Why does gravitational radiation produce vorticity?

L Herrera\(^1\), W Barreto\(^2\), J Carot\(^3\), A Di Prisco\(^1\)

\(^1\) Escuela de Física, Facultad de Ciencias, Universidad Central de Venezuela, Caracas, Venezuela.

\(^2\) Centro de Física Fundamental, Facultad de Ciencias, Universidad de los Andes, Mérida, Venezuela.

\(^3\) Departament de Física, Universitat Illes Balears, E-07122 Palma de Mallorca, Spain.

March 27, 2007

Abstract

We calculate the vorticity of world–lines of observers at rest in a Bondi–Sachs frame, produced by gravitational radiation, in a general Sachs metric. We claim that such an effect is related to the super–Poynting vector, in a similar way as the existence of the electromagnetic Poynting vector is related to the vorticity in stationary electrovacuum spacetimes.

1 Introduction

The theoretical description and the experimental observation of gravitational radiation are among the most relevant challenges confronting general relativity.

A great deal of work has been done so far in order to provide a consistent framework for the study of such phenomenon. Also, important collaboration efforts have been carried on, and are now under consideration, to put in evidence gravitational waves.

Therefore it is clear that any specific phenomenon caused by gravitational radiation and which might be observed, could in principle lead to the detection of such elusive waves and are of utmost relevance.

In this respect, particular attention deserves the link between gravitational radiation and vorticity of world–lines of observers at rest in a Bondi space–time \(^1\)\(^2\)\(^3\). Specifically, it has been shown that the leading term in the vorticity (in an expansion of powers of \(1/r\)) is expressed through the news function in such a way that it will vanish if and only if there is no news (no radiation). This suggests the possibility of detecting gravitational waves by means of gyroscopes \(^1\)\(^4\).
The issue we want to address here is: What is the mechanism by means of which gravitational radiation produces vorticity?

To answer to such a question it is worth recalling a result obtained by Bonnor [5] concerning the dragging of inertial frames by a charged magnetic dipole. To explain the appearance of vorticity in such space–times, Bonnor notices that the corresponding electromagnetic Poynting vector has a non–vanishing component, describing a flow of electromagnetic energy round in circles where frame–dragging occurs [6]. He then suggests that such a flow of energy affects inertial frames by producing vorticity of congruences of particles, relative to the compass of inertia. This conjecture has been recently confirmed to be valid for general axially symmetric stationary electrovacuum metrics [7].

One is then tempted to speculate that a similar mechanism is working in the case of gravitational radiation, i.e. a flow of gravitational “energy” would produce vorticity of congruence of observers. For testing such a conjecture we have available a tensor quantity which allows to define a covariant (super)–energy density and a (super)–Poynting vector, namely the Bel–Robinson tensor [8].

When the super–Poynting vector \((P_\mu)\), based on the Bel–Robinson tensor, as defined in [9], is calculated for the Bondi metric [10], we obtain that the contravariant azimuthal component \((P_\phi)\) of such vector vanishes [3], as expected from the the reflection symmetry of the Bondi metric, which, intuitively, seems to be incompatible with the presence of a circular flow of energy in the \(\phi\) direction. However, the vorticity vector, which is orthogonal to the plane of rotation, has in the Bondi spacetime only one non–vanishing contravariant component (\(\phi\)). Implying thereby that the plane of the associated rotation is orthogonal to the \(\phi\) direction. Therefore, it is not the \(\phi\) component of \(P_\mu\) (as mistakenly stated in [3]) the possible source of vorticity, but the \(\theta\) component, the cause for such an effect, in the Bondi case.

In order to strenght further the case for the super-Poynting vector as the cause of the mentioned vorticity, we shall consider here the general radiative metric without axial and reflection symmetry [11, 12]. In this latter case we shall obtain a non–vanishing \(P_\phi\), which we will identify as the cause of the \(\theta\) component of the vorticity vector. The discussion on these results is presented in the last section. However, before considering the general (Sachs) case it is instructive to review very briefly the basic ideas of the Bondi formalism in the reflection symmetric case.

2 The Bondi’s formalism

The general form of an axially and reflection symmetric asymptotically flat metric given by Bondi is [10]

\[
\begin{align*}
ds^2 &= \left(\frac{V}{r}e^{2\beta} - U^2 r^2 e^{2\gamma}\right) du^2 + 2e^{2\beta} du dr \\
&
+ 2Ur^2 e^{\gamma} du d\theta - r^2 \left(e^{2\gamma} d\theta^2 + e^{-2\gamma} \sin^2 \theta d\phi^2\right)
\end{align*}
\] (1)
where $V, \beta, U$ and $\gamma$ are functions of $u, r$ and $\theta$.

