E_6 and the bipartite entanglement of three qutrits

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Recent investigations have established an analogy between the entropy of four-dimensional supersymmetric black holes in string theory and entanglement in quantum information theory. Examples include: (1) $N = 2$ STU black holes and the tripartite entanglement of three qubits (2-state systems), where the common symmetry is $[SL(2)]^3$ and (2) $N = 8$ black holes and the tripartite entanglement of seven qubits where the common symmetry is $E_7 \supset [SL(2)]^7$. Here we present another example: $N = 8$ black holes (or black strings) in five dimensions and the bipartite entanglement of three qutrits (3-state systems), where the common symmetry is $E_6 \supset [SL(3)]^3$. Both the black hole (or black string) entropy and the entanglement measure are provided by the Cartan cubic E_6 invariant. Similar analogies exist for magic $N = 2$ supergravity black holes in both four and five dimensions.

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I. $D = 4$ BLACK HOLES AND QUBITS

It sometimes happens that two very different areas of theoretical physics share the same mathematics. This may eventually lead to the realization that they are, in fact, dual descriptions of the same physical phenomena, or it may not. Either way, it frequently leads to new insights in both areas. Recent papers [1–6] have established an analogy between the entropy of certain four-dimensional supersymmetric black holes in string theory and entanglement measures in quantum information theory. In this paper we extend the analogy from four dimensions to five which also involves going from two-state systems (qubits) to three-state systems (qutrits).

We begin by recalling the four-dimensional examples:

A. $N = 2$ black holes and the tripartite entanglement of three qubits

The three qubit system (Alice, Bob, Charlie) is described by the state

$$|\Psi\rangle = a_{ABC}|ABC\rangle, \quad (1.1)$$

where $A = 0, 1$ so the Hilbert space has dimension $2^3 = 8$. The complex numbers a_{ABC} transforms as a $(2, 2, 2)$ under $SL(2, C)_A \times SL(2, C)_B \times SL(2, C)_C$. The tripartite entanglement is measured by the 3-tangle [7,8]

$$\tau_3(ABC) = 4|\text{Deta}_{ABC}|, \quad (1.2)$$

where Deta_{ABC} is Cayley's hyperdeterminant [9]

$$\begin{aligned} \text{Deta} = & -\frac{1}{2}\epsilon^{A_1A_2}\epsilon^{B_1B_2}\epsilon^{A_3A_4}\epsilon^{B_3B_4}\epsilon^{C_1C_4}\epsilon^{C_2C_3}a_{A_1B_1C_1}a_{A_2B_2C_2} \\ & \times a_{A_3B_3C_3}a_{A_4B_4C_4}. \end{aligned} \quad (1.3)$$

The hyperdeterminant is invariant under $SL(2)_A \times SL(2)_B \times SL(2)_C$ and under a triality that interchanges A , B , and C .

In the context of stringy black holes the 8 a_{ABC} are the 4 electric and 4 magnetic charges of the $N = 2$ STU black hole [10] and hence take on real (integer) values. The STU model corresponds to $N = 2$ supergravity coupled to three vector multiplets, where the symmetry is $[SL(2, Z)]^3$. The Bekenstein-Hawking entropy of the black hole, S , was first calculated in [11]. The connection to quantum information theory arises by noting [1] that it can also be expressed in terms of Cayley's hyperdeterminant

$$S = \pi\sqrt{|\text{Deta}_{ABC}|}. \quad (1.4)$$

One can then establish a dictionary between the classification of various entangled states (separable A-B-C; bipartite entangled A-BC, B-CA, C-AB; tripartite entangled W; tripartite entangled GHZ) and the classification of various “small” and “large” BPS and non-BPS black holes [1–6]. For example, the GHZ state [12]

$$|\Psi\rangle \sim |111\rangle + |000\rangle \quad (1.5)$$

with $\text{Deta}_{ABC} \geq 0$ corresponds to a large non-BPS 2-charge black hole; the W-state

$$|\Psi\rangle \sim |100\rangle + |010\rangle + |001\rangle \quad (1.6)$$

with $\text{Deta}_{ABC} = 0$ corresponds to a small BPS 3-charge black hole; the GHZ state

$$|\Psi\rangle = -|000\rangle + |011\rangle + |101\rangle + |110\rangle \quad (1.7)$$

corresponds to a large BPS 4-charge black hole.

