Analytic solutions for marginal deformations in open superstring field theory

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ABSTRACT: We extend the calculable analytic approach to marginal deformations recently developed in open bosonic string field theory to open superstring field theory formulated by Berkovits. We construct analytic solutions to all orders in the deformation parameter when operator products made of the marginal operator and the associated superconformal primary field are regular.

KEYWORDS: String Field Theory, Superstrings and Heterotic Strings
1. Introduction

Ever since the analytic solution for tachyon condensation in open bosonic string field theory \cite{Schnabl2007} was constructed by Schnabl \cite{Schnabl2007}, new analytic technologies have been developed \cite{Schnabl2007, Bergshoeff2008, Bergshoeff2009, Bergshoeff2010, Bergshoeff2011, Bergshoeff2012, Bergshoeff2013}, and analytic solutions for marginal deformations were recently constructed \cite{Berkovits2007, Berkovits2008}.

We believe that we are now in a new phase of research on open string field theory.\footnote{For earlier study of marginal deformations in string field theory and related work, see \cite{Bergshoeff2008, Bergshoeff2009}.}

Extension of these new technologies to closed string field theory, however, does not seem straightforward. The star product \cite{Schnabl2007} used in open string field theory has a simpler description in the conformal field theory (CFT) formulation when we use a coordinate called the sliver frame which was originally introduced in \cite{Berkovits2007}. It has been an important ingredient in recent developments. Closed bosonic string field theory \cite{Bergshoeff2008, Bergshoeff2009, Bergshoeff2010, Bergshoeff2011, Bergshoeff2012, Bergshoeff2013, Bergshoeff2014} and heterotic string field theory \cite{Berkovits2007, Berkovits2008}, however, use infinitely many non-associative string products, and we have not found any coordinate where simple descriptions of these string products are possible.

On the other hand, extension to open superstring field theory formulated by Berkovits \cite{Berkovits2007} is promising because the string product used in the theory is the same as that in open bosonic string field theory. In this paper we construct analytic solutions for marginal deformations in open superstring field theory.

We first review the solutions for marginal deformations in open bosonic string field theory. The solutions take the form of an expansion in terms of the deformation parameter $\lambda$, and analytic expressions to all order in $\lambda$ have been derived when operator products made...
of the marginal operator are regular [16, 17]. When the operator product of the marginal operator with itself is singular, solutions were constructed to \( O(\lambda^3) \) by regularizing the singularity and by adding counterterms [17].

The goal of this paper is to construct analytic solutions in open superstring field theory when operator products made of the marginal operator and the associated superconformal primary field of dimension 1/2 are regular. It will be a starting point for constructing analytic solutions when these operators have singular operator products. We first simplify the equation of motion for open superstring field theory by field redefinition. We then make an ansatz motivated by the structure of the solutions in the bosonic case and solve the equation of motion analytically. The solutions in the superstring case turn out to be remarkably simple and similar to those in the bosonic case. The final section of the paper is devoted to conclusions and discussion.

We learned that T. Erler independently found analytic solutions for marginal deformations in open superstring field theory [47] prior to our construction.

2. Solutions in open bosonic string field theory

In this section, we review the analytic solutions for marginal deformations constructed in [16, 17] for the open bosonic string. The equation of motion for open bosonic string field theory [1] is given by

\[
Q_B \Psi + \Psi^2 = 0 ,
\]

where \( \Psi \) is the open string field and \( Q_B \) is the BRST operator. All the string products in this paper are defined by the star product [1]. The open bosonic string field \( \Psi \) has ghost number 1 and is Grassmann odd. The BRST operator is Grassmann odd and is nilpotent: \( Q_B^2 = 0 \). It is a derivation with respect to the star product:

\[
Q_B (\varphi_1 \varphi_2) = (Q_B \varphi_1) \varphi_2 + (-1)^{\varphi_1} \varphi_1 (Q_B \varphi_2)
\]

for any states \( \varphi_1 \) and \( \varphi_2 \), where \( (-1)^{\varphi_1} = 1 \) when \( \varphi_1 \) is Grassmann even and \( (-1)^{\varphi_1} = -1 \) when \( \varphi_1 \) is Grassmann odd.

The deformation of the boundary CFT for the open string by a matter primary field \( V \) of dimension 1 is marginal to linear order in the deformation parameter. When the deformation is exactly marginal, we expect a solution of the form

\[
\Psi_\lambda = \sum_{n=1}^{\infty} \lambda^n \Psi(n),
\]

where \( \lambda \) is the deformation parameter, to the nonlinear equation of motion [2,1]. When operator products made of \( V \) are regular, analytic expressions of \( \Psi(n)'s \) were derived in [16, 17], and the BPZ inner product \( \langle \varphi, \Psi(n) \rangle \) for a state \( \varphi \) in the Fock space is given by

\[
\langle \varphi, \Psi(n) \rangle = \int_0^1 dt_1 \int_0^1 dt_2 \cdots \int_0^1 dt_{n-1} \left\{ f \circ \varphi(0) cV(1) B cV(1 + t_1) B cV(1 + t_1 + t_2) \cdots \times B cV(1 + t_1 + t_2 + \cdots + t_{n-1}) \right\} \prod_{i=1}^{n-1} \lambda^{i+1}. \]

