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BATALIN-VILKOVISKY APPROACH TO THE METRIC
INDEPENDENCE OF TQFT

Mario Abud

Dipartimento di Scienze Fisiche, Università di Napoli
Mostra d'Oltremare Pad 19, 80125 Napoli, Italy

and

Istituto Nazionale di Fisica Nucleare, Sezione di Napoli

Gaetano Fiore

SISSA, Strada Costiera 11, 34014 Trieste, Italy

and

Istituto Nazionale di Fisica Nucleare, Sezione di Trieste

ABSTRACT

We address the problem of metric independence of the partition function of TQFT in the BV formalism. This generalizes results which are valid only when the Faddeev-Popov type gauge-fixing works. Our treatment includes, without difficulties, the interesting cases of theories with open BRST algebras. We study theories of p-forms and discuss the construction of improved functional measures. A number of examples are analysed and proved to lead to metric independent partition functions. The construction of observables is discussed as well.

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1. After Witten's works on TQFT (Topological Quantum Field Theory) [1] some effort has been dedicated to some extensions of these beautiful constructions to include theories where the basic BRST symmetry may be non-nilpotent. Such theories, usually said to have open (or on-shell reducible) BRST algebras, present a number of difficulties related to the fact that generally the gauge-fixing sector of the action cannot be written in the form $\{Q, \text{something}\}$. The technique of quantization which can be profitably used in this kind of theories is the one introduced by Batalin, Vilkovisky [2] and later developed by the authors of [3]. The main obstacle to obtain a construction which reproduces sensible results in a TQFT is that the gauge-fixing procedure inevitably introduces a BRST non-exact metric dependence into the action. Maybe the simplest example of this kind of theories is the nonabelian BF theory [4],[5]. Several authors [6], [7] tried to give convincing proofs that the partition function of the latter is, despite these difficulties, metric independent.

Another important point needed in the proof of metric independence of the partition function is the construction of an improved functional measure. Recently some authors [8],[9] attacked the problem mainly using Fujikawa's prescription [10].

In the present work we demonstrate that a judicious use of the Batalin-Vilkovisky formalism is enough to prove in great generality the metric independence of the partition function of any theory with a metric independent classical action S_{inv} . We realize that the metric dependence of the full gauge-fixed action amounts to a dependence on the gauge-fixing functional ψ . Hence it is not surprising (disregarding the problem of the measure at this stage) that the metric independence relies on the gauge independence of the partition function (a property enjoyed by all honest non-anomalous gauge theories). In fact the use of the gauge independence as formulated in Ref. [3] leads to an elegant proof of the result.

The problem of the construction of improved measures (here specialized to p-form theories, which are natural candidates in providing metric independent S_{inv} 's) is fully treated by defining some suitable functional "volumes". We establish that for all known TQFT the improved measures are metric independent. Unfortunately, even if it seems natural that this fact should be related to the absence of degrees of freedom of the theory, we did not succeed in finding a general proof directly intertwining these two properties.

Finally we discuss the case of the observables of the abelian BF theory (the nonabelian case has recently been discussed along completely different lines in Ref. [7]) and conjecture that some non derivative quadratic polynomials of the B -fields could provide for a complete characterization of the topology of the base manifold.

The paper is organized as follows. In section 2. after a brief presentation of the B.V. formalism we deal with the proof of the metric independence of the partition function without being concerned with the functional measure. In section 3. we construct the

improved measure for theories of p-forms. In section 4 we check the metric independence of the improved measures for Topological Yang-Mills, Chern-Simons, abelian BF and nonabelian BF theories. Here the problem of the observables of BF theories is also addressed. The conclusions are presented in section 5.

