Instability of current sheets and formation of plasmoid chains

N. F. Loureiro,1,2 A. A. Schekochihin,3,4 and S. C. Cowley5,3

1Center for Multiscale Plasma Dynamics, University of Maryland, College Park, Maryland 20742-3511, USA
2Plasma Physics Laboratory, Princeton University, Princeton, New Jersey 08543, USA
3Blackett Laboratory, Imperial College, London SW7 2BW, United Kingdom
4King’s College, University of Cambridge, Cambridge CB2 1ST, United Kingdom
5Department of Physics and Astronomy, UCLA, Los Angeles, California 90095-1547, USA

(Dated: March 27, 2007)

Current sheets formed in magnetic reconnection events are found to be unstable to high-wavenumber perturbations. The instability is very fast: its maximum growth rate scales as $S^{1/4}/v_A$, where $L_{CS}$ is the length of the sheet, $v_A$ the Alfvén speed and $S$ the Lundquist number. As a result, a chain of plasmoids (secondary islands) is formed, whose number scales as $S^{3/8}$.

PACS numbers: 52.35.Vd, 52.35.Py, 94.30.cp, 96.60.Iv

Magnetic reconnection is a plasma phenomenon in which oppositely directed magnetic field lines are driven together, break and rejoin in a topologically different configuration. It is an essential element in our understanding of the solar flares and the geotail [1, 2, 3], where it is directly observed [4, 5], as well as of other astrophysical plasmas. On Earth, it plays a crucial role in the dynamics of magnetically confined plasmas in fusion devices [6].

There are two standard reconnection models: the Sweet-Parker (SP) model [7, 8], and the Petscheck model [9]. The latter is very appealing as it predicts fast reconnection rates similar to the observed ones. However, numerical simulations have consistently failed to reproduce this ideal scenario, which implies a resistively slow instability growth rate [23]. In this Letter, we show that, while the resistivity is essential for the instability, the growth rate is, in fact, much faster even than the ideal rate.

Theoretical estimates for the speed of magnetic reconnection in natural systems are usually based on the idea that a current sheet is formed (Fig. 1), whose length $L_{CS}$ is determined by the system’s global properties, while the width is $L_{CS}/\sqrt{S}$, where $S = v_A L_{CS}/\eta$ is the Lundquist number, $v_A$ is the upstream Alfvén speed, and $\eta$ is the magnetic diffusivity. Let us consider such a current sheet, and ask if it can be stable over any significant period of time. The SP reconnection time is $(L_{CS}/v_A)\sqrt{S}$. If we consider times much shorter than this time, we can determine the structure of the current sheet by seeking a stationary solution of the induction equation

$$\partial_t B + u \cdot \nabla B = B \cdot \nabla u - B \nabla \cdot u + \eta \nabla^2 B,$$

where $u$ is the velocity field and $B$ the magnetic field, which we will measure in velocity units. While the desired resistive equilibrium is stationary, it is not static. It is a consistent feature of current sheets, both measured [12] and simulated [24], that they support linear outflows along themselves. If incompressibility is assumed and the reconnection is considered in two dimensions ($x$, $y$) with a current sheet along the $y$ axis, then, inside the current

![FIG. 1: Schematic drawing of the current sheet with in- and outflows and a forming plasmoid chain.](https://example.com/f1.png)
sheet, \( u_x = -\Gamma_0 x \) and \( u_y = \Gamma_0 y \). Here \( \Gamma_0 = 2v_A/L_{CS} \), so the outflows are Alfvénic.

