_0 \sim D$. If $j_0 \geq 3 \log \left(\frac{3 \cdot 12}{2^{-\delta(k)}}\right)$ then the protocol stops and returns no answer. Otherwise Alice sends to Bob $j_0$ lexicographically first element from $x$, denoted by $(x_1, \ldots, x_{j_0})$.

2. Bob sends to Alice any two indices $i_1$ and $i_2$, such that $I_x \stackrel{\text{def}}{=} \{x_i | i = 1 \leq i \leq j_0 \setminus \{i_1, i_2\}\}$ and $y$ are disjoint, followed by $j_0$ lexicographically first element from $y$, denoted by $(y_1, \ldots, y_{j_0-1})$.

3. Let $i_3$ and $i_4$ be any two indices, such that $I_y \stackrel{\text{def}}{=} \{y_i | i = 1 \leq i \leq j_0 \setminus \{i_3, i_4\}\}$ and $x$ are disjoint, denote $x' \stackrel{\text{def}}{=} (x \cup I_y) \setminus I_x$.

4. Alice and Bob use public randomness to choose a random permutation $\sigma$ over the elements of $X_{n^2}$.

5. Alice and Bob run the protocol $S$ on the input $(\sigma(x), \sigma(y))$. Let $A$ be the rectangle defined by $S$, where $(\sigma(x), \sigma(y))$ belongs. If there exists no pair of distinct elements
\{a, b\} \subset y \text{ such that } \mathbf{Pr}_{x \sim \mathcal{U}_A} \left[ \{a, b\} \subset x \right] \geq \delta(k) \text{ then the protocol stops and returns no answer, otherwise let } (a', b') \text{ be any such pair.}

6. If \{\sigma^{-1}(a'), \sigma^{-1}(b')\} \subset x \cap y \text{ then the protocol stops and Alice outputs those two elements. Otherwise the protocol stops and returns no answer.}

It is clear that the protocol is 0-error and its communication cost is \( O(k + j_0 \cdot \log n) \leq O \left( k + \log^2 \left( \frac{n}{\gamma} \right) \right) \), let us calculate the probability that it produces an answer.

For the purpose of analysis we consider an “idealized” protocol \( S'' \), similar to \( S' \) but having no halting condition in stage 1, i.e., \( S'' \) continues to run regardless of the value of \( j_0 \).

Define three events:
- Let \( E_1 \) be the event that in the stage 5 of \( S'' \) a pair \((a', b')\) has been chosen and \( \{\sigma^{-1}(a'), \sigma^{-1}(b')\} \subset \tilde{x} \cap y \).
- Let \( E_2 \) be the event that \( j_0 \leq 3 \log \left( \frac{312}{\gamma \cdot \delta(k)} \right) \) and \( E_1 \) occurs.
- Let \( E_3 \) be the event that \( j_0 \leq 3 \log \left( \frac{312}{\gamma \cdot \delta(k)} \right) \) and \( S'' \) is successful, i.e., \((a', b')\) has been chosen and \( \{\sigma^{-1}(a'), \sigma^{-1}(b')\} \subset x \cap y \).

Obviously, the probability that \( S' \) is successful is equal to the probability that \( E_3 \) occurs.

\( E_1 \) occurs if \( \left( (\sigma(\tilde{x}), \sigma(y)) \right) \text{ belongs to a } \delta(k)\text{-labeled rectangle} \) and \( \text{ for some } \{a', b'\} \subset y \text{ it holds that } \mathbf{Pr}_{x \sim \mathcal{U}_A} \left[ \{a', b'\} \subset x \right] \geq \delta(k) \) \text{ and } \{a', b'\} \subset x \}; \text{ let us denote these events by } \mathcal{E}_1(1), \mathcal{E}_1(2) \text{ and } \mathcal{E}_1(3). \text{ Note that because } \sigma \text{ is a uniformly random permutation and } j_0 \sim D, \text{ it holds that } (\sigma(\tilde{x}), \sigma(y)) \sim \mathcal{U}(n^{2+})_{1 \times 1}, \text{ and therefore } \mathbf{Pr} \left[ \mathcal{E}_1(1) \right] \geq \frac{\gamma}{37}. \text{ By the definition of a } \delta(k)\text{-labeled rectangle, } \mathbf{Pr} \left[ \mathcal{E}_1(2) \mid \mathcal{E}_1(1) \right] \geq \frac{1}{7}. \text{ Clearly, } \mathbf{Pr} \left[ \mathcal{E}_1(3) \mid \mathcal{E}_1(1) \right] \geq \delta(k). \text{ Therefore, } \mathbf{Pr} \left[ E_1 \right] \geq \gamma \cdot \frac{\delta(k)}{156}.

