Hamilton-Jacobi formulation of systems within Caputo’s fractional derivative

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Abstract

In this paper we develop a fractional Hamilton-Jacobi formulation for discrete systems in terms of fractional Caputo derivatives. The fractional action function is obtained and the solutions of the equations of motion are recovered. An example is studied in details.

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1 Introduction

Fractional calculus generalized the classical calculus and it has many important applications in various fields of science and engineering [1-10]. These applications include classical and quantum mechanics, field theory, and optimal control [11-30] formulated mostly in terms of Riemann-Liouville (RL) and Caputo fractional derivatives. In contrast with RL derivative, Caputo derivative of a constant is zero, and for a fractional differential equation defined in terms of Caputo derivatives the standard boundary conditions are well defined. Therefore, this kind of fractional derivative gained importance among engineers and scientists.

Recently, the fractional Hamiltonian formulations were investigated for fractional discrete and continuous systems in terms of RL and Caputo derivatives [22-37]. Even more recently, a fractional Hamilton-Jacobi formalism within RL derivative was proposed in [38].

As it well known the classical Hamilton-Jacobi equation represents a reformulation of classical mechanics which equivalent to other formulations such as Newton’s laws of motion, Lagrangian mechanics and Hamiltonian mechanics. In addition, the Hamilton-Jacobi equation is useful in finding the conserved quantities for mechanical systems, which may be possible even when the mechanical problem itself cannot be solved completely. The Hamilton-Jacobi theory represents the only formulation of mechanics in which the motion of a particle can be represented as a wave.

On the other hand in the area of fractional mechanics the investigation of the fractional Hamilton-Jacobi equation is still at the beginning of its development.

Due to the above mentioned reasons the formulation of a fractional Hamilton-Jacobi within Caputo fractional derivatives is an interesting issue to be investigated.

The plan of this paper is as follows:

In section two the basic mathematical tools are briefly described. In section three the fractional Hamilton equations within Caputo’s derivative are mentioned. Section four contains the fractional Hamilton-Jacobi formulation by using Caputo’s derivative. An illustrative example is analyzed in section five. Finally, section six contains our conclusions.
2 Basic definitions

In the following we briefly present some fundamental definitions used in the previous section.

The left and the right Riemann-Liouville and Caputo fractional derivatives are defined as follows:

**The Left Riemann-Liouville Fractional Derivative**

\[
a D^\alpha_t f(t) = \frac{1}{\Gamma(n - \alpha)} \left( \frac{d}{dt} \right)^n \int_a^t (t - \tau)^{n-\alpha-1} f(\tau)d\tau,
\]

(1)

**The Right Riemann-Liouville Fractional Derivative**

\[
_t D^\alpha_b f(t) = \frac{1}{\Gamma(n - \alpha)} \left( -\frac{d}{dt} \right)^n \int_t^b (\tau - t)^{n-\alpha-1} f(\tau)d\tau,
\]

(2)

The corresponding Caputo’s fractional derivatives are defined as follows:

**The Left Caputo Fractional Derivative**

\[
C_a D^\alpha_t f(t) = \frac{1}{\Gamma(n - \alpha)} \int_a^t (t - \tau)^{n-\alpha-1} \left( \frac{d}{d\tau} \right)^n f(\tau)d\tau,
\]

(3)

and

**The Right Caputo Fractional Derivative**

\[
C_t D^\alpha_b f(t) = \frac{1}{\Gamma(n - \alpha)} \int_t^b (\tau - t)^{n-\alpha-1} \left( -\frac{d}{d\tau} \right)^n f(\tau)d\tau,
\]

(4)

where \( \alpha \) represents the order of the derivative such that \( n - 1 < \alpha < n \).

In [22] the fractional Euler-Lagrange equations are obtained. We present briefly the main result obtained in the following (for more details see [22]).