A simple equilibrium solution \( B_0 \) of Eq. \((1)\) exists, which accommodates this flow pattern: \( B_{0x} = 0 \), and \( B_{0y} = B_{0y}(x) \) satisfies
\[
\delta_{CS}^2 \partial_x^2 B_{0y} + \partial_x (x B_{0y}) = 0, \tag{2}
\]
where \( \delta_{CS} = (\eta/\Gamma_0)^{1/2} \) is the characteristic width of the current sheet. The solution of this equation that vanishes and changes sign at \( x = 0 \) (the center of the sheet) is \( B_{0y} = v_A f(\xi) \), where \( \xi = x/\delta_{CS} \) and
\[
f(\xi) = \alpha e^{-\xi^2/2} \int_0^\xi dz \ e^{z^2/2}. \tag{3}
\]
The integration constant \( \alpha \) is chosen by matching this solution with the magnetic field outside the sheet: \( B_{0y} = \pm v_A \), which is a solution of Eq. \((1)\) with constant inflows \( u_x = \mp u_0, \ u_y = 0 \). In order to complete our simple model of the current sheet, we must choose a suitable point \( x = \pm x_0 \) at which the inside and outside solutions could be matched. The flow is discontinuous at this point: \( \mathbf{u} = (u_0,0) \) for \( x < -x_0 \), \( \mathbf{u} = (-\Gamma_0 x,\Gamma_0 y) \) for \( -x_0 < x < x_0 \) and \( \mathbf{u} = (-u_0,0) \) for \( x > x_0 \), where \( u_0 = \Gamma_0 x_0 \). We consider this to be an acceptable simplification of the real, more complicated, and continuous, flow profile. The natural matching point is the point where \( B_{0y}(x) \) has its maximum (minimum), \( B_{0y}(\pm x_0) = 0 \), and where, therefore, the current vanishes. This gives \( x_0 = \xi_0 \delta_{CS} \), where \( \xi_0 \approx 1.31 \). We then require \( f(\pm \xi_0) = \pm 1 \), so \( \alpha = \xi_0 \). The equilibrium magnetic field and current are plotted in Fig. 2.

We shall now show that this equilibrium is susceptible to a very fast linear instability. We consider the two-dimensional case and solve the Reduced MHD equations
\[
\begin{align*}
\partial_t \nabla_\perp^2 \phi + \{ \phi, \nabla_\perp^2 \phi \} &= \{ \psi, \nabla_\perp^2 \psi \}, \tag{4} \\
\partial_t \psi + \{ \phi, \psi \} &= \eta \nabla_\perp^2 \psi + E_0. \tag{5}
\end{align*}
\]
Here \( \phi \) and \( \psi \) are the stream and flux functions of the in-plane velocity and magnetic field, so \( \mathbf{u} = (-\partial_t \phi, \partial_x \phi) \), \( \mathbf{B} = (-\partial_t \psi, \partial_x \psi) \). Eq. \((4)\) is the curl of the (inviscid) equation for an incompressible conducting fluid, Eq. \((5)\) is the induction equation \((1)\) uncurled. \( E_0 \) is the equilibrium electric field, which must satisfy \( \partial_x E_0 = 0 \). The model of equilibrium flows described above corresponds to \( \phi_0 = \Gamma_0 x y \) for \( |x| < x_0 \) and \( \phi_0 = \pm \Gamma_0 x_0 y \) for \( |x| > x_0 \), which satisfies Eq. \((4)\). The magnetic-field profile \((3)\) satisfies Eq. \((5)\), provided we choose \( E_0 = -v_A \Gamma_0 \delta_{CS} \alpha \) for \( |x| < x_0 \) and \( E_0 = -v_A \Gamma_0 x_0 \) for \( |x| > x_0 \).