2. Following Ref. [2], let $\mathcal{S}_{inv}(\phi)$ be a (local) action depending on the fields $\{\phi_i\}$ and $R_\alpha^i(\phi)$ corresponding (local) gauge generators:

$$\frac{\delta \mathcal{S}_{inv}}{\delta \phi_i} R_\alpha^i = 0 \quad (2.1)$$

We employ a condensed notation such that an unique index represents both discrete indices and the continuous space-time variables x . A repeated index implies summation over discrete indices and integration over x .

In order to quantize the theory the set of fields $\{\phi_i\}$ is enlarged to a set $\{\Phi_A\}$ by the introduction of ghosts, antighosts and Stueckelberg fields with definite Grassman parity $\varepsilon(\Phi_A)$ and ghost number $gh(\Phi_A)$. To each field Φ_A one associates an antifield Φ_A^* with

$$\varepsilon(\Phi_A^*) = \varepsilon(\Phi_A) + 1 \equiv \varepsilon_A + 1, \quad gh(\Phi_A^*) + gh(\Phi_A) = -1 \quad (2.2)$$

One introduces the antibracket $(\ , \)$ for any couple of functionals X, Y of Φ, Φ^* through

$$(X, Y) \equiv \frac{\delta_r X}{\delta \Phi_A} \frac{\delta_l Y}{\delta \Phi_A^*} - \frac{\delta_r X}{\delta \Phi_A^*} \frac{\delta_l Y}{\delta \Phi_A} \quad (2.3)$$

Indices l, r above denote left and right derivatives respectively, which are related in the following way:

$$\frac{\delta_r X(\sigma)}{\delta \sigma_A} = (-1)^{\varepsilon(\sigma_A)(\varepsilon(X)+1)} \frac{\delta_l X(\sigma)}{\delta \sigma_A}, \quad \sigma_A = \Phi_A, \Phi_A^* \quad (2.4)$$

The antibracket obeys the relations below

$$\begin{aligned} \varepsilon[(X, Y)] &= \varepsilon(X) + \varepsilon(Y) + 1 \\ (X, Y) &= -(-1)^{(\varepsilon(X)+1)(\varepsilon(Y)+1)} (Y, X) \\ (XY, Z) &= X(Y, Z) + (-1)^{\varepsilon(Y)(\varepsilon(Z)+1)} (X, Z)Y \\ (X, (Y, Z)) &+ (-1)^{(\varepsilon(X)+1)(\varepsilon(Z)+1)} + \text{cycl. perm. of } (X, Y, Z) = 0 \end{aligned} \quad (2.5)$$

In analogy with the Poisson bracket of the Hamiltonian formalism one can introduce the notion of a canonical transformation by means of a (fermionic) generating functional. It is defined as usual by requiring that the transformation preserves the antibrackets $(\Phi_A, \Phi_B) = 0 = (\Phi_A^*, \Phi_B^*)$, $(\Phi_A, \Phi_B^*) = \delta_{AB}$. The infinitesimal canonical transformation with generating (fermionic) functional $\delta G(\Phi, \Phi^*)$ is defined through

$$\begin{aligned}\Phi_A &\rightarrow \Phi'_A = \Phi_A + \delta\Phi_A, & \delta\Phi_A &\equiv (\Phi_A, \delta G(\Phi, \Phi^*)) = \frac{\delta_l(\delta G)}{\delta\Phi_A^*} \\ \Phi_A^* &\rightarrow \Phi_{A'}^* = \Phi_A^* + \delta\Phi_A^*, & \delta\Phi_A^* &\equiv (\Phi_A^*, \delta G(\Phi, \Phi^*)) = -\frac{\delta_r(\delta G)}{\delta\Phi_A}\end{aligned}\quad (2.6)$$

The first step of the quantization consists of finding the so-called proper solution S of the B.V.'s Master Equation

$$2\frac{\delta_r S}{\delta\Phi_A}\frac{\delta_l S}{\delta\Phi_A^*} = (S, S) = 0 \quad (2.7)$$

obeying the boundary conditions $S(\Phi, \Phi^*)|_{\Phi^*=0} = S_{inv}(\phi)$. Note that canonical transformations (2.6) preserve the Master Equation, as one can easily check by means of properties (2.5). A theorem [11] guarantees the existence of a solution which is, in fact, unique up to canonical transformations (see [3])

The BRST transformation of any functional $A(\Phi, \Phi^*)$ is defined to be the canonical transformation (2.6.b) generated by S :

$$sA \equiv (A, S), \quad s^2 = 0, \quad (2.8)$$

s being nilpotent as a consequence of relations (2.5), (2.7).