\( - 2 - \)
We follow the notation used in [3, 10, 17]. In particular, see the beginning of section 2 of [3] for the relation to the notation used in [2]. Here and in what follows we use $\varphi$ to denote a generic state in the Fock space and $\varphi(0)$ to denote its corresponding operator in the state-operator mapping. We use the doubling trick in calculating CFT correlation functions. As in [10], we define the oriented straight lines $V_\pm$ by

$$
V_\pm^\alpha = \{ z \mid \text{Re}(z) = \pm \frac{1}{2} (1 + \alpha) \},
$$

orientation : $\pm \frac{1}{2} (1 + \alpha) - i \infty \rightarrow \pm \frac{1}{2} (1 + \alpha) + i \infty , \tag{2.5}
$$

and the surface $W_\alpha$ can be represented as the region between $V_0^-$ and $V_2^+$, where $V_0^-$ and $V_2^+$ are identified by translation. The function $f(z)$ is

$$
f(z) = \frac{2}{\pi} \arctan z , \tag{2.6}
$$

and $f \circ \varphi(z)$ denotes the conformal transformation of $\varphi(z)$ by the map $f(z)$. The operator $B$ is defined by

$$
B = \int \frac{dz}{2\pi i} b(z) , \tag{2.7}
$$

and when $B$ is located between two operators at $t_1$ and $t_2$ with $1/2 < t_1 < t_2$, the contour of the integral can be taken to be $-V_0^+$ with $2 t_1 - 1 < \alpha < 2 t_2 - 1$. The anticommutation relation of $B$ and $c(z)$ is

$$\{B, c(z)\} = 1 , \tag{2.8}
$$

and $B^2 = 0$.

The solution can be written more compactly as

$$
\langle \varphi, \Psi^{(n)} \rangle = \int_0^1 dt_1 \int_0^1 dt_2 \ldots \int_0^1 dt_{n-1} \langle f \circ \varphi(0) \prod_{i=0}^{n-2} [cV(1 + \ell_i)B] cV(1 + \ell_{n-1}) \rangle_{W_{1+\ell_{n-1}}} , \tag{2.9}
$$

where

$$
\ell_0 = 0 , \quad \ell_i \equiv \sum_{k=1}^i t_k \quad \text{for} \quad i = 1, 2, 3, \ldots . \tag{2.10}
$$

It can be further simplified as

$$
\Psi_{\lambda} = \frac{1}{1 - \lambda X_b J_b} \lambda X_b , \tag{2.11}
$$

where

$$
\frac{1}{1 - \lambda X_b J_b} \equiv 1 + \sum_{n=1}^{\infty} (\lambda X_b J_b)^n . \tag{2.12}
$$

The state $X_b$ is the same as $\Psi^{(1)}$:

$$
\langle \varphi, X_b \rangle = \langle f \circ \varphi(0) cV(1) \rangle_{W_1} . \tag{2.13}
$$

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It solves the linearized equation of motion: $Q_B X_b = 0$. The definition of $J_b$ is a little involved. It is defined when it appears as $\varphi_1 J_b \varphi_2$ between two states $\varphi_1$ and $\varphi_2$ in the Fock space. The string product $\varphi_1 J_b \varphi_2$ is given by

$$
\langle \varphi, \varphi_1 J_b \varphi_2 \rangle = \int_0^1 dt \langle f \circ \varphi(0) f_1 \circ \varphi_1(0) B f_{1+t} \circ \varphi_2(0) \rangle_{W_1+t},
$$

(2.14)

where $\varphi_1(0)$ and $\varphi_2(0)$ are the operators corresponding to the states $\varphi_1$ and $\varphi_2$, respectively. The map $f_\alpha(z)$ is a combination of $f(z)$ and translation:

$$
f_\alpha(z) = \frac{2}{\pi} \arctan \pi z + \alpha.
$$

(2.15)

The string product $\varphi_1 J_b \varphi_2$ is well defined if $f_1 \circ \varphi_1(0) B f_{1+t} \circ \varphi_2(0)$ is regular in the limit $t \to 0$. In the definition of $\Psi_\lambda$, $J_b$ always appears between two $X_b$’s. Since $c(1) B c(1+t) = c(1)$ in the limit $t \to 0$, the ghost part of $X_b J_b X_b$ is finite.\(^3\) Therefore, $X_b J_b X_b$ is well defined if the operator product $V(1) V(1+t)$ is regular in the limit $t \to 0$. The ghost part of the state $\Psi^{(n)} = (X_b J_b)^{n-1} X_b$ is also finite because $B c(z) B = B$ and $c(1) B c(1+\ell_{n-1}) = c(1)$ in the limit $\ell_{n-1} \to 0$. Therefore, $\Psi^{(n)}$ is well defined if the operator product in the matter sector

$$
\int_0^1 dt_1 \int_0^1 dt_2 \ldots \int_0^1 dt_{n-1} \prod_{i=0}^{n-1} \left[ V(1 + \ell_i) \right]
$$

(2.16)

is finite. For example, the marginal deformation associated with the rolling tachyon and the deformations in the light-cone directions satisfy the regularity condition \(^{[13, 17]}\).