Let us consider small perturbations to this equilibrium, so \( \psi = \psi_0 + \delta \psi, \ \phi = \phi_0 + \delta \phi \), and linearize Eqs. \((1)\)-(\(2)\). If we seek solutions in the form \( \delta \phi(x, y, t) = \phi_1(x,t) \exp[ik(t)y] \) and \( \delta \psi(x, y, t) = \psi_1(x,t) \exp[ik(t)y] \), where \( k(t) = k_0 \exp(-\Gamma_0 t) \), then \( \phi_1 \) and \( \psi_1 \) satisfy
\[
(\partial_x^2 - k^2) \partial_t \phi_1 = -\Gamma_0 x \partial_x (\partial_x^2 - k^2) \phi_1 + 2\Gamma_0 k^2 \phi_1 - [B_{0y}(x)(\partial_x^2 - k^2) - B_{0y}^0(x)] ik \psi_1, \tag{6}
\]
\[
\partial_t \psi_1 = -\Gamma_0 x \partial_x \psi_1 - B_{0y}(x) ik \phi_1 = \eta(\partial_x^2 - k^2) \psi_1. \tag{7}
\]
We now seek exponentially growing solutions, \( \phi_1(x,t) = -i \Phi(x) \exp(\gamma t) \) and \( \psi_1(x,t) = \Psi(x) \exp(\gamma t) \). A solvable eigenvalue problem for \( \gamma \) can be obtained if we assume \( \gamma \gg \Gamma_0 \), so the terms proportional to \( \Gamma_0 \) can be neglected and \( k(t) = k_0 \). Note that the presence of the linear in-and outflows in this approximation is only felt via the equilibrium profile \( B_{0y}(x) \). Rewriting Eqs. \((6)\)-(\(7)\) in terms of the dimensionless variable \( \xi = x/\delta_{CS} \) and denoting \( \kappa = k_0 v_A/\Gamma_0 = k_0 L_{CS}/2, \ \epsilon = (\eta \Gamma_0)^{1/2}/v_A = 2\delta_{CS}/L_{CS}, \ \lambda = \gamma/\Gamma_0 \kappa \), we get
\[
\lambda(\Phi'' - \kappa^2 c^2 \Phi) = -f(\xi)(\Phi'' - \kappa^2 c^2 \Psi) + f''(\xi) \Psi, \tag{8}
\]
\[
\lambda \Psi - f(\xi) \Phi = \frac{1}{\kappa} (\Phi'' - \kappa^2 c^2 \Psi), \tag{9}
\]
where the derivatives are with respect to \( \xi \).

The eigenvalue problem given by Eqs. \((8)\)-(\(9)\) is mathematically similar to the standard tearing mode problem \((\text{22})\), except the role of resistivity is played by \( 1/\kappa \). We shall assume that this parameter is small, i.e., we shall look for high-wavenumber perturbations of the current sheet. Another small parameter is \( \epsilon = (2/S)^{1/2}, \) which is the inverse aspect ratio of the sheet. We shall assume that \( \kappa \ll 1 \). All these assumptions will prove to be correct for the fastest growing modes.

We now proceed as in the standard tearing mode calculation, considering first the outer region, \( \xi \sim 1 \), and then the inner region, \( \xi \ll 1 \).

**Outer region.** The behavior here is “ideal.” Assuming that \( (1/\kappa) \ll \lambda \ll 1 \), we get from Eq. \((8)\) \( \Phi = \lambda \Psi/f(\xi) \) and then from Eq. \((8)\)
\[
\Psi'' = \left[ \frac{f''(\xi)}{f(\xi)} + \kappa^2 c^2 \right] \Psi. \tag{10}
\]
Given the functional form of the equilibrium \( f(\xi) \), solving this equation exactly is difficult. Instead, we shall solve
it perturbatively, using $\kappa^2 \epsilon^2 \ll 1$. Neglecting $\kappa^2 \epsilon^2$ to lowest order, we have an equation whose one solution is $f(0)$. It is then easy to find the second solution, so the general solution can be written as follows

$$
\Psi^\pm(\xi) = C^\pm_1 f(0) + C^\pm_2 f(0) \int_{\pm \xi_0}^\xi \frac{dz}{f^2(z)},$$

(11)

where $C^\pm_1$ and $C^\pm_2$ are constants of integration and $\pm$ refers to the solution at positive and negative values of $\xi$. We ask for the solution (but not its derivative) to be continuous at $\xi = 0$. At small $\xi$, we have $f(\xi) \approx \alpha \epsilon$, so the integral in Eq. (11) is dominated by its upper limit and we find that $C^+_1 = C^-_1 = -\alpha \Psi(0)$.

The constants $C^\pm_1$ are found by matching the solution (11) to the outer solution at $|\xi| > \xi_0$ (outside the current sheet). There we have, instead of Eq. (11),

$$
\Psi'' = \kappa^2 \epsilon^2 \Psi,
$$

(12)

so $\kappa^2 \epsilon^2$ can no longer be neglected. The solution that decays at $\xi \to \pm \infty$ is $\Psi^\pm = C^\pm_3 \exp(\mp \kappa \epsilon \xi)$. Matching this solution and its derivative to the solution (11) and using $f(\pm \xi_0) = \pm 1$ and $f'(\pm \xi_0) = 0$, we get $C^\pm_3 = \pm \alpha \Psi(0)/\kappa \epsilon$ and $C^\pm_3 = \alpha \Psi(0) \exp(\pm \kappa \epsilon \xi)/\kappa \epsilon$. Thus, for $|\xi| < \xi_0$,

$$
\Psi^\pm(\xi) = \pm \frac{\alpha \Psi(0)}{\kappa \epsilon} f(\xi) \int_{\pm \xi_0}^\xi \frac{dz}{f^2(z)},
$$