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B. $N = 2$ black holes and the bipartite entanglement of two qubits

An even simpler example [2] is provided by the two qubit system (Alice and Bob) described by the state

$$|\Psi\rangle = a_{AB}|AB\rangle, \quad (1.8)$$

where $A = 0, 1$, and the Hilbert space has dimension $2^2 = 4$. The a_{AB} transforms as a $(2, 2)$ under $SL(2, C)_A \times SL(2, C)_B$. The entanglement is measured by the 2-tangle

$$\tau_2(AB) = C^2(AB) \quad (1.9)$$

where

$$C(AB) = 2|\text{deta}_{AB}| \quad (1.10)$$

is the *concurrence*. The determinant is invariant under $SL(2, C)_A \times SL(2, C)_B$ and under a duality that interchanges A and B . Here it is sufficient to look at $N = 2$ supergravity coupled to just one vector multiplet and the 4 a_{AB} are the 2 electric and 2 magnetic charges of the axion-dilaton black hole with entropy

$$S = \pi|\text{deta}_{AB}|. \quad (1.11)$$

For example, the Bell state

$$|\Psi\rangle \sim |11\rangle + |00\rangle \quad (1.12)$$

with $\text{deta}_{AB} \geq 0$ corresponds to a large non-BPS 2-charge black hole.

C. $N = 8$ black holes and the tripartite entanglement of seven qubits

We recall that in the case of $D = 4$, $N = 8$ supergravity, the 28 electric and 28 magnetic charges belong to the 56 of $E_{7(7)}$. The black hole entropy is [13,14]

$$S = \pi\sqrt{|J_4|}, \quad (1.13)$$

where J_4 is Cartan's quartic E_7 invariant [15,16]. It may be written

$$\begin{aligned} J_4 = & P^{ij}Q_{jk}P^{kl}Q_{li} - \frac{1}{4}P^{ij}Q_{ij}P^{kl}Q_{kl} \\ & + \frac{1}{96}(\epsilon^{ijklmnop}Q_{ij}Q_{kl}Q_{mn}Q_{op} \\ & + \epsilon_{ijklmnop}P^{ij}P^{kl}P^{mn}P^{op}), \end{aligned} \quad (1.14)$$

where P^{ij} and Q_{jk} are 8×8 antisymmetric matrices.

The qubit interpretation [4] relies on the decomposition

$$E_7(C) \supset [SL(2, C)]^7 \quad (1.15)$$

under which

$$\begin{aligned} 56 \rightarrow & (2, 2, 1, 2, 1, 1, 1) + (1, 2, 2, 1, 2, 1, 1) \\ & + (1, 1, 2, 2, 1, 2, 1) + (1, 1, 1, 2, 2, 1, 2) \\ & + (2, 1, 1, 1, 2, 2, 1) + (1, 2, 1, 1, 1, 2, 2) \\ & + (2, 1, 2, 1, 1, 1, 2) \end{aligned} \quad (1.16)$$

FIG. 1. The E_7 entanglement diagram. Each of the seven vertices A,B,C,D,E,F,G represents a qubit and each of the seven triangles ABD, BCE, CDE, DEF, EFG, FGA, GAB describes a tripartite entanglement.

suggesting the tripartite entanglement of seven qubits (Alice, Bob, Charlie, Daisy, Emma, Fred, and George) described by the state

$$\begin{aligned} |\Psi\rangle = & a_{ABD}|ABD\rangle + b_{BCE}|BCE\rangle + c_{CDF}|CDF\rangle \\ & + d_{DEG}|DEG\rangle + e_{EFA}|EFA\rangle + f_{FGB}|FGB\rangle \\ & + g_{GAC}|GAC\rangle, \end{aligned} \quad (1.17)$$

where $A = 0, 1$, so the Hilbert space has dimension $7 \cdot 2^3 = 56$. The a, b, c, d, e, f, g transform as a 56 of $E_7(C)$. The entanglement may be represented by a heptagon where the

FIG. 2. The Fano plane has seven points, representing the seven qubits, and seven lines (the circle counts as a line) with three points on every line, representing the tripartite entanglement, and three lines through every point.

vertices A,B,C,D,E,F,G represent the seven qubits and the seven triangles ABD, BCE, CDF, DEG, EFA, FGB, GAC represent the tripartite entanglement. See Fig. 1. Alternatively, we can use the Fano plane. See Fig. 2. The Fano plane also corresponds to the multiplication table of the octonions.¹