An important property of $J_b$ is

$$
\varphi_1 (Q_B J_b) \varphi_2 = \varphi_1 \varphi_2
$$

(2.17)

when $f_1 \circ \varphi_1(0) f_{1+t} \circ \varphi_2(0)$ vanishes in the limit $t \to 0$. Since the BRST transformation of $b(z)$ is the energy-momentum tensor $T(z)$, the inner product $\langle \varphi, \varphi_1 (Q_B J_b) \varphi_2 \rangle$ is given by

$$
\langle \varphi, \varphi_1 (Q_B J_b) \varphi_2 \rangle = \int_0^1 dt \langle f \circ \varphi(0) f_1 \circ \varphi_1(0) \mathcal{L} f_{1+t} \circ \varphi_2(0) \rangle_{W_1+t},
$$

(2.18)

where

$$
\mathcal{L} = \int \frac{dz}{2\pi i} T(z),
$$

(2.19)

and the contour of the integral is the same as that of $B$. As discussed in \([3]\), an insertion of $\mathcal{L}$ is equivalent to taking a derivative with respect to $t$. It is analogous to the relation $L_0 e^{-tL_0} = -\partial_t e^{-tL_0}$ in the standard strip coordinates, where $L_0$ is the zero mode of the energy-momentum tensor. We thus have

$$
\langle \varphi, \varphi_1 (Q_B J_b) \varphi_2 \rangle = \int_0^1 dt \partial_t \langle f \circ \varphi(0) f_1 \circ \varphi_1(0) f_{1+t} \circ \varphi_2(0) \rangle_{W_1+t}
$$

$$
= \langle f \circ \varphi(0) f_1 \circ \varphi_1(0) f_2 \circ \varphi_2(0) \rangle_{W_2}
$$

(2.20)

\(^3\)Note that $f_\alpha \circ cV(0) = cV(\alpha)$ because $cV$ is a primary field of dimension 0.
when $f_1 \circ \varphi_1(0) f_{1+t} \circ \varphi_2(0)$ vanishes in the limit $t \to 0$. This completes the proof of (2.17). When $\varphi_1 = \varphi_2 = X_b$, the operator product $cV(1)cV(1+t)$ vanishes in the limit $t \to 0$ if $V(1)V(1+t)$ is regular in the limit $t \to 0$. In the language of [17], $\varphi_1 J_b \varphi_2$ is

$$\varphi_1 J_b \varphi_2 = \int_0^1 dt \varphi_1 e^{-(t-1)L_L^+ (B_L^+)} \varphi_2,$$  \hfill (2.21)

and the relation (2.17) follows from $\{Q_B, B_L^+\} = L_L^+$.

To summarize, when operator products made of $V$ are regular, the solution (2.11) is well defined, and we can safely use the relations

$$Q_B X_b = 0, \quad Q_B J_b = 1$$  \hfill (2.22)

for the Grassmann-odd states $X_b$ and $J_b$ when we calculate the BRST transformation of $\Psi_\lambda$. It is now straightforward to calculate $Q_B \Psi_\lambda$, and the result is

$$Q_B \Psi_\lambda = -\frac{1}{1 - \lambda X_b J_b} \lambda X_b \frac{1}{1 - \lambda X_b J_b} \lambda X_b.$$  \hfill (2.23)

We have thus shown that $\Psi_\lambda$ in (2.11) satisfies the equation of motion (2.1).

3. Equation of motion for open superstring field theory

The equation of motion for open superstring field theory [46] is

$$\eta_0 (e^{-\Phi} Q_B e^\Phi) = 0,$$  \hfill (3.1)

where $\Phi$ is the open superstring field. It is Grassmann even and has ghost number 0 and picture number 0. The superghost sector is described by $\eta, \xi$, and $\phi$ [48, 49], and the zero modes of $\eta$ and $\xi$ are included in the Hilbert space. The operator $\eta_0$ is the zero mode of $\eta$ and a derivation with respect to the star product. For any states $\varphi_1$ and $\varphi_2$, we have

$$\eta_0 (\varphi_1 \varphi_2) = (\eta_0 \varphi_1) \varphi_2 + (-1)^{\varphi_1} \varphi_1 (\eta_0 \varphi_2),$$  \hfill (3.2)

as in the case of $Q_B$, where $(-1)^{\varphi_1} = 1$ when $\varphi_1$ is Grassmann even and $(-1)^{\varphi_1} = -1$ when $\varphi_1$ is Grassmann odd. The Grassmann-odd operator $\eta_0$ is nilpotent and anticommutes with $Q_B$:

$$Q_B^2 = 0, \quad \eta_0^2 = 0, \quad \{Q_B, \eta_0\} = 0.$$  \hfill (3.3)

Since $\eta_0 (e^{-\Phi} Q_B e^\Phi) = e^{-\Phi} [Q_B (e^\Phi \eta_0 e^{-\Phi})] e^\Phi$, the equation of motion can also be written as follows:

$$Q_B (e^\Phi \eta_0 e^{-\Phi}) = 0.$$  \hfill (3.4)