\( E_2 \) occurs if \( E_1 \) occurs and \( j_0 \leq 3 \log \left( \frac{312}{\gamma \cdot \delta(k)} \right) \), therefore

\[
\mathbf{Pr} \left[ E_2 \right] \geq \frac{\gamma \cdot \delta(k)}{156} - \mathbf{Pr} \left. \left[ \geq \frac{\gamma \cdot \delta(k)}{312} \right] \right] \geq \frac{\gamma \cdot \delta(k)}{312},
\]

where the second inequality follows from Claim 3.1.

Finally, \( E_3 \) occurs if \( E_2 \) occurs and the points \( \sigma^{-1}(a') \) and \( \sigma^{-1}(b') \) belong to \( x \cap y \). Given \( j_0 \), the randomized mapping of \((x, y)\) to \((\sigma(\tilde{x}), \sigma(y))\) produces a uniformly random instance according to \( \mathcal{U}(n^{2+})_{1 \times 1}(\mathcal{X}_j) \). Moreover, the two elements of \( x \cap y \) are mapped to uniformly random elements of \( \sigma(\tilde{x}) \cap \sigma(y) \). Therefore, the probability that \( \{\sigma^{-1}(a'), \sigma^{-1}(b')\} = x \cap y \) is equal to \( \frac{1}{j_0} \). But \( E_2 \) guarantees that \( j_0 \leq 3 \log \left( \frac{312}{\gamma \cdot \delta(k)} \right) \), therefore

\[
\mathbf{Pr} \left[ E_3 \mid E_2 \right] \in \Omega \left( \frac{1}{j_0} \log^2 \left( \frac{1}{\gamma \cdot \delta(k)} \right) \right). \text{ Recalling that } \delta(k) \in \Omega \left( \frac{1}{\gamma^2} \right), \text{ we conclude that } \mathbf{Pr} \left[ E_3 \right] \in \Omega \left( \frac{\gamma}{k^2 \log^2 (n/\gamma)} \right).
\]
5.3 Solving $\tilde{P}_{I \times 1}^{(n)}$ is expensive

It is not hard to see that a protocol of communication cost $k$ can solve $\text{DISJ}_n$ only with probability $O\left(\left(\frac{k}{2}\right)^t\right)$. In this section we will prove the following generalization of this statement.

**Theorem 5.8.** Let $1 \leq t \leq \frac{n}{2}$, then any 0-error protocol of cost $k \in \Omega(t \log n)$ solving $\tilde{P}_{I \times 1}^{(n)}$ w.r.t. $U_{1 \times 1}^{(n:t)}$ can succeed with probability $O\left(\left(\frac{kt}{n}\right)^t\right)$.

**Proof of Theorem 5.8.** Let $S$ be a 0-error protocol of cost $k$ solving $\tilde{P}_{I \times 1}^{(n)}$ w.r.t. $U_{1 \times 1}^{(n:t)}$ with probability $p_t^{(t)}$. Let us define $p_i^{(t)}$ for $i > t$ to be the probability that $S$ outputs $t$ elements from $x \cap y$ when $(x, y) \sim U_{1 \times 1}^{(n:i)}$.