The measure of the tripartite entanglement of the seven qubits is provided by the *3-tangle*

$$\tau_3(ABCDEFG) = 4|J_4| \quad (1.18)$$

with

$$J_4 \sim a^4 + b^4 + c^4 + d^4 + e^4 + f^4 + g^4 + 2[a^2b^2 + b^2c^2 + c^2d^2 + d^2e^2 + e^2f^2 + f^2g^2 + g^2a^2 + a^2c^2 + b^2d^2 + c^2e^2 + d^2f^2 + e^2g^2 + f^2a^2 + g^2b^2 + a^2d^2 + b^2e^2 + c^2f^2 + d^2g^2 + e^2a^2 + f^2b^2 + g^2c^2] + 8[bcdf + cdeg + defa + efgb + fgac + gabd + abce] \quad (1.19)$$

where products like

$$a^4 = (ABD)(ABD)(ABD)(ABD) = \epsilon^{A_1A_2}\epsilon^{B_1B_2}\epsilon^{D_1D_4}\epsilon^{A_3A_4}\epsilon^{B_3B_4}\epsilon^{D_2D_3}a_{A_1B_1D_1}a_{A_2B_2D_2}a_{A_3B_3D_3}a_{A_4B_4D_4} \quad (1.20)$$

exclude four individuals (here Charlie, Emma, Fred, and George), products like

$$a^2b^2 = \frac{1}{2}(ABD)(ABD)(BCE)(BCE) = \frac{1}{2}\epsilon^{A_1A_2}\epsilon^{B_1B_3}\epsilon^{D_1D_2}\epsilon^{B_2B_4}\epsilon^{C_3C_4}\epsilon^{E_3E_4}a_{A_1B_1D_1}a_{A_2B_2D_2}b_{B_3C_3E_3}b_{B_4C_4E_4} \quad (1.21)$$

exclude two individuals (here Fred and George) and products like

$$abce = (ABD)(BCE)(CDF)(EFA) = \epsilon^{A_1A_4}\epsilon^{B_1B_2}\epsilon^{C_2C_3}\epsilon^{D_1D_3}\epsilon^{E_2E_4}\epsilon^{F_3F_4}a_{A_1B_1D_1}b_{B_2C_2E_2}c_{C_3D_3F_3}e_{E_4F_4A_4} \quad (1.22)$$

exclude one individual (here George).²

Once again large non-BPS, small BPS, and large BPS black holes correspond to states with $J_4 > 0$, $J_4 = 0$, and $J_4 < 0$, respectively.

D. Magic supergravities in $D = 4$

The black holes described by Cayley's hyperdeterminant are those of $N = 2$ supergravity coupled to three vector multiplets, where the symmetry is $[SL(2, Z)]^3$. In [4] the following four-dimensional generalizations were considered:

- (1) $N = 2$ supergravity coupled to l vector multiplets where the symmetry is $SL(2, Z) \times SO(l-1, 2, Z)$ and the black holes carry charges belonging to the $(2, l+1)$ representation ($l+1$ electric plus $l+1$ magnetic).
- (2) $N = 4$ supergravity coupled to m vector multiplets where the symmetry is $SL(2, Z) \times SO(6, m, Z)$ where the black holes carry charges belonging to the $(2, 6+m)$ representation ($m+6$ electric plus $m+6$ magnetic).
- (3) $N = 8$ supergravity where the symmetry is the non-compact exceptional group $E_{7(7)}(Z)$ and the black holes carry charges belonging to the fundamental 56-dimensional representation (28 electric plus 28 magnetic).

In all three case there exist quartic invariants akin to Cayley's hyperdeterminant whose square root yields the corresponding black hole entropy. In [4] we succeeded in giving a quantum theoretic interpretation in the $N = 8$ case together with its truncations to $N = 4$ (with $m = 6$) and $N = 2$ (with $l = 3$, the case we already knew [1]).

However, as suggested by Levay [5], one might also consider the "magic" supergravities [17–19]. These correspond to the R, C, H, O (real, complex, quaternionic, and octonionic) $N = 2$, $D = 4$ supergravity coupled to 6, 9, 15, and 27 vector multiplets with symmetries $Sp(6, Z)$, $SU(3, 3)$, $SO^*(12)$, and $E_{7(-25)}$, respectively. Once again, as has been shown just recently [20], in all cases there are quartic invariants whose square root yields the corresponding black hole entropy.