We number the coordinates $x^{0,1,2,3} = u, r, \theta, \phi$ respectively. $u$ is a timelike coordinate such that $u = \text{constant}$ defines a null surface. In flat spacetime this surface coincides with the null light cone open to the future. $r$ is a null coordinate ($g_{rr} = 0$) and $\theta$ and $\phi$ are two angle coordinates (see [10] for details).

The four metric functions are assumed to be expanded in series of $1/r$, then using the field equations Bondi gets

$$\gamma = cr^{-1} + \left( C - \frac{1}{6} c^3 \right) r^{-3} + ... \tag{2}$$

$$U = - \left( c_\theta + 2c \cot \theta \right) r^{-2} + \left[ 2N + 3cc_\theta + 4c^2 \cot \theta \right] r^{-3} ... \tag{3}$$

$$V = r - 2M$$

$$- \left( N_\theta + N \cot \theta - c_\theta^2 - 4cc_\theta \cot \theta - \frac{1}{2} c^2 (1 + 8 \cot^2 \theta) \right) r^{-1} + ... \tag{4}$$

$$\beta = -\frac{1}{4} c^2 r^{-2} + ... \tag{5}$$

where $c, C, N$ and $M$ are functions of $u$ and $\theta$, letters as subscripts denote derivatives.

Bondi shows that field equations allow to determine the $u$-derivatives of all functions in $\beta$ and $\gamma$ with the exceptions of news functions $c_u$.

Thus the main conclusions emerging from Bondi’s approach may be summarized as follows.

If $\gamma, M$ and $N$ are known for some $u = a$(constant) and $c_u$ (the news function) is known for all $u$ in the interval $a \leq u \leq b$, then the system is fully determined in that interval. In other words, whatever happens at the source, leading to changes in the field, it can only do so by affecting $c_u$ and viceversa. In the light of this comment the relationship between news function and the occurrence of radiation becomes clear.

Now, for an observer at rest in the frame of (1), the four-velocity vector has components $u^\alpha = A^{-1} \delta_\alpha^0$, with

$$A \equiv \left( \frac{V}{r} e^{2\beta} - U^2 r^2 e^{2\gamma} \right)^{1/2}. \tag{6}$$

Then, it can be shown that for such an observer the vorticity vector

$$\omega^\alpha = \frac{1}{2\sqrt{-g}} \eta^{\alpha\mu\lambda} u_\mu u_{\nu,\lambda}, \quad \eta_{\alpha\beta\gamma\delta} \equiv \sqrt{-g} \epsilon_{\alpha\beta\gamma\delta} \tag{7}$$

where $\eta_{\alpha\beta\gamma\delta} = +1$ for $\alpha, \beta, \gamma, \delta$ in even order, $-1$ for $\alpha, \beta, \gamma, \delta$ in odd order and $0$ otherwise, may be written as (see [11] for details)

$$\omega^\alpha = (0, 0, 0, \omega^\phi) \tag{8}$$

giving for the absolute value of $\omega^\alpha$ (keeping only the leading term)
\[ \Omega = -\frac{1}{2r}(c_{u\theta} + 2c_{u} \cot \theta). \] (9)

Therefore, up to order \(1/r\), a gyroscope at rest in \(\Omega\) will precess as long as the system radiates \((c_u \neq 0)\).

Next, the super–Poynting vector based on the Bel–Robinson tensor, as defined in Maartens and Basset [9], is

\[ P_\alpha = \eta_{\alpha\beta\gamma\delta} E_\beta^{\rho} H^{\gamma\rho\delta}, \] (10)

where \(E_{\mu\nu}\) and \(H_{\mu\nu}\), are the electric and magnetic parts of Weyl tensor, respectively, formed from Weyl tensor \(C_{\alpha\beta\gamma\delta}\) and its dual \(\tilde{C}_{\alpha\beta\gamma\delta}\) by contraction with the four velocity vector given by

\[ E_{\alpha\beta} = C_{\alpha\gamma\beta\delta} u^\gamma u^\delta \] (11)

\[ H_{\alpha\beta} = \tilde{C}_{\alpha\gamma\beta\delta} u^\gamma u^\delta = \frac{1}{2} \eta_{\alpha\gamma\epsilon\delta} C_{\beta\epsilon\delta} u^\gamma u^\rho, \] (12)

The electric and magnetic parts of Weyl tensor have been explicitly calculated for the Bondi metric in [3], using those expressions the reader can easily verify that in this case \(P^\phi = 0\). As mentioned before, this is to be expected due to the reflection symmetry of the metric.