We further simplify the equation of motion by field redefinition. Since the open superstring field $\Phi$ has vanishing ghost and picture numbers, there is a natural class of field redefinitions given by

$$\Phi_{\text{new}} = \sum_{n=1}^\infty a_n \Phi_{\text{odd}}.$$  \hfill (3.5)
where $a_n$’s are constants. The map from $\Phi_{\text{old}}$ to $\Phi_{\text{new}}$ is well defined at least perturbatively. We choose

$$1 - \Phi_{\text{new}} = e^{-\Phi_{\text{old}}},$$

(3.6)

and the equation of motion (3.4) written in terms of $\Phi_{\text{new}}$ is

$$-Q_B \left( \frac{1}{1 - \Phi} \eta_0 \Phi \right) = - \frac{1}{1 - \Phi} \left[ Q_B \eta_0 \Phi + (Q_B \Phi) \frac{1}{1 - \Phi} (\eta_0 \Phi) \right] = 0,$$

(3.7)

where

$$\frac{1}{1 - \Phi} \equiv 1 + \sum_{n=1}^{\infty} \Phi^n.$$  

(3.8)

In the following sections, we solve the equation of motion of the form

$$Q_B \eta_0 \Phi + (Q_B \Phi) \frac{1}{1 - \Phi} (\eta_0 \Phi) = 0,$$

(3.9)

or

$$Q_B \eta_0 \Phi + (Q_B \Phi) (\eta_0 \Phi) + \sum_{n=1}^{\infty} (Q_B \Phi) \Phi^n (\eta_0 \Phi) = 0.$$  

(3.10)

4. Solutions to second order

For any marginal deformation of the boundary CFT for the open superstring, there is an associated superconformal primary field $V_{1/2}$ of dimension 1/2, and the marginal operator $V_1$ of dimension 1 is the supersymmetry transformation of $V_{1/2}$. For example, $V_{1/2}$ is the fermionic coordinate $\psi^\mu(z)$ when $V_1$ is the derivative of the bosonic coordinate $i \partial X^\mu(z)$ up to a normalization constant. In the RNS formalism, the unintegrated vertex operator in the $-1$ picture is $c e^{-\phi} V_{1/2}$, and the unintegrated vertex operator in the 0 picture is $c V_1$. In open superstring field theory [4], the solution to the linearized equation of motion $Q_B \eta_0 \Phi^{(1)} = 0$ associated with the marginal deformation is given by $\Phi^{(1)} = X$, where $X$ is the state corresponding to the operator $V(0) = c \xi e^{-\phi} V_{1/2}(0)$:

$$\langle \varphi, X \rangle = \langle f \circ \varphi(0) V(1) \rangle_{W_1} = \langle f \circ \varphi(0) c \xi e^{-\phi} V_{1/2}(1) \rangle_{W_1}.$$  

(4.1)

See [28] for some explicit calculations in open superstring field theory when $V_{1/2}(z) = \psi^\mu(z)$.

When the deformation is exactly marginal, we expect a solution of the form

$$\Phi_{\lambda} = \sum_{n=1}^{\infty} \lambda^n \Phi^{(n)}.$$  

(4.2)

where $\lambda$ is the deformation parameter, to the nonlinear equation of motion [4]. The equation for $\Phi^{(2)}$ is

$$Q_B \eta_0 \Phi^{(2)} = - (Q_B \Phi^{(1)}) (\eta_0 \Phi^{(1)}) = - (Q_B X) (\eta_0 X).$$  

(4.3)
The right-hand side is annihilated by $Q_B$ and by $\eta_0$ because $Q_B\eta_0 X = 0$. In order to solve the equation for $\Phi^{(2)}$, we introduce a state $J$ by replacing $b(z)$ in $J_b$ for the bosonic case with $\xi b(z)$. Since
\[
\eta_0 \cdot \xi b(z) \equiv \int \frac{dw}{2\pi i} \eta(w) \xi b(z) = b(z) \tag{4.4}
\]
and the BRST transformation of $b(z)$ gives the energy-momentum tensor, we expect that $\xi b(z)$ in the superstring case plays a similar role of $b(z)$ in the bosonic case. In fact, the zero mode of $\xi b(z)$ divided by $L_0$ was used in the calculation of on-shell four-point amplitudes in [50].

We again define $J$ when it appears as $\varphi_1 J \varphi_2$ between two states $\varphi_1$ and $\varphi_2$ in the Fock space. The string product $\varphi_1 J \varphi_2$ is given by
\[
\langle \varphi, \varphi_1 J \varphi_2 \rangle = \int_{t_1}^{t_2} dt \langle f \circ \varphi(0) f_1 \circ \varphi_1(0) J f_1+ t \circ \varphi_2(0) \rangle_{W_{t_1 t_2}}, \tag{4.5}
\]
where $\varphi_1(0)$ and $\varphi_2(0)$ are the operators corresponding to the states $\varphi_1$ and $\varphi_2$, respectively. The operator $J$ is defined by
\[
J = \int \frac{dz}{2\pi i} \xi b(z), \tag{4.6}
\]
and when $J$ is located between two operators at $t_1$ and $t_2$ with $1/2 < t_1 < t_2$, the contour of the integral can be taken to be $-V_\alpha^+$ with $2t_1 - 1 < \alpha < 2t_2 - 1$. As in the case of $J_b$, the string product $\varphi_1 J \varphi_2$ is well defined if $f_1 \circ \varphi_1(0) J f_1+ t \circ \varphi_2(0)$ is regular in the limit $t \to 0$. We also have an important relation
\[
\varphi_1 (Q_B \eta_0 J) \varphi_2 = \varphi_1 \varphi_2 \tag{4.7}
\]
if $f_1 \circ \varphi_1(0) f_1+ t \circ \varphi_2(0)$ vanishes in the limit $t \to 0$. The proof of this relation follows from that of (2.17) after we use (4.4) in calculating $\eta_0 J$. We will discuss these regularity conditions later and proceed for the moment assuming they are satisfied. Namely, we assume that states involving $J$ are well defined and that we can use the relations
\[
Q_B\eta_0 X = 0, \quad Q_B\eta_0 J = 1 \tag{4.8}
\]
for the Grassmann-even states $X$ and $J$.