**Proposition.** There exists an absolute constant $c$, such that for $i$, $t \leq i \leq \frac{n}{2}$,

$$p_i^{(t)} \leq \min\left\{\left(\frac{k}{n}\right)^t, \left(1 + \frac{ck}{n}\right) \cdot \left(1 - \frac{t}{i+1}\right) \cdot p_{i+1}^{(t)}\right\}.$$

Before we prove it let us see how the proposition implies the theorem. Let $n$ be sufficiently large such that $t + \frac{n}{2ck} < \frac{n}{2}$. If for any $i$, $t \leq i \leq t + \frac{n}{2ck}$, it holds that $p_i^{(t)} \leq \left(\frac{k}{n}\right)^t$ then let $i_0$ be the smallest value like that and $p_t^{(t)} \leq (1 + \frac{ck}{n})^{i_0-t} p_{i_0}^{(t)} \in O\left(\left(\frac{k}{n}\right)^t\right)$. Otherwise

$$p_t^{(t)} \leq \left(1 + \frac{ck}{n}\right)^{\frac{t}{2ck}} \cdot \prod_{i=t}^{t+\frac{n}{2ck} - 1} \frac{i+1}{i+1} \cdot p_t^{(t)} \cdot \frac{n}{2ck} \leq 2 \frac{\prod_{i=1}^{t} i}{\prod_{j=1}^{i_0+t/n - 1} j} \in O\left(\left(\frac{kt}{n}\right)^t\right),$$

as required.

Now let us prove the proposition. Let $i_0 \geq t$ be such that $p_{i_0}^{(t)} > \left(1 - \frac{t}{i_0+t}\right) p_{i_0+1}^{(t)}$ and $p_{i_0}^{(t)} > \left(\frac{k}{n}\right)^t$, our goal is to show that $p_{i_0}^{(t)} \leq (1 + \frac{ck}{n}) \left(1 - \frac{t}{i_0+t}\right) p_{i_0+1}^{(t)}$. We will use the properties of $p_{i_0}^{(t)}$ to build a protocol solving $\text{DISJ}_m$ for $m \defeq n^2 - i_0$ w.r.t. some “nontrivial” distribution.

Consider the following public coin protocol $S'$ running on input $(x', y')$, such that $x' \subset [m], |x'| = n/2 - i_0, y' \subset [m], |y'| = n - i_0$.

1. Let $x'_0 \defeq x' \cup \{j\}_{j=m+1}^{n^2} \text{ and } y'_0 \defeq y' \cup \{j\}_{j=m+1}^{n^2}$. Alice and Bob use public randomness to choose a random permutation $\sigma$ over the elements of $[n^2]$.

2. Alice and Bob run the protocol $S$ on the input $(\sigma(x'_0), \sigma(y'_0))$. If $S$ does not outputs $t$ elements then $S'$ refuses to answer. Otherwise if the $t$ produced elements belong to $\sigma(\{j\}_{m < j \leq n^2})$ then $S'$ outputs 0, else $S'$ refuses to answer.

Let us assume that we know that either $(x', y') \in X_0$ or $(x', y') \in X_1$ and our goal is to distinguish the two cases. In the first case the pair $(\sigma(x'_0), \sigma(y'_0))$ is distributed according to $U_{1 \times 1}^{(n:i_0)}$ and therefore $S'$ outputs 0 with probability $p_{i_0}^{(t)}$. In the second case the pair $(\sigma(x'_0), \sigma(y'_0))$ is distributed according to $U_{1 \times 1}^{(n:i_0+1)}$ and therefore $S'$ outputs 0 with probability

---

4We believe that this theorem might be of independent interest.
\[ p_{i0+1} \cdot \left( \frac{t}{i0} \right) \left( \frac{t}{i0+1} \right) = \left( 1 - \frac{t}{i0+1} \right) p_{i0+1}. \] According to our assumption the former is higher than the latter, therefore if \( S' \) outputs 0, that can be viewed as a probabilistic evidence for \( (x', y') \in X_0. \)

Define \( D \) to be the uniform distribution over all pairs \( (x', y') \) satisfying \( x' \subset [m], |x'| = n/2 - i_0, y' \subset [m], |y'| = n - i_0. \) Then \( D(X_0) \geq \frac{1}{3}, \) as can be seen by analogy to the proof of Claim 5.1. Note that \( D \) satisfies the requirements of Lemma 5.2 if we chose \( \alpha_1 = \frac{4}{3} \) and \( \alpha_2 = 1, \) the lemma implies that for \( \delta = \frac{1}{2000}, \) some absolute constant \( c_0 \) and any rectangle \( A \) it holds that

\[ D(A \cap X_1) \geq \delta \cdot D(A \cap X_0) - 2^{-c_0 \cdot n}. \] (12)