Indeed, rotation singles out a specific direction of time, which is at variance with the equivalence of both directions of time, implicit in reflection symmetry. This is the reason why the Bondi metric in the time independent limit becomes the static Weyl metric and not the general stationary spacetime.

On the other hand, since the associated rotation in this case, takes place on the plane orthogonal to the \(\phi\) direction, it is \(P^\phi\) the quantity to be associated with vorticity. Indeed, the vorticity vector describes the rate of rotation, on its orthogonal plane, with respect to proper time of the set of neighboring particles, relative to the local compass of inertia [14]. As it can be seen from expressions (29–31) below, in the Bondi limit, such a component is not vanishing.

Let us now analyze the general case.

## 3 The general radiative metric

Shortly after the publication of the seminal paper by Bondi and coworkers, Sachs [11] presented a generalization of the Bondi formalism relaxing the conditions of axial and reflection symmetry. In this case the line element reads (we have found more convenient to follow the notation given in [12] [13] which is slightly different from the original Sachs paper)

\[
\begin{align*}
 ds^2 &= \left( V r^{-1} e^{2\beta} - r^2 e^{2\gamma} U^2 \cosh 2\delta - r^2 e^{-2\gamma} W^2 \times 
 \cosh 2\delta - 2r^2 U W \sinh 2\delta \right) du^2 + 2e^{2\beta} du dr + 2r^2 \times 
\end{align*}
\]
\[(e^{2\gamma}U \cosh 2\delta + W \sinh 2\delta) du d\theta + 2r^2(e^{-2\gamma}W \times \\
\cosh 2\delta + U \sinh 2\delta) \sin \theta du d\phi - r^2(e^{2\gamma} \cosh 2\delta \sinh 2\delta \\
+ e^{-2\gamma} \cosh 2\delta \sin^2 \theta \phi^2 + 2 \sinh 2\delta \sin \theta d\theta d\phi), \tag{13}\]

where \(\beta, \gamma, \delta, U, W, V\) are functions of \(x^0 = u, x^1 = r, x^2 = \theta, x^3 = \phi\). Observe that, unlike [13], we adopt the signature \(-2\) as in [12].

The general analysis of the field equations is similar to the one in [10], but of course expressions are far more complicated (see [11, 12, 13] for details). In particular, there are now two new functions.

The asymptotic expansion of metric functions read in this case (see [12] for details)

\[\gamma = c r^{-1} + (C - c^3/6 - 3cd^2/2) r^{-3} + \ldots \tag{14}\]

\[\delta = d r^{-1} + (H + c^2 d/2 - d^3/6) r^{-3} + \ldots \tag{15}\]

\[U = -(c_\phi + 2c \cot \theta + d_\phi \csc \theta) r^{-2} \]
\[+ [2N + 3(cco + ddu) + 4(c^2 + d^3) \cot \theta \\
- 2(c_\phi d - cd_\phi) \csc \theta] r^{-3} + \ldots \tag{16}\]

\[W = -(d_\theta + 2d \cot \theta - c_\phi \csc \theta) r^{-2} \]
\[+ [2Q + 2(c_\phi d - cd_\phi) + 3(cco + ddu) \csc \theta] r^{-3} + \ldots \tag{17}\]

\[V = r - 2M - (N_\theta + \cot \theta + Q_\phi \csc \theta - (c^2 + d^2)/2 \\
-(c_\theta^2 + d_\theta^2) - 4(cco + ddu) \cot \theta - 4(c^2 + d^2) \cot^2 \theta \\
-(c_\phi^2 + d_\phi^2) \csc^2 \theta + 4(c_\phi d - cd_\phi) \csc \theta \cot \theta \\
+ 2(c_\phi d_\theta - c_\phi d_\phi) \csc \theta] r^{-1} + \ldots \tag{18}\]

\[\beta = -(c^2 + d^2) r^{-2}/4 + \ldots \tag{19}\]

where \(c, C, d, H, N, Q\) and \(M\) are now functions of \(u, \theta\) and \(\phi\). It can be shown [11, 12, 13] that field equations allow to determine the \(u\)-derivatives of all functions in \(\beta\) and \(\gamma\) with the exceptions of news functions \(c_u\) and \(d_u\), which remain arbitrary and whose existence represents a clear-cut criterium for the presence of gravitational radiation.