Let \( J[q] \) be a functional of the form

\[
J[q] = \int_a^b L(t, q, _a C^\alpha D^\alpha_t q, _t C^\beta D^\beta_b q)dt
\]

(5)

where \( 0 < \alpha, \beta < 1 \) and defined on the set of functions \( y(x) \) which have continuous LCFD of order \( \alpha \) and RCFD of order \( \beta \) in \([a, b]\). Then a necessary
condition for $J[q]$ to have an extremum for a given function $q(t)$ is that $q(t)$ satisfy the generalized Euler-Lagrange equation given by

$$\frac{\partial L}{\partial q} + t D_b^\alpha \frac{\partial L}{\partial a^C D_b^\alpha q} + a D_t^\beta \frac{\partial L}{\partial t^C D_b^\beta q} = 0, \quad t \in [a, b]$$

(6)

and the transversality conditions given by

$$\left[ t D_b^\alpha - 1 \left( \frac{\partial L}{\partial a^C D_t^\alpha q} \right) - a D_t^\beta - 1 \left( \frac{\partial L}{\partial t^C D_b^\beta q} \right) \right] \eta(t)|_a^b = 0.$$

(7)

### 3 Fractional Hamiltonian formulation

In this section we briefly present the Hamiltonian formulation within Caputo’s fractional derivatives.

Let us consider the fractional Lagrangian as given below

$$L(q, C_a^C D_t^\alpha q, C_t^C D_b^\beta q, t), \quad 0 < \alpha, \beta < 1.$$ 

(8)

By using (8) we define the canonical momenta $p_\alpha$ and $p_\beta$ as follows

$$p_\alpha = \frac{\partial L}{\partial C_a^C D_t^\alpha q}, \quad p_\beta = \frac{\partial L}{\partial C_t^C D_b^\beta q}.$$ 

(9)

Making use of (8) and (9) we define the fractional canonical Hamiltonian as

$$H = p_\alpha C_a^C D_t^\alpha q + p_\beta C_t^C D_b^\beta q - L.$$ 

(10)

The fractional Hamilton equations are obtained as follows [37]

$$\frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}, \quad \frac{\partial H}{\partial p_\alpha} = C_a^C D_t^\alpha q, \quad \frac{\partial H}{\partial p_\beta} = C_t^C D_b^\beta q, \quad \frac{\partial H}{\partial q} = t D_b^\alpha p_\alpha + a D_t^\beta p_\beta.$$ 

(11)

### 4 Fractional Hamilton-Jacobi formulation

In this section, we construct the Hamilton-Jacobi formulation within Caputo fractional derivatives. According to [37], the fractional Hamiltonian is defined in equation (10). Now, let consider the canonical transformations
with the generating function $F_2(C_a D_t^{\alpha-1} q, C_b D_b^{\beta-1} q, P_\alpha, P_\beta, t) = S$. The new Hamiltonian will take the form

$$K = P_\alpha C_a D_t^{\alpha} Q + P_\beta C_t D_b^{\beta} Q - L(Q, C_a D_t^{\alpha} Q, C_b D_b^{\beta} Q).$$

(12)

The variations of the following integrals

$$\delta \int_{t_1}^{t_2} (p_\alpha C_a D_t^{\alpha} q + p_\beta C_t D_b^{\beta} q - H) dt = 0, \quad \delta \int_{t_1}^{t_2} (p_\alpha C_a D_t^{\alpha} q + p_\beta C_t D_b^{\beta} q - H) dt = 0,$$

(13)

are vanishing, then we obtain the relation between the Hamiltonians $H$ and $K$ as follows

$$p_\alpha C_a D_t^{\alpha} q + p_\beta C_t D_b^{\beta} q - H = P_\alpha C_a D_t^{\alpha} Q + P_\beta C_t D_b^{\beta} Q - K + \frac{dF}{dt},$$

(14)

where the function $F$ is given as

$$F = S(C_a D_t^{\alpha-1} q, C_b D_b^{\beta-1} q, P_\alpha, P_\beta, t) - P_\alpha C_a D_t^{\alpha-1} Q - P_\beta C_b D_b^{\beta-1} Q.$$

(15)

Substituting this function $F$ in equation (14), we obtain the following equations

$$C_a D_t^{\alpha-1} Q = \frac{\partial S}{\partial P_\alpha}, \quad C_b D_b^{\beta-1} Q = \frac{\partial S}{\partial P_\beta},$$

(16)

$$p_\alpha = \frac{\partial S}{\partial C_a D_t^{\alpha-1} q}, \quad p_\beta = \frac{\partial S}{\partial C_t D_b^{\beta-1} q},$$

(17)

$$K = H + \frac{\partial S}{\partial t}.$$  

(18)