Motivated by the structure of the solutions in the bosonic case, we look for a solution which consists of $X J X$, $Q_B$, and $\eta_0$ to the equation (4.3) for $\Phi^{(2)}$. There are nine possible states:
\[
(Q_B\eta_0 X) J X = 0, \quad (Q_B X)(\eta_0 J) X, \quad (Q_B X) J (\eta_0 X), \quad (\eta_0 X)(Q_B J) X, \quad X (Q_B\eta_0 J) X = X^2, \quad X (Q_B J)(\eta_0 X), \quad (\eta_0 X) J (Q_B X), \quad X (\eta_0 J)(Q_B X), \quad X J (Q_B\eta_0 X) = 0. \tag{4.9}
\]
Two of them vanish and one of them reduces to \( X^2 \). We then calculate the action of \( Q_B \eta_0 \) on the nonvanishing states:

\[
Q_B \eta_0 \left[ (Q_B X) (\eta_0 J) X \right] = - (Q_B X) (\eta_0 X) , \\
Q_B \eta_0 \left[ (Q_B X) J (\eta_0 X) \right] = (Q_B X) (\eta_0 X) , \\
Q_B \eta_0 \left[ (\eta_0 X) (Q_B J) X \right] = - (\eta_0 X) (Q_B X) , \\
Q_B \eta_0 \left[ X (Q_B \eta_0 J) X \right] = - (\eta_0 X) (Q_B X) + (Q_B X) (\eta_0 X) , \\
Q_B \eta_0 \left[ X (Q_B J) (\eta_0 X) \right] = - (Q_B X) (\eta_0 X) , \\
Q_B \eta_0 \left[ (\eta_0 X) J (Q_B X) \right] = (\eta_0 X) (Q_B X) , \\
Q_B \eta_0 \left[ X (\eta_0 J) (Q_B X) \right] = - (\eta_0 X) (Q_B X) .
\]

We thus find that \( (Q_B X) (\eta_0 J) X, - (Q_B X) J (\eta_0 X), \) and \( X (Q_B J) (\eta_0 X) \) solve the equation (4.13) for \( \Phi^{(2)} \). We can also take an appropriate linear combination of the seven states, and different solutions should be related by gauge transformations. We choose

\[
\Phi^{(2)} = (Q_B X) (\eta_0 J) X 
\]

and consider its extension to \( \Phi^{(n)} \) in the next section.

5. Solutions in open superstring field theory

Remarkably, a simple extension of \( \Phi^{(2)} \) in (4.11) solves the equation of motion (3.9) to all orders in \( \lambda \). A solution is given by

\[
\Phi^{(3)} = (Q_B X) (\eta_0 J) (Q_B X) (\eta_0 J) X , \\
\Phi^{(4)} = (Q_B X) (\eta_0 J) (Q_B X) (\eta_0 J) (Q_B X) (\eta_0 J) X , \\
\vdots \\
\Phi^{(n)} = \left[ (Q_B X) (\eta_0 J) \right]^{n-1} X ,
\]

or

\[
\Phi_\lambda = \frac{1}{1 - \lambda (Q_B X) (\eta_0 J)} \lambda X , 
\]

where

\[
\frac{1}{1 - \lambda (Q_B X) (\eta_0 J)} = 1 + \sum_{n=1}^{\infty} \left[ \lambda (Q_B X) (\eta_0 J) \right]^n .
\]

Let us now show that \( \Phi_\lambda \) given by (5.2) satisfies the equation of motion (3.9). Since \( Q_B X \) and \( \eta_0 J \) are annihilated by \( \eta_0 \), the state \( \eta_0 \Phi_\lambda \) is given by

\[
\eta_0 \Phi_\lambda = \frac{1}{1 - \lambda (Q_B X) (\eta_0 J)} \lambda (\eta_0 X) . 
\]