Let us first calculate the vorticity for the congruence of observers at rest in [13], whose four-velocity vector is given by \(u^\alpha = A^{-1} \delta^\alpha_u\), where now \(A\) is given by

\[A = (V r^{-1} e^{2\beta} - r^2 e^{2\gamma} U^2 \cosh 2\delta \\
- r^2 e^{-2\gamma} W^2 \cosh 2\delta - 2r^2 UW \sinh 2\delta)^{1/2}. \tag{20}\]
Thus, (7) lead us to

$$\omega^\alpha = (\omega^u, \omega^r, \omega^\theta, \omega^\phi),$$

(21)

where

$$\omega^u = \frac{1}{2A^2 \sin \theta} \{2r^2 A^{-2} [(U^2 e^{2\gamma} + W^2 e^{-2\gamma}) \sinh 2\delta \cosh 2\delta
$$

$$+ W \cosh^2 2\delta) \gamma_u] + (W^2 e^{-2\gamma} - U^2 e^{2\gamma}) \delta_u + \frac{1}{2}(W U_r - U W_u)\}
$$

$$+ A^2 [A^{-2} (W e^{-2\gamma} \cosh 2\delta + U \sinh 2\delta)] \theta
$$

$$- A^2 [A^{-2} (W \sinh 2\delta + U e^{2\gamma} \cosh 2\delta)] \phi\}$$

(22)

$$\omega^r = \frac{1}{r \sin \theta} \{2r^2 A^{-2} [\{(U^2 e^{2\gamma} + W^2 e^{-2\gamma}) \sinh 2\delta \cosh 2\delta
$$

$$+ W \cosh^2 2\delta) \gamma_u] + (W^2 e^{-2\gamma} - U^2 e^{2\gamma}) \delta_u + \frac{1}{2}(W U_r - U W_u)\}
$$

$$+ A^2 [A^{-2} (W e^{-2\gamma} \cosh 2\delta + U \sinh 2\delta)] \theta
$$

$$- A^2 [A^{-2} (W \sinh 2\delta + U e^{2\gamma} \cosh 2\delta)] \phi\}$$

(23)

$$\omega^\theta = \frac{1}{2r^2 \sin \theta} \{A^2 e^{-2\gamma} [r^2 A^{-2} (W \sinh 2\delta + U e^{-2\gamma} \cosh 2\delta)]_r
$$

$$- e^{2\beta} A^{-2} [r^2 e^{-2\gamma} (W \sinh 2\delta + U e^{-2\gamma} \cosh 2\delta)]_u
$$

$$+ e^{2\beta} A^{-2} (e^{-2\beta} A^2)\phi\},$$

(24)

and

$$\omega^\phi = \frac{1}{2r^2 \sin \theta} \{A^2 e^{-2\gamma} [r^2 A^{-2} (W \sinh 2\delta + U e^{2\gamma} \cosh 2\delta)]_r
$$

$$- e^{2\beta} A^{-2} [r^2 e^{-2\gamma} (W \sinh 2\delta + U e^{2\gamma} \cosh 2\delta)]_u
$$

$$+ A^{-2} e^{2\beta} (A^2 e^{-2\beta})\phi\}.$$ 

(25)

Thus, for the leading term of the absolute value of $\omega^u$ we get

$$\Omega = \frac{1}{2r} [(c_{\theta u} + 2c_u \cot \theta + d_{\phi u} \csc \theta)^2
$$

$$+ (d_{\theta u} + 2d_u \cot \theta - c_{\phi u} \csc \theta)^2]^{1/2},$$

(26)

which of course reduces to (9) in the Bondi (axially and reflection symmetric) case ($d = c_\phi = 0$).