Here we demonstrate that the black hole/qubit correspondence does indeed continue to hold for magic supergravities. The crucial observation is that, although the black hole charges a_{ABC} are real (integer) numbers and the entropy (1.13) is invariant under $E_{7(7)}(Z)$, the coefficients a_{ABC} that appear in the qutrit state (1.17) are complex. So the three tangle (1.18) is invariant under $E_7(C)$ which contains both $E_{7(7)}(Z)$ and $E_{7(-25)}(Z)$ as subgroups. To find a supergravity correspondence, therefore, we could equally well have chosen the magic octonionic $N = 2$ supergravity rather than the conventional $N = 8$ supergravity. The fact that

$$E_{7(7)}(Z) \supset [SL(2)(Z)]^7 \quad (1.23)$$

but

¹Not the "split" octonions as was incorrectly stated in the published version of [4].

²This corrects the corresponding equation in the published version of [4] which had the wrong index contraction.

$$E_{7(-25)}(Z) \not\supset [SL(2)(Z)]^7 \quad (1.24)$$

is irrelevant. All that matters is that

$$E_7(C) \supset [SL(2)(C)]^7. \quad (1.25)$$

The same argument holds for the magic real, complex and quaternionic $N = 2$ supergravities which are, in any case, truncations of $N = 8$ (in contrast to the octonionic).

Having made this observation, one may then revisit the conventional $N = 2$ and $N = 4$ cases (1) and (2) above. When we looked at the seven qubit subsector $E_7(C) \supset SL(2, C) \times SO(12, C)$, we gave an $N = 4$ supergravity interpretation with symmetry $SL(2, R) \times SO(6, 6)$ [4], but we could equally have given an interpretation in terms of $N = 2$ supergravity coupled to 11 vector multiplets with symmetry $SL(2, R) \times SO(10, 2)$.

Moreover, $SO(l - 1, 2)$ is contained in $SO(l + 1, C)$ and $SO(6, m)$ is contained in $SO(12 + m, C)$ so we can give a qubit interpretation to more vector multiplets for both $N = 2$ and $N = 4$, at least in the case of $SO(4n, C)$ which contains $[SL(2, C)]^{2n}$.

II. FIVE-DIMENSIONAL SUPERGRAVITY

In five dimensions we might consider:

- (1) $N = 2$ supergravity coupled to $l + 1$ vector multiplets where the symmetry is $SO(1, 1, Z) \times SO(l, 1, Z)$ and the black holes carry charges belonging to the $(l + 1)$ representation (all electric).
- (2) $N = 4$ supergravity coupled to m vector multiplets where the symmetry is $SO(1, 1, Z) \times SO(m, 5, Z)$ where the black holes carry charges belonging to the $(m + 5)$ representation (all electric).
- (3) $N = 8$ supergravity where the symmetry is the non-compact exceptional group $E_{6(6)}(Z)$ and the black

holes carry charges belonging to the fundamental 27-dimensional representation (all electric).

The electrically charged objects are pointlike and the magnetic duals are one-dimensional, or stringlike, transforming according to the contragredient representation. In all three cases above there exist cubic invariants akin to the determinant which yield the corresponding black hole or black string entropy.

In this section we briefly describe the salient properties of the maximal $N = 8$ case, following [21]. We have 27 Abelian gauge fields which transform in the fundamental representation of $E_{6(6)}$. The first invariant of $E_{6(6)}$ is the cubic invariant [14,15,21–23]

$$J_3 = q_{ij} \Omega^{jl} q_{lm} \Omega^{mn} q_{np} \Omega^{pi}, \quad (2.1)$$

where q_{ij} is the charge vector transforming as a 27 which can be represented as traceless $Sp(8)$ matrix. The entropy of a black hole with charges q_{ij} is then given by

$$S = \pi \sqrt{|J_3|}. \quad (2.2)$$

We will see that a configuration with $J_3 \neq 0$ preserves $1/8$ of the supersymmetries. If $J_3 = 0$ and $\frac{\partial J_3}{\partial q^i} \neq 0$ then it preserves $1/4$ of the supersymmetries, and finally if $\frac{\partial J_3}{\partial q^i} = 0$ (and the charge vector q^i is nonzero), the configuration preserves $1/2$ of the supersymmetries. We will show this by choosing a particular basis for the charges, the general result following by U-duality.