Next step is the gauge-fixing of the action. One defines

$$S_\psi(\Phi, \bar{\Phi}^*) \equiv S(\Phi, \bar{\Phi}^* + \frac{\delta\psi}{\delta\bar{\Phi}}) \quad (2.9)$$

where ψ is some fermionic local gauge-fixing functional usually called gauge fermion and has to satisfy some admissibility conditions; the latter guarantee nondegeneracy of $S_\psi(\Phi, 0)$, from which we can build a generating functional $Z(J)$ leading to a sensible perturbation theory.

The *finite* canonical transformation with generating functional $F(\Phi_A, \Phi_a^*)$ can be defined by

$$\begin{aligned}\Phi_A &\rightarrow \Phi'_A, & \Phi'_A &\equiv \frac{\delta_l F}{\delta\Phi_A^*} \\ \Phi_A^* &\rightarrow \Phi_{A'}^*, & \Phi_{A'}^* &\equiv \frac{\delta_r F}{\delta\Phi_A}\end{aligned}$$

As S_ψ can be obtained from S through a finite canonical transformation with generating functional $F = \Phi_A^* \Phi_A - \psi(\Phi)$, it satisfies the M. E.

$$(S_\psi, S_\psi) = 0 \quad (2.10)$$

as well. One can define a new off-shell nilpotent (because of (2.5),(2.10)) BRST transformation $s^{(\psi)}$ as the canonical transformation generated by $S_\psi(\Phi, \bar{\Phi}^*)$

$$s^{(\psi)} A \equiv (A, S_\psi) \quad (2.11)$$

and assign to $\bar{\Phi}^*$ the role of the BRST sources. It is immediate to realize that the BRST transformation introduced by B.V. [2] is obtained from the latter by setting $\bar{\Phi}^* = 0$ after evaluation of the antibracket and, consequently, it is nilpotent only on-shell.

Introduce a generalized generating functional $Z_\psi(J, \bar{\Phi}^*)$ (depending on the external sources J , the BRST sources $\bar{\Phi}^*$ and the choice of ψ) through

$$Z_\psi(J, \bar{\Phi}^*) \equiv \int \mathcal{D}\Phi e^{i(S_\psi(\Phi, \bar{\Phi}^*) + J_A \Phi_A)} \quad (2.12)$$

As proved in Ref. [3] $Z_\psi(J, \bar{\Phi}^*)$ obeys the Ward Identities

$$J_A \frac{\delta_l Z_\psi(J, \bar{\Phi}^*)}{\delta \bar{\Phi}_A^*} = 0 \quad (2.13)$$

Let us recall the proof of this relation. In (2.12) one performs the change of integration variables $\Phi_A \rightarrow \Phi_A + (\Phi_A, S_\psi)\mu$ (μ being an anticommuting parameter). As (disregarding anomalies) the jacobian can be set equal to one (see a similar argument below), direct use of eq. (2.10) and the property $\mu^2 = 0$ imply relation (2.13).