For the calculation of \( Q_B \Phi_\lambda \), we use \( Q_B \left[ (Q_B X) (\eta_0 J) \right] = - Q_B X \) to find

\[
Q_B \left[ \frac{1}{1 - \lambda (Q_B X) (\eta_0 J)} \right] = - \frac{1}{1 - \lambda (Q_B X) (\eta_0 J)} \lambda (Q_B X) \frac{1}{1 - \lambda (Q_B X) (\eta_0 J)} .
\]
The state $Q_B \Phi_\lambda$ is given by
\[
Q_B \Phi_\lambda = -\frac{1}{1 - \lambda (Q_B X)(\eta_0 J)} \lambda (Q_B X) \Phi_\lambda + \frac{1}{1 - \lambda (Q_B X)(\eta_0 J)} \lambda X
\]
\[
= \frac{1}{1 - \lambda (Q_B X)(\eta_0 J)} \lambda (Q_B X) \left[ 1 - \frac{1}{1 - \lambda (Q_B X)(\eta_0 J)} \lambda X \right].
\]
(5.6)

Note that
\[
(Q_B \Phi_\lambda)_\frac{1}{1 - \Phi_\lambda} = \frac{1}{1 - \lambda (Q_B X)(\eta_0 J)} \lambda (Q_B X).
\]
(5.7)

Finally, $Q_B \eta_0 \Phi_\lambda$ is given by
\[
Q_B \eta_0 \Phi_\lambda = -\frac{1}{1 - \lambda (Q_B X)(\eta_0 J)} \lambda (Q_B X) \Phi_\lambda + \frac{1}{1 - \lambda (Q_B X)(\eta_0 J)} \lambda (\eta_0 X).
\]
(5.8)

We have thus shown that $\Phi_\lambda$ given by (5.2) satisfies the equation of motion (3.9).

An explicit expression of $\Phi^{(n)}$ in the CFT formulation is given by
\[
\langle \varphi, \Phi^{(n)} \rangle = \int_0^1 dt_1 \int_0^1 dt_2 \ldots \int_0^1 dt_{n-1} \left\langle f \circ \varphi(0) \prod_{i=0}^{n-2} Q_B \cdot V(1+\ell_i) \right\rangle_{W_{i+\ell_{n-1}}},
\]
where the BRST transformation of $V$ is
\[
Q_B \cdot V(z) = cV_1(z) + \eta e^\phi V_1/2(z).
\]
(5.10)

Note that $J$ in $J$ has been replaced by $B$ in $\eta_0 J$ because of (4.4). The term $\eta e^\phi V_1/2(1+\ell_i)$ in $Q_B \cdot V(1+\ell_i)$ does not contribute when $i = 1, 2, \ldots, n-2$ because $B^2 = 0$. By repeatedly using $Bc(z)B = B$, we find
\[
\langle \varphi, \Phi^{(n)} \rangle = \int d^{n-1}t \left\langle f \circ \varphi(0) cV_1(1) B \prod_{i=1}^{n-2} V_1(1+\ell_i) c \xi e^{-\phi} V_1/2(1+\ell_{n-1}) \right\rangle_{W_{i+\ell_{n-1}}}
\]
\[
+ \int d^{n-1}t \left\langle f \circ \varphi(0) \eta \eta e^\phi V_1/2(1) B \prod_{i=1}^{n-2} V_1(1+\ell_i) c \xi e^{-\phi} V_1/2(1+\ell_{n-1}) \right\rangle_{W_{i+\ell_{n-1}}},
\]
(5.11)

where we have defined
\[
\int d^{n-1}t \equiv \int_0^1 dt_1 \int_0^1 dt_2 \ldots \int_0^1 dt_{n-1}.
\]
(5.12)

We can also construct a different solution if we choose $\Phi^{(2)}$ to be $X (Q_B J)(\eta_0 X)$. It is easy to show that $\Phi_\lambda$ given by
\[
\Phi_\lambda = \lambda X \frac{1}{1 - \lambda (Q_B J)(\eta_0 X)}
\]
(5.13)
satisfies the equation of motion (3.9). It is also straightforward to construct analytic solutions based on star-algebra projectors other than the silver state using the method in [10].
6. Regularity conditions

In the proof that the solution \((5.2)\) satisfies the equation of motion \((3.9)\), we used the following relations:

\[
\begin{align*}
(Q_B X) (Q_B \eta_0 J) X &= (Q_B X) X, \\
(Q_B X) (Q_B \eta_0 J) (Q_B X) &= (Q_B X) (Q_B X), \\
(Q_B X) (Q_B \eta_0 J) (\eta_0 X) &= (Q_B X) (\eta_0 X).
\end{align*}
\] (6.1)

Let us study the conditions for these relations to hold. Since

\[
\begin{align*}
\eta_0 \cdot V(z) &= \eta_0 \cdot [c \xi e^{-\phi} V_{1/2}(z)] = -ce^{-\phi} V_{1/2}(z), \\
Q_B \cdot V(z) &= Q_B \cdot [c \xi e^{-\phi} V_{1/2}(z)] = cV_1(z) + \eta e^{\phi} V_{1/2}(z),
\end{align*}
\] (6.2)

and \(V, Q_B \cdot V,\) and \(\eta_0 \cdot V\) are all primary fields of dimension 0, the condition for \((6.1)\) gives

\[
\begin{align*}
\lim_{w \to z} [cV_1(z) + \eta e^{\phi} V_{1/2}(z)] c\xi e^{-\phi} V_{1/2}(w) &= 0, \\
\lim_{w \to z} [cV_1(z) + \eta e^{\phi} V_{1/2}(z)] [cV_1(w) + \eta e^{\phi} V_{1/2}(w)] &= 0, \\
\lim_{w \to z} [cV_1(z) + \eta e^{\phi} V_{1/2}(z)] c\xi e^{-\phi} V_{1/2}(w) &= 0.
\end{align*}
\] (6.3)