Let \( l \in \mathbb{N} \) and \( S'_l \) be a protocol that runs \( S' \) as a subroutine \( l \) times and outputs 0 if all the instantiations of \( S' \) return 0 (otherwise \( S'_l \) refuses to answer). Denote by \( E_0 \) the event that \( S'_l \) outputs 0. If \( (x', y') \in X_0 \) then \( E_0 \) occurs with probability \( \left( p_{i0}^{(l)} \right)^l, \) if \( (x', y') \in X_1 \) then \( E_0 \) occurs with probability \( \left( 1 - \frac{t}{i0+1} \right) \cdot p_{i0+1}^{(l)} \). Therefore,

\[ \Pr[\mathcal{X}_0 \text{ and } E_0] \geq \frac{1}{3} \left( p_{i0}^{(l)} \right)^l \]

and

\[ \Pr[\mathcal{X}_1 \text{ and } E_0] \leq \left( 1 - \frac{t}{i0+1} \right) \cdot \left( \frac{t}{i0+1} \right)^l. \]

Suppose that \( S'_l \) uses \( s \) random bits. For any \( r \in \{0, 1\}^s, \) let \( S'_l(r) \) be the deterministic protocol obtained from \( S'_l \) by using the bits of \( r \) instead of the random bits. Note that \( S'_l(r) \) is a protocol of communication cost \( kl, \) therefore it partitions the domain into \( 2^{kl} \) rectangles, denote them by \( A_i^{(r)} \) for \( i \in [2^{kl}] \).

Let

\[ \beta(l) \overset{\text{def}}{=} \frac{1}{3} \cdot \left( \frac{p_{i0}^{(l)}}{1 - \frac{t}{i0+1}} \right)^l, \]

then

\[ \Pr[\mathcal{X}_0 \text{ and } E_0] \geq \beta(l) \cdot \Pr[\mathcal{X}_1 \text{ and } E_0] \]

and

\[ \sum_{\substack{r \in \{0, 1\}^s \\ i \in [2^{kl}]} \left( D \left( A_i^{(r)} \cap X_0 \right) - \beta(l) \cdot D \left( A_i^{(r)} \cap X_1 \right) \right) \geq 0. \]

Denote \( B \overset{\text{def}}{=} \left\{ A_i^{(r)} \mid r \in \{0, 1\}^s, i \in [2^{kl}] \right\}. \) Let \( \mu \overset{\text{def}}{=} \mathbb{E}_{A \in B} \left[ D(A \cap X_0) \right], \) denote \( B' \overset{\text{def}}{=} \left\{ A \in B \mid D(A \cap X_0) \geq \frac{\mu}{2} \right\}. \) Then

\[ \sum_{A \in B'} D(A \cap X_0) \geq \frac{1}{2} \sum_{A \in B} D(A \cap X_0) \geq \frac{\beta(l)}{2} \sum_{A \in B} D(A \cap X_1) \geq \frac{\beta(l)}{2} \sum_{A \in B'} D(A \cap X_1), \]

and therefore there exists \( A_0 \in B', \) such that \( D(A_0 \cap X_0) \geq \frac{\beta(l)}{2} D(A_0 \cap X_1). \)
It holds that
\[
\mu \geq \frac{1}{2kl} \cdot \Pr_D [X_0 \text{ and } E_0] \geq \left( \frac{P_{l0}}{P_{t0}} \right)^{t} \geq \frac{k^{2l}}{3 \cdot 2^{2l} \cdot n^l} > 2^{-kl-tl \log n - 2}.
\]