(13)

and for $|\xi| > \xi_0$,

$$
\Psi^\pm(\xi) = \frac{\alpha \Psi(0)}{\kappa \epsilon} \exp[\kappa \epsilon (\xi_0 \mp \xi)].
$$

(14)

This outer solution is plotted in Fig. 3. It has a discontinuous derivative at $\xi = 0$ and is a classic unstable eigenfunction known from the tearing-mode problem. The instability parameter $\Delta' = [\Psi'(0) - \Psi'(-0)]/\Psi(0)$ is

$$
\Delta' = \frac{2 \alpha^2}{\kappa \epsilon} + \alpha^2 \left[ \int_{-\xi_0}^{\xi_0} \frac{dz}{f^2(z)} \right],
$$

(15)

where $[\ldots]$ denotes the nonsingular part of the integral. The second term in Eq. (15), which is equal to 0.57, can be dropped compared to the first, so to lowest order we have simply $\Delta' = 2 \alpha^2/\kappa \epsilon$.

Inner region. Here $\xi \ll 1$ and we can assume $f(\xi) \approx \alpha \epsilon$. Eqs. (8–9), assuming $\partial \xi \gg 1$, become

$$
\lambda \Psi'' = -\alpha \epsilon \Psi'',
$$

(16)

$$
\lambda \Psi - \alpha \epsilon \Psi = \frac{1}{\kappa} \Psi''.
$$

(17)

Since $\Delta' = 2 \alpha^2/\kappa \epsilon \gg 1$, this eigenvalue problem is mathematically similar to the one solved by Coppi et al. [20] for large-$\Delta'$ tearing modes. It reduces to solving the following transcendental equation for the growth rate

$$
-\frac{\pi}{8} (\kappa \epsilon)^{1/3} \Lambda^{5/4} \Gamma ((\Lambda^{3/2} - 1)/4) = \Delta' - \frac{2 \alpha^2}{\kappa \epsilon},
$$

(18)

where $\Gamma$ is the gamma function and $\Lambda = \lambda \alpha^{-2/3} \kappa^{1/3}$. The width of the inner region is given by $\delta = (\Lambda/\kappa)^{1/4} \alpha^{-1/2}$. Recall that $\lambda = \gamma/\Gamma_0$ and $\kappa = \kappa_0 v_A/\Gamma_0$.

Eq. (18) has two interesting limits: assuming $\Lambda \ll 1$,

$$
\frac{\gamma}{\Gamma_0} \approx 1.63 \kappa^{-2/5} \epsilon^{-4/5},
$$

(19)

assuming $\Lambda \rightarrow 1$ (from below),

$$
\frac{\gamma}{\Gamma_0} \approx (\alpha \kappa)^{2/3} - \sqrt{\pi/3} \kappa^2 \epsilon.
$$

(20)

The maximum growth rate $\gamma_{\text{max}}$ lies between these two asymptotics. Comparing the two terms in Eq. (20) shows that it is attained for $\kappa_{\text{max}} \sim \epsilon^{-3/4} \gg 1$ and, therefore, $\gamma_{\text{max}}/\Gamma_0 \sim \epsilon^{-1/2} \gg 1$. We note also that $\kappa_{\text{max}} \epsilon \sim \epsilon^{1/4}$, $\lambda \sim \epsilon^{1/4}$ and the inner layer width $\delta \sim \epsilon^{1/4}$ for the fastest growing mode. This confirms all of the ordering assumptions we made in our calculation.

Fig. 4 shows the dependence $\gamma(\kappa)$ resulting from the numerical solution of Eq. (18) for two different values of the current sheet aspect ratio $\epsilon^{-1}$. Both scalings [19] and [20] are manifest. The vertical lines identify the value of $\kappa$ above which the assumption $\kappa^2 \epsilon^2 \ll 1$ breaks down and, strictly speaking, our calculation is no longer valid. This is, however, of secondary importance, as the maximum of the growth rate lies well within the region of validity of our asymptotic ordering for $\epsilon^{-1} \gtrsim 10^3$.