In five dimensions the compact group H is $USp(8)$. We choose our conventions so that $USp(2) = SU(2)$. In the commutator of the supersymmetry generators we have a central charge matrix Z_{ab} which can be brought to a normal form by a $USp(8)$ transformation. In the normal form the central charge matrix can be written as

$$e_{ab} = \begin{pmatrix} s_1 + s_2 - s_3 & 0 & 0 & 0 \\ 0 & s_1 + s_3 - s_2 & 0 & 0 \\ 0 & 0 & s_2 + s_3 - s_1 & 0 \\ 0 & 0 & 0 & -(s_1 + s_2 + s_3) \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad (2.3)$$

and we can order s_i so that $s_1 \geq s_2 \geq |s_3|$. The cubic invariant, in this basis, becomes

$$J_3 = s_1 s_2 s_3. \quad (2.4)$$

Even though the eigenvalues s_i might depend on the moduli, the invariant (2.4) only depends on the quantized values of the charges. We can write a generic charge configuration as UeU^t , where e is the normal frame as above, and the invariant will then be (2.4). There are three distinct possibilities:

$$\begin{aligned} J_3 \neq 0 & \quad s_1, s_2, s_3 \neq 0; \\ J_3 = 0, \quad \frac{\partial J_3}{\partial q^i} \neq 0 & \quad s_1, s_2 \neq 0, \quad s_3 = 0; \\ J_3 = 0, \quad \frac{\partial J_3}{\partial q^i} = 0 & \quad s_1 \neq 0, \quad s_2, s_3 = 0. \end{aligned} \quad (2.5)$$

Taking the case of type II on T^5 we can choose the rotation in such a way that, for example, s_1 corresponds to the solitonic five-brane charge, s_2 to the fundamental string winding charge along some direction, and s_3 to the Kaluza-Klein (KK) momentum along the same direction. We can see that in this specific example the three possibilities in

(2.5) break $1/8$, $1/4$, and $1/2$ supersymmetries. The respective orbits are

$$\frac{E_6(6)}{F_{4(4)}} \frac{E_6(6)}{SO(5,4) \ltimes T_{16}} \frac{E_6(6)}{SO(5,5) \ltimes T_{16}}. \quad (2.6)$$

This also shows that one can generically choose a basis for the charges so that all others are related by U-duality. The basis chosen here is the S-dual of the D -brane basis usually chosen for describing black holes in type IIB on T^5 . All others are related by U-duality to this particular choice. Note that, in contrast to the four-dimensional case where flipping the sign of J_4 (1.14) interchanges BPS and non-BPS black holes, the sign of the J_3 (2.4) is not important since it changes under a CPT transformation. There is no non-BPS orbit in five dimensions.

In five dimensions there are also stringlike configurations which are the magnetic duals of the configurations considered here. They transform in the contragredient $27'$ representation and the solutions preserving $1/2$, $1/4$, $1/8$ supersymmetries are characterized in an analogous way. We could also have configurations where we have both pointlike and stringlike charges; the pointlike charge is uniformly distributed along the string, it is more natural to consider this configuration as a pointlike object in $D = 4$ by dimensional reduction.

It is useful to decompose the U-duality group into the T-duality group and the S-duality group. The decomposition reads $E_6 \rightarrow SO(5,5) \times SO(1,1)$, leading to

$$27 \rightarrow 16_1 + 10_{-2} + 1_4. \quad (2.7)$$

The last term in (2.7) corresponds to the Neveu-Schwarz (NS) five-brane charge. The **16** corresponds to the D -brane charges and the **10** corresponds to the 5 directions of KK momentum and the 5 directions of fundamental string winding, which are the charges that explicitly appear in string perturbation theory. The cubic invariant has the decomposition

$$(27)^3 \rightarrow 10_{-2}10_{-2}1_4 + 16_116_110_{-2}. \quad (2.8)$$

This is saying that in order to have a nonzero area black hole we must have three NS charges (more precisely some “perturbative” charges and a solitonic five-brane); or we can have 2 D -brane charges and one NS charge. In particular, it is not possible to have a black hole with a nonzero horizon area with purely D -brane charges.

Notice that the noncompact nature of the groups is crucial in this classification.

III. $D = 5$ BLACK HOLES AND QUTRITS

So far, all the quantum information analogies involve four-dimensional black holes and qubits. In order to find an analogy with five-dimensional black holes we invoke three-state systems called *qutrits*.