Therefore, \( D(A_0 \cap X_0) \geq \frac{\delta}{\beta} > 2^{-kl-tl \log n - 3} \). Now we use (12), which gives us
\[
\frac{2}{\beta(l)} \cdot D(A_0 \cap X_0) \geq D(A_0 \cap X_1) > \delta \cdot D(A_0 \cap X_0) = 2^{-c_0 \cdot n},
\]
\[
2^{-c_0 \cdot n} \geq \left( \frac{\delta - \frac{2}{\beta(l)}}{D(A_0 \cap X_0)} \right) \cdot D(A_0 \cap X_0) \geq \left( \delta - \frac{2}{\beta(l)} \right) \cdot 2^{-kl-tl \log n - 3},
\]
\[
\delta - \frac{2}{\beta(l)} \leq 2^{k(t+\log n) + 3 - c_0 \cdot n}.
\]

Recall that \( k \in \Omega(t \log n) \), so there exists an absolute constant \( c_1 \), such that \( 2^{k(t+\log n) + 3 - c_0 \cdot n} < \frac{\delta}{2} \) as long as \( l \leq \frac{c_1 n}{k} \). Consequently,
\[
\frac{4}{\delta} \geq \beta \left( \frac{c_1 n}{k} \right) = \frac{1}{3} \cdot \left( \frac{P_{l0}}{P_{t0}} \right)^{t} \cdot \left( \frac{1 - \frac{t}{t_0 + 1}}{P_{t0}^{(t)}} \cdot \frac{P_{t0}^{(t)}}{P_{t0 + 1}^{(t)}} \right)^{\frac{c_1 n}{k}},
\]
which implies that for some absolute constant \( c \),
\[
\left( \frac{P_{l0}}{P_{t0}} \right)^{t} \cdot \left( \frac{1 - \frac{t}{t_0 + 1}}{P_{t0}^{(t)}} \cdot \frac{P_{t0}^{(t)}}{P_{t0 + 1}^{(t)}} \right)^{\frac{c_1 n}{k}} \leq c \Rightarrow \left( \frac{P_{l0}}{P_{t0}} \right)^{t} \cdot \left( \frac{1 - \frac{t}{t_0 + 1}}{P_{t0}^{(t)}} \cdot \frac{P_{t0}^{(t)}}{P_{t0 + 1}^{(t)}} \right) \leq 1 + \frac{ck}{n},
\]
as required. \(\square\) 

**Theorem 5.8**

**5.4 Lower bound on the classical 2-way communication complexity of \( P(n) \)**

**Claim 5.9.** Solving \( P(n) \) in the classical 2-way setting with bounded error requires a protocol of cost \( \Omega \left( \frac{n^{1/4}}{\sqrt[3]{\log n}} \right) \).

**Proof of Claim 5.9.** Assume that a protocol \( S \) of communication cost \( k \in o(n) \) solves \( P(n) \) with error bounded by \( \frac{1}{3 \cdot 10^{10}} \).

Then Lemma 5.3 implies that there exists a protocol \( S' \) of communication cost \( O(k) \) that solves \( P'_{I_{1 \times 1}} \) w.r.t. \( U_{1 \times 1}^{(n/2)} \) with probability \( \frac{1}{2n} \) with error bounded by \( \frac{1}{10^{10}} \).

By Theorem 5.4, there exists a protocol \( S'' \) of communication cost \( O(k + \log^2(n)) \) solving \( P''_{I_{1 \times 1}} \) in 0-error setting w.r.t. \( U_{1 \times 1}^{(n/2)} \) with probability \( \Omega \left( \frac{1}{nk^2 \log^4(n)} \right) \).

Choose \( t = 2 \), Theorem 5.8 implies that \( S'' \) can succeed only with probability \( \Omega \left( \frac{k^2 + \log^4(n)}{n^2} \right) \), therefore \( k \in \Omega \left( \frac{n^{1/4}}{\sqrt[3]{\log n}} \right) \), as required. \(\square\) 

**Claim 5.9**
6 Conclusions and further work.

The protocol described in Section 4 together with Claim 5.9 imply Theorem 1.1.

It would be interesting to strengthen this result. Is it possible to find a functional problem that requires exponentially more expensive protocol in $R$ than in $Q^1$? How about simultaneous protocols?

In other words, give a separation that would logically imply as many results mentioned in the Introduction as possible.

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References

[B] H. Buhrman - Personal communication.


[C] R. Cleve - Personal communication.