Next, calculation of the super–Poynting gives the following result

$$P_\mu = (0, P_r, P_\theta, P_\phi),$$

(27)

where the explicit terms are too long to be written at this point.
Although algebraic manipulation by hand is feasible for the Bondi metric, for the Bondi–Sachs one is quite cumbersome. Thus, we calculated the super–Poynting components using two different sets of Maple scripts (available upon request). One, uses the Maple intrinsic procedures to deal with tensors; the electric and magnetic parts of the Weyl tensor were manipulated not explicitly up to the leading term output for the super–Poynting vector components. The most important feature for this procedure, which leads to considerable simplification, is the use of the shift vector \( U^A = (U, W/\sin \theta) \) and the 2–surfaces metric of constant \( u^h_{AB} = \left( e^{2\gamma} \cosh 2\delta \sin 2\delta \sin \theta, e^{-2\gamma} \cosh 2\delta \sin^2 \theta \right) \), where \( A, B \) runs from 2 to 3. In our calculations we keep these auxiliary variables, \( U^A \) and \( h^h_{AB} \), as far as was possible. The other set of Maple scripts uses the GR T ensor II computer algebra package (running on Maple); in this case, all the relevant objects (electric and magnetic parts of the Weyl tensor, super–Poynting vector \( P^\alpha \)) were calculated exactly for the metric (13) performing next a series expansion (at \( r \to \infty \)) for the components of \( P^\alpha \), taking into account the expressions (14–19) for the metric functions and keeping just the leading term in the series. Both approaches lead us to the same results.

The leading terms for each super–Poynting component are

\[
\begin{align*}
P_r &= -2r^{-2}(d^2_{uu} + c^2_{uu}), \\
P_\theta &= -\frac{2}{r^2 \sin \theta} \{ [2(d^2_{uu} + c^2_{uu})c + c_{uu}c_u + d_{uu}d_u] \cos \theta \\
&\quad + [c_{uu}c_{\theta u} + d_{uu}d_{\theta u} + (c^2_{uu} + d^2_{uu})c_{\theta}] \sin \theta \\
&\quad + c_{uu}d_{\phi u} - d_{uu}c_{\phi u} + (d^2_{uu} + c^2_{uu})d_{\phi},
\end{align*}
\]

and

\[
\begin{align*}
P_\phi &= \frac{2}{r^2} \{ [2c_{uu}d_u - d_{uu}c_u - (d^2_{uu} + c^2_{uu})d] \cos \theta \\
&\quad + [c_{uu}d_{\theta u} - d_{uu}c_{\theta} - (c^2_{uu} + d^2_{uu})d_{\theta}] \sin \theta \\
&\quad + (d^2_{uu} + c^2_{uu})c_{\phi} - (c_{uu}c_{\phi u} + d_{uu}d_{\phi u})\}
\end{align*}
\]

from which it follows

\[
P^\phi = -\frac{2}{r^4 \sin^2 \theta} \{ \sin \theta [d_{uu}c_{uu} - d_{uu}c_{\theta}] + 2 \cos \theta \times \\
[c_{uu}d_u - d_{uu}c_u] - [c_{uu}c_{\phi u} + d_{uu}d_{\phi u}]\}
\]

which of course vanishes in the Bondi case.

4 Discussion

We have seen so far that gravitational radiation produces vorticity in the congruence of observers at rest in the frame of (13). We conjecture that such
vorticity is caused by the presence of a flow of super–energy, as described by the super–Poynting vector \[ \mathbf{P} \].

In the case of axial and reflection symmetry (Bondi), the plane of rotation is orthogonal to the unit vector \( \hat{e}_\phi \), in the \( \phi \) direction, and accordingly it is the \( P^\theta \) component the responsible for such vorticity, whereas the azimuthal component \( P^\phi \) vanishes, in agreement with the fact that the symmetry of the Bondi metric excludes rotation along \( \phi \) direction.

In the general case, we have shown that the vorticity vector has also components along \( \hat{e}_r \) and \( \hat{e}_\theta \), implying thereby rotations on the corresponding orthogonal planes. In particular, rotations in the \( \phi \) direction are now allowed \( (\omega^\theta \neq 0) \), and therefore we should expect a non–vanishing \( P^\phi \), which is in fact what happens.

Thus, we have shown that there is always a non–vanishing component of \( P^\mu \), on the plane orthogonal to a unit vector along which there is a non–vanishing component of vorticity. Inversely, \( P^\mu \) vanishes on a plane orthogonal to a unit vector along which the component of vorticity vector vanishes (Bondi). This supports further our conjecture about the link between the super–Poynting vector and vorticity.

**Acknowledgments**

JC gratefully acknowledges financial support from the Spanish Ministerio de Educación y Ciencia through the grant FPA2004-03666. This work was supported partially by FONACIT under grants S1–98003270 and F2002000426; by CDCHT–ULA under grant C–1267–04–05–A. LH wishes to thank Roy Maartens for pointing out the super–Poynting vector as defined in [9]. WB is grateful for the hospitality received at the Pittsburgh Supercomputing Center, where this work was finished.

**References**