A. $N = 2$ black holes and the bipartite entanglement of two qutrits

The two qutrit system (Alice and Bob) is described by the state

$$|\Psi\rangle = a_{AB}|AB\rangle,$$

where $A = 0, 1, 2$, so the Hilbert space has dimension $3^2 = 9$. The a_{AB} transforms as a $(3, 3)$ under $SL(3)_A \times SL(3)_B$. The bipartite entanglement is measured by the concurrence [24]

$$C(AB) = 3^{3/2} |\det a_{AB}|. \quad (3.1)$$

The determinant is invariant under $SL(3, C)_A \times SL(3, C)_B$ and under a duality that interchanges A and B .

The black hole interpretation is provided by $N = 2$ supergravity coupled to 8 vector multiplets with symmetry $SL(3, C)$ where the black hole charges transform as a 9. The entropy is given by

$$S = \pi |\det a_{AB}|. \quad (3.2)$$

B. $N = 8$ black holes and the bipartite entanglement of three qutrits

As we have seen in Sec. II in the case of $D = 5$, $N = 8$ supergravity, the black hole charges belong to the 27 of $E_{6(6)}$ and the entropy is given by (2.2).

The qutrit interpretation now relies on the decomposition

$$E_6(C) \supset SL(3, C)_A \times SL(3, C)_B \times SL(3, C)_C \quad (3.3)$$

under which

$$27 \rightarrow (3, 3, 1) + (3', 1, 3) + (1, 3', 3') \quad (3.4)$$

suggesting the bipartite entanglement of three qutrits (Alice, Bob, Charlie). However, the larger symmetry requires that they undergo at most bipartite entanglement of a very specific kind, where each person has bipartite entanglement with the other two:

$$|\Psi\rangle = a_{AB}|AB\rangle + b^B{}_C|BC\rangle + c^C{}_A|CA\rangle, \quad (3.5)$$

where $A = 0, 1, 2$, so the Hilbert space has dimension $3 \cdot 3^2 = 27$. The three states transforms as a pair of triplets under two of the $SL(3)$'s and singlets under the remaining one. Individually, therefore, the bipartite entanglement of each of the three states is given by the determinant (3.1). Taken together, however, we see from (3.4) that they transform as a complex 27 of $E_6(C)$. The entanglement diagram is a triangle with vertices ABC representing the qutrits and the lines AB, BC, and CA representing the entanglements. See Fig. 3. The $N = 2$ truncation of Sec. III A is represented by just the line AB with endpoints A and B.

Note that:

- (1) Any pair of states has an individual in common.

FIG. 3. The entanglement diagram is a triangle with vertices ABC representing the qutrits and the lines AB, BC, and CA representing the entanglements.

- (2) Each individual is excluded from one out of the three states.

The entanglement measure will be given by the concurrence

$$C(ABC) = 3^{3/2}|J_3| \quad (3.6)$$

J_3 being the singlet in $27 \times 27 \times 27$:

$$J_3 \sim a^3 + b^3 + c^3 + 6abc, \quad (3.7)$$

where the products

$$a^3 = \epsilon^{A_1 A_2 A_3} \epsilon^{B_1 B_2 B_3} a_{A_1 B_1} a_{A_2 B_2} a_{A_3 B_3}, \quad (3.8)$$

$$b^3 = \epsilon_{B_1 B_2 B_3} \epsilon^{C_1 C_2 C_3} b^{B_1}_{C_1} b^{B_2}_{C_2} b^{B_3}_{C_3}, \quad (3.9)$$

$$c^3 = \epsilon_{C_1 C_2 C_3} \epsilon_{A_1 A_2 A_3} c^{C_1 A_1} c^{C_2 A_2} c^{C_3 A_3} \quad (3.10)$$

exclude one individual (Charlie, Alice, and Bob, respectively), and the product

$$abc = a_{AB} b^B_{C^C} c^{CA} \quad (3.11)$$

excludes none.

C. Magic supergravities in $D = 5$

Just as in four dimensions, one might also consider the magic supergravities [17–19]. These correspond to the R, C, H, O (real, complex, quaternionic, and octonionic) $N = 2$, $D = 5$ supergravity coupled to 5, 8, 14, and 26 vector multiplets with symmetries $SL(3, R)$, $SL(3, C)$, $SU^*(6)$ and $E_{6(-26)}$, respectively. Once again, in all cases there are cubic invariants whose square root yields the corresponding black hole entropy [20].