These are satisfied if the operator products \(V_1(z) V_{1/2}(w)\) and \(V_1(z) V_1(w)\) are regular in the limit \(w \to z\), and \(V_{1/2}(z) V_{1/2}(w)\) vanishes in the limit \(w \to z\). The vertex operator \(V_{1/2}(z)\) is Grassmann odd so that the last condition is satisfied if the operator product \(V_{1/2}(z) V_{1/2}(w)\) is not singular. To summarize, the equation of motion is satisfied if the operator products \(V_1(z) V_{1/2}(w), V_1(z) V_1(w),\) and \(V_{1/2}(z) V_{1/2}(w)\) are regular in the limit \(w \to z\).

Let us next consider if the solution itself is finite and if any intermediate steps in the proof are well defined. The expressions can be divergent when two or more operators collide, but if the states

\[
[ (Q_B X) (\eta_0 J) ]^{n-1} X, \quad [ (Q_B X) (\eta_0 J) ]^{n-1} (Q_B X), \quad [ (Q_B X) (\eta_0 J) ]^{n-1} (\eta_0 X)
\] (6.4)

for any positive integer \(n\) are finite, the solution and any intermediate steps in the proof are well defined. An explicit expression of \(\Phi^{(n)} = [ (Q_B X) (\eta_0 J) ]^{n-1} X\) has been presented in \((5.11)\). Expressions of \([ (Q_B X) (\eta_0 J) ]^{n-1} (Q_B X)\) and \([ (Q_B X) (\eta_0 J) ]^{n-1} (\eta_0 X)\) can be obtained from \((5.11)\) by replacing \(c \xi e^{-\phi} V_{1/2}(1+\ell_{n-1})\) with \(cV_1(1+\ell_{n-1}) + \eta e^{\phi} V_{1/2}(1+\ell_{n-1})\) and with \(-ce^{-\phi} V_{1/2}(1+\ell_{n-1})\), respectively. The bc ghost sector is finite because \(c(z) B c(w)\) is finite in the limit \(w \to z\). The superghost sector is also finite because \(\eta e^{\phi}(1) \xi e^{-\phi}(1+\ell_{n-1})\) and \(\eta e^{\phi}(1) \eta e^{\phi}(1+\ell_{n-1})\) are finite in the limit \(\ell_{n-1} \to 0\). Therefore, all the expressions are
well defined if the contributions from the matter sector listed below are finite:

\[
\int_0^1 dt_1 \int_0^1 dt_2 \ldots \int_0^1 dt_{n-1} \prod_{i=0}^{n-1} \left[ V_1(1 + \ell_i) \right], \\
\int_0^1 dt_1 \int_0^1 dt_2 \ldots \int_0^1 dt_{n-1} V_{1/2}(1) \prod_{i=0}^{n-1} \left[ V_1(1 + \ell_i) \right], \\
\int_0^1 dt_1 \int_0^1 dt_2 \ldots \int_0^1 dt_{n-1} \prod_{i=0}^{n-2} \left[ V_1(1 + \ell_i) \right] V_{1/2}(1 + \ell_{n-1}), \\
\int_0^1 dt_1 \int_0^1 dt_2 \ldots \int_0^1 dt_{n-1} V_{1/2}(1) \prod_{i=1}^{n-2} \left[ V_1(1 + \ell_i) \right] V_{1/2}(1 + \ell_{n-1}),
\]

(6.5)

where \(\ell_i\) was defined in (2.10). To summarize, if operator products of an arbitrary number of \(V_1\)'s and at most two \(V_{1/2}\)'s are regular, the solution (5.2) is well defined and satisfies the equation of motion (3.9).

7. Conclusions and discussion

We have constructed analytic solutions for marginal deformations in open superstring field theory when operator products made of \(V_1\)'s and \(V_{1/2}\)'s are regular. Our solutions are very simple and remarkably similar to the solutions in the bosonic case [16, 17]. We expect that there will be further progress of analytic methods in open superstring field theory.

It would be interesting to study the rolling tachyon in open superstring field theory, and we expect that marginal deformations associated with the rolling tachyon solutions satisfy the regularity conditions discussed in the preceding section. However, deformations we are interested in typically have singular operator products of the marginal operator. In the bosonic case, solutions to third order in \(\lambda\) have been constructed when the operator product of the marginal operator is singular [17]. We hope that a procedure similar to the one developed in the bosonic case will work in the superstring case, and it is important to carry out the program to all orders in the deformation parameter.