In the case that the new variables $(Q, P_\alpha, P_\beta)$ are constants in time, then $K = 0$. One can easily show that action function can be put in the form.

$$S = \int_{t_1}^{t_2} L dt.$$

(19)

For time-independent Hamiltonian $H$, the action function $S$ can be put in the form

$$S = W_1(C_a D_t^{\alpha-1} q, E_1) + W_2(C_b D_b^{\beta-1} q, E_2) + f(E_1, E_2, t),$$

(20)

here, $W$ is called the Hamilton’s characteristic function and $f(E_1, E_2, t) = -Et$, where $E_1 = P_\alpha$, $E_2 = P_\beta$ and $E = E_1 + E_2$.  

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Making use of equation (22), and after some simple manipulations, we obtain

\[ C_a D_t^{\alpha-1} Q = \frac{\partial S}{\partial P_{\alpha}} = \lambda_1, \quad C_b D_t^{\beta-1} Q = \frac{\partial S}{\partial P_{\beta}} = \lambda_2 \]  \hspace{1cm} (21)

\[ p_{\alpha} = \frac{\partial W_1}{\partial C_a D_t^{\alpha-1} q}, \quad p_{\beta} = \frac{\partial W_2}{\partial C_t D_b^{\beta-1} q}, \]  \hspace{1cm} (22)

\[ \frac{\partial S}{\partial t} = -H = -E. \]  \hspace{1cm} (23)

and the Hamilton-Jacobi partial differential equation for fractional system reads as

\[ \frac{\partial S}{\partial t} + H = 0. \]  \hspace{1cm} (24)

Now the solutions of equation (20), give the values of \( W_1 \) and \( W_2 \) as

\[ W_1 = \int p_{\alpha} C_a D_t^{\alpha-1} q, \quad W_2 = \int p_{\beta} C_t D_b^{\beta-1} q. \]  \hspace{1cm} (25)

5 Illustrative example

In this section, we shall present an example to demonstrate the application of the formulation developed in the previous section. Let us consider the following Hamiltonian [36]

\[ H = \frac{1}{2} p_{\alpha}^2 + q. \]  \hspace{1cm} (26)

The fractional Hamilton-Jacobi equation is given by

\[ \frac{1}{2} p_{\alpha}^2 + q - E_1 = 0. \quad p_{\beta} = 0. \]  \hspace{1cm} (27)

Making use of equation (22), we get

\[ \left[ \frac{\partial W_1}{\partial C_a D_t^{\alpha-1} q} \right]^2 + q - E_1 = 0, \]  \hspace{1cm} (28)

which has the following solution

\[ W_1 = \sqrt{2(E_1 - q)} C_a D_t^{\alpha-1} q. \]  \hspace{1cm} (29)
this equation leads us to obtain $p_\alpha$ and $S$ respectively as

\[ p_\alpha = \sqrt{2(E_1 - q)} \]  
(30)
\[ S = \sqrt{2(E_1 - q)} C^C D_t^{\alpha - 1} q - E_1 t. \]  
(31)

Again making use of equations (21), (31), we arrive at

\[ C^C D_t^{\alpha - 1} Q = \frac{\partial S}{\partial t} = \frac{1}{\sqrt{2(E_1 - q)}} C^C D_t^{\alpha - 1} q - t = \lambda_1. \]  
(32)

Equation (32), can be easily solved to obtain

\[ C^C D_t^{\alpha - 1} q = \sqrt{2(E_1 - q)}(t + \lambda_1), \]  
(33)

or

\[ C^C D_t^{\alpha} q = \sqrt{2(E_1 - q)} = p_\alpha. \]  
(34)

Taking the Riemann-Liouville derivative of equation (34), we obtain

\[ t D_t^C D_t^{\alpha} q = 1. \]  
(35)

This result is in exact agreement with that obtained if we use the fractional Hamiltonian formulation.

6 Conclusions

In this paper, we have obtained the Hamilton-Jacobi partial differential equation within Caputo’s fractional derivative. Finding the action function leads us to obtain the solutions of equations of motion. An example was investigated and the solutions of the equations of motion are in exact agreement with those obtained by using the Hamiltonian formulation. The advantages of using the method presented in this paper, is that we can easily obtain the action function, which is the essential part to obtain the path integral quantization for any mechanical fractional system, and this topic is now under investigation.
References