Here we demonstrate that the black hole/qubit correspondence continue to hold for these $D = 5$ magic supergravities, as well as $D = 4$. Once again, the crucial observation is that, although the black hole charges a_{AB}

are real (integer) numbers and the entropy (2.2) is invariant under $E_{6(6)}(Z)$, the coefficients a_{AB} that appear in the wave function (3.5) are complex. So the 2-tangle (3.6) is invariant under $E_6(C)$ which contains both $E_{6(6)}(Z)$ and $E_{6(-26)}(Z)$ as subgroups. To find a supergravity correspondence therefore, we could equally well have chosen the magic octonionic $N = 2$ supergravity rather than the conventional $N = 8$ supergravity. The fact that

$$E_{6(6)}(Z) \supset [SL(3)(Z)]^3 \quad (3.12)$$

but

$$E_{6(-26)}(Z) \not\supset [SL(3)(Z)]^3 \quad (3.13)$$

is irrelevant. All that matters is that

$$E_6(C) \supset [SL(3)(C)]^3. \quad (3.14)$$

The same argument holds for the magic real, complex and quaternionic $N = 2$ supergravities which are, in any case, truncations of $N = 8$ (in contrast to the octonionic). In fact, the example of Sec. III A corresponds to the complex case.

Having made this observation, one may then revisit the conventional $N = 2$ and $N = 4$ cases (1) and (2) of Sec. II. $SO(l, 1)$ is contained in $SO(l + 1, C)$ and $SO(m, 5)$ is contained in $SO(5 + m, C)$, so we can give a qutrit interpretation to more vector multiplets for both $N = 2$ and $N = 4$, at least in the case of $SO(6n, C)$ which contains $[SL(3, C)]^n$.

IV. CONCLUSIONS

We note that the 27-dimensional Hilbert space given in (3.4) and (3.5) is not a subspace of the 3^3 -dimensional three qutrit Hilbert space given by (3, 3, 3), but rather a direct sum of three 3^2 -dimensional Hilbert spaces. It is, however, a subspace of the 7^3 -dimensional three 7-dit Hilbert space given by (7, 7, 7). Consider the decomposition

$$SL(7)_A \times SL(7)_B \times SL(7)_C \rightarrow SL(3)_A \times SL(3)_B \times SL(3)_C$$

under which

$$\begin{aligned} (7, 7, 7) \rightarrow & (3', 3', 3') + (3', 3', 3) + (3', 3, 3') + (3, 3', 3') \\ & + (3', 3, 3) + (3, 3', 3) + (3, 3, 3') + (3, 3, 3) \\ & + (3', 3', 1) + (3', 1, 3') + (1, 3', 3') + (3', 1, 3) \\ & + (3', 3, 1) + (1, 3, 3') + (3, 3, 1) + (3, 1, 3) \\ & + (1, 3, 3) + (3, 1, 3') + (3, 3', 1) + (1, 3', 3) \\ & + (3', 1, 1) + (1, 3', 1) + (1, 1, 3') + (3, 1, 1) \\ & + (1, 3, 1) + (1, 1, 3) + (1, 1, 1). \end{aligned}$$

This contains the subspace that describes the bipartite entanglement of three qutrits, namely,

$$(3', 3, 1) + (3, 1, 3) + (1, 3', 3').$$

So the triangle entanglement we have described fits within conventional quantum information theory.

Our analogy between black holes and quantum information remains, for the moment, just that. We know of no physics connecting them.

Nevertheless, just as the exceptional group E_7 describes the tripartite entanglement of seven qubits [4,5], we have seen in this paper that the exceptional group E_6 describes the bipartite entanglement of three qutrits. In the E_7 case, the quartic Cartan invariant provides both the measure of entanglement and the entropy of the four-dimensional $N = 8$ black hole, whereas in the E_6 case, the cubic Cartan invariant provides both the measure of entanglement and the entropy of the five-dimensional $N = 8$ black hole.

Moreover, we have seen that similar analogies exist not only for the $N = 4$ and $N = 2$ truncations, but also for the magic $N = 2$ supergravities in both four and five dimensions. (In the four-dimensional case, this had previously been conjectured by Levay [3,5].) Gunaydin has suggested [25] that the appearance of octonions implies a connection to quaternionic and/or octonionic quantum mechanics. This was not apparent (at least to us) in the four-

dimensional $N = 8$ case [4], but the appearance in the five-dimensional magic $N = 2$ case of $SL(3, R)$, $SL(3, C)$, $SL(3, H)$, and $SL(3, O)$ is more suggestive.

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