Our choice of \(\Phi^{(2)}\) in (4.11) was based on a technical reason, and it is not clear if this gauge choice is physically suitable. In particular, the solution \(\Phi_{\lambda}\) in (5.2) does not satisfy the reality condition in the string field. However, it is difficult for us to imagine that there are two inequivalent solutions generated by a single marginal operator which coincide to linear order in \(\lambda\), and we expect that our solution is related to a real one by a gauge transformation. In fact, we can explicitly confirm this at \(O(\lambda^2)\). In order to see this, it is useful to write the solution in the original definition of the string field by inverting the field redefinition (3.6):

\[
\Phi_{\text{old}} = - \ln (1 - \Phi_{\text{new}}) = \sum_{n=1}^{\infty} \frac{1}{n} \Phi_{\text{new}}^n.
\]

(7.1)

We expand \(\Phi_{\text{old}}\) in powers of \(\lambda\) as

\[
\Phi_{\text{old}} = \sum_{n=1}^{\infty} \lambda^n \Phi_{\text{old}}^{(n)}.
\]

(7.2)
and then $\Phi^{(2)}_{\text{old}}$ is given by

$$\Phi^{(2)}_{\text{old}} = \Phi^{(2)}_{\text{new}} + \frac{1}{2} (\Phi^{(1)}_{\text{new}})^2 = (Q_B X) (\eta_0 J) X + \frac{1}{2} X^2. \quad (7.3)$$

The string field $\Phi^{(2)}_{\text{old}}$ does not satisfy the reality condition. However, there is another solution which satisfies the reality condition given by

$$\frac{1}{2} [(Q_B X) (\eta_0 J) X + X (\eta_0 J) (Q_B X)], \quad (7.4)$$

and the difference between (7.3) and (7.4) is

$$(Q_B X) (\eta_0 J) X + \frac{1}{2} X^2 - \frac{1}{2} [(Q_B X) (\eta_0 J) X + X (\eta_0 J) (Q_B X)] = \frac{1}{2} Q_B [X (\eta_0 J) X] \quad (7.5)$$

and can be eliminated by a gauge transformation. The open superstring field theory formulated by Berkovits can also be used to describe the $N = 2$ string by replacing $Q_B$ and $\eta_0$ with the generators in the $N = 2$ string \[\text{[46]}\], but the reality condition on the string field for the $N = 2$ string does not seem to be satisfied for $\Phi^{(2)}_{\lambda}$ in (5.2) either.\footnote{A string field within our ansatz satisfies the reality condition when it is odd under the conjugation given by replacing $X \rightarrow -X$ and by reversing the order of string products. Signs from anticommuting Grassmann-odd string fields have to be taken care of in reversing the order of string products.}

The conjugation in \[\text{[46]}\] seems to map $\Phi^{(2)}_{\lambda}$ in (5.2) to $\Phi^{(2)}_{\lambda}$ in (5.13). We again expect that our solution is related to a solution satisfying the reality condition by a gauge transformation. For example, $- (Q_B X) J (\eta_0 X)$, which is another solution to the equation for $\Phi^{(2)}_{\lambda}$, seems to satisfy the reality condition, and the difference between $- (Q_B X) J (\eta_0 X)$ and $\Phi^{(2)}_{\lambda}$ in (4.11) is $\eta_0 [(Q_B X) J X]$ and can be eliminated by a gauge transformation generated by $\eta_0$. We have also found that $(Q_B X) (Q_B J) X (\eta_0 J) (\eta_0 X)$, which seems to satisfy the reality condition, solves the equation for $\Phi^{(3)}_{\lambda}$ when $\Phi^{(2)}_{\lambda} = - (Q_B X) J (\eta_0 X)$, but we have not been able to extend the solution to all orders in $\lambda$. We think that there is a good chance that solutions satisfying the reality condition for the ordinary superstring or for the $N = 2$ string can be found within our ansatz, and it would be desirable to have their explicit expressions. On the other hand, we believe that the solution in (5.2) has an advantage because the actions of $Q_B$ and $\eta_0$ on (5.2) are very simple.

It has been expected that the moduli space of D-branes are reproduced by the moduli space of solutions to open string field theory, and we think that our approach provides a concrete setup to address this question. We have seen a one-to-one correspondence between the condition for exact marginality in boundary CFT \[\text{[51]}\] and the absence of obstruction in solving the equation of motion for string field theory at $O(\lambda^2)$ in the bosonic case \[\text{[17]}\]. It would be important to study the correspondence at higher orders and in the superstring.

\footnote{Our understanding is that the conjugation in \[\text{[46]}\] is given by replacing $X \rightarrow X, J \rightarrow -J, Q_B \rightarrow \eta_0$, and $\eta_0 \rightarrow Q_B$ and by reversing the order of string products, and the string field should be even under the conjugation. Again signs from anticommuting Grassmann-odd string fields have to be taken care of in reversing the order of string products. The string field $\Phi_{\text{new}}$ in (3.6) is real when $\Phi_{\text{old}}$ is real with respect to this reality condition, while this is not the case for the reality condition for the ordinary superstring discussed earlier.}
case, and a better understanding of the correspondence might help us complete the program of constructing solutions when the operator product of the marginal operator is singular. We hope that further developments in this subject will shed light on more conceptual issues in string theory such as background independence or the question why the condition that the $\beta$ function vanishes in the world-sheet theory gives the equation of motion in the spacetime theory.

**Note added.** After the first version of this paper was submitted to arXiv, we found analytic solutions satisfying the reality condition [52]. We also learned that T. Erler independently constructed analytic solutions satisfying the reality condition, which were presented in the second version of [47].

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