We have shown analytically that SP current sheets with large aspect ratios are intrinsically unstable to high-wave-number perturbations. Since $\epsilon^{-1} = (S/2)^{1/2}$, the maximum growth rate scales with the Lundquist number as $\gamma_{\text{max}} \sim S^{1/4} \Gamma_0$. Typical Lundquist numbers in plasmas of interest are extremely large (e.g., $S \approx 10^{12}$ in the solar corona), so the instability of the current sheet is extremely fast compared to the Alfvén time $\sim \Gamma_0^{-1}$. One immediate consequence of this is that stable current sheets with aspect ratios above some critical value cannot exist. Numerical simulations suggest that this value...
is \( \sim 10^2 \) \cite{12, 13}, corresponding to \( S_c \sim 10^4 \). Above this value, the sheet breaks up and a chain of plasmoids is formed. Their number scales as 

\[
S_{\gamma} \sim \kappa_{\text{max}} \sim S^{3/8}
\]

and their length is \( \sim S^{1/8} \) larger than the current-sheet width \( \delta_{\text{CS}} \).

Since the growth rate we have found is much larger than the inverse Alfvén time, the plasmoid width should become comparable to the inner-layer width \( \delta \sim S^{-1/8}\delta_{\text{CS}} \) before the plasmoids can be expelled from the current sheet by the Alfvénic outflows. The plasmoid evolution then becomes nonlinear. In order to have a quantitative theory of how the plasmoids affect the reconnection, one needs to understand what happens in the nonlinear regime. The dynamics in this regime will be dictated by a competition between three processes: the nonlinear growth and saturation of the plasmoids due to reconnection, the plasmoid coalescence \cite{27, 28}, and the expulsion of the plasmoids along the current sheet by the Alfvénic outflows. Numerical results on the stalling of the coalescence instability at large \( S \) \cite{29} suggest that multiple plasmoids can survive in the nonlinear regime rather than coalescing into a single plasmoid. The formation of multiple large plasmoids is also reported in \cite{18, 20}. If the width of the plasmoids can indeed become bigger than the width of the current sheet before they coalesce or are expelled, estimates of the SP reconnection time must no longer be based on the parameters of the original current sheet but rather on some effective reconnection region whose width is determined by the saturated plasmoid chain. A nonlinear study of the current sheet instability and plasmoid dynamics is currently underway.

Discussions with D. A. Uzdensky are gratefully acknowledged. N.F.L. was supported by the Center for Multiscale Plasma Dynamics, DOE Fusion Science Center Cooperative Agreement ER54785. A.A.S. was supported by a PPARC Advanced Fellowship.

[31] Going to the next order in our outer solution results in an additional subdominant contribution to \( \Delta' \) comparable to the second term in Eq. (13). Note that while the contribution of the second term in Eq. (13) to \( \Delta' \) is negligible, the term itself cannot be dropped because it ensures that the solution has a finite value at \( \xi = 0 \). Note also that there exists an unstable solution that is entirely confined inside the current sheet, so that \( \Psi(\pm \xi_0) = 0 \). For this solution, \( C^+_{\Delta'} = 0 \) in Eq. (11) and to the lowest order in \( \kappa^2 \gamma^2 \), \( \Delta' \) is given just by the second term in Eq. (15). Solving Eq. (10) to the second order in \( \kappa^2 \gamma^2 \), we find \( \Delta'(\kappa) \approx 0.57 - 1.47 \kappa^2 \gamma^2 \). The growth rate then is obtained from Eq. (15). The result is \( \gamma/\Gamma_0 \approx 0.61 \kappa^{2/5} \Delta'(\kappa)^{4/5} \), whence \( \kappa_{\text{max}} \approx 0.28 \kappa^{-1} \) and \( \gamma_{\text{max}} \sim \epsilon^{-2/5} \). We will not consider this solution because it grows slower than the unconfined mode and its validity hinges on the numerical smallness of \( \kappa_{\text{max}} \gamma^2 \approx 0.08 \).