By taking the first variation with respect to J_A and putting $J_A = 0$ one finds:

$$\frac{\delta Z_\psi(0, \bar{\Phi}^*)}{\delta \bar{\Phi}_A^*} = 0. \quad (2.14)$$

To prove the gauge independence of the partition function $Z_\psi(\bar{\Phi}^*) = Z_\psi(0, \bar{\Phi}^*)$ consider $Z_{\psi+\delta\psi}$ defined according to relation (2.12); to first order in $\delta\psi$:

$$\begin{aligned} Z_{\psi+\delta\psi}(\bar{\Phi}^*) &= \int \mathcal{D}\Phi e^{iS_\psi(\Phi, \bar{\Phi}^*)} \left(1 + i \frac{\delta_r \delta_g \psi}{\delta \Phi_A} \frac{\delta_l S_\psi}{\delta \bar{\Phi}_A^*}\right) \\ &= Z_\psi + i \int \mathcal{D}\Phi e^{iS_\psi(\Phi, \bar{\Phi}^*)} (\delta\psi, S_\psi). \end{aligned} \quad (2.15)$$

Perform only the (canonical) change of integration variables $\Phi_A \rightarrow \Phi_A + \delta\Phi_A$ with $\delta\Phi_A = (\Phi_A, S_\psi \delta\psi)$ (whereas the antifields are not to be transformed)⁽¹⁾. The jacobian of the transformation reads

$$J = 1 + \frac{\delta(\delta\Phi_A)}{\delta\Phi_A}(-1)^{\varepsilon_A} = 1 - (\delta\psi, S_\psi) + \left(\frac{\delta_l^2 \delta S_\psi}{\delta\Phi_A \delta\Phi_A^*}(-1)^{(\varepsilon_A)}\right)\delta\psi; \quad (2.16)$$

If the theory is anomaly-free one can find a gauge invariant regularization which eliminates terms proportional to $\delta(0)$ such as the last term of (2.16). The term in $(\delta\psi, S_\psi)$ in the jacobian cancels the variation of the action and one gets

$$Z_{\psi+\delta\psi}(\bar{\Phi}^*) = Z_\psi(\bar{\Phi}^*) \quad (2.17)$$

Let us stress that the term $(\delta\psi, S_\psi)$ appearing in formula (2.15) reduces to the usual BRST transform $\{Q, \delta\psi\}$ for theories with an irreducible closed gauge algebra. The result $\langle (\delta\psi, S_\psi) \rangle = 0$ is the generalization of the well known result $\langle \{Q, \delta\psi\} \rangle = 0$ valid for these theories. In fact one can easily show in an analogous way that

$$\langle (W(\Phi), S_\psi) \rangle = 0 \quad (2.18)$$

for any functional W depending only on the fields.

As mentioned before we shall be concerned with gauge theories (over a Riemannian manifold) where $S_{inv}(\phi)$ is metric independent. The corresponding solution S of the Master Equation (2.7) is determined by S_{inv} and its symmetries, therefore is metric independent as well. A dependence of the partition function on the manifold metric $g_{\mu\nu}$ can arise only from two sources: first, the gauge-fixing procedure (i.e. an explicit $g_{\mu\nu}$ dependence of ψ); second, from the functional measure. Let us postpone to the next section the second point and deal for the moment with a naive measure. Using relation (2.17) we can prove in full generality that Z_ψ is metric independent.

Denote the infinitesimal variation of any functional A with respect to the metric $g_{\mu\nu}$ by $\delta_g A (= \frac{\delta A}{\delta g_{\mu\nu}} \delta g_{\mu\nu})$. To first order in $\delta g_{\mu\nu}$ relations (2.12), (2.17) imply:

$$\delta_g Z_\psi(\bar{\Phi}^*) = \int \mathcal{D}\Phi e^{iS(\Phi, \bar{\Phi}^* + \frac{\delta\psi}{\delta\bar{\Phi}} + \frac{\delta\delta_g\psi}{\delta\bar{\Phi}})} - e^{iS(\Phi, \bar{\Phi}^* + \frac{\delta\psi}{\delta\bar{\Phi}})} = Z_{\psi+\delta_g\psi}(\bar{\Phi}^*) - Z_\psi(\bar{\Phi}^*) = 0 \quad (2.19)$$

From the above argument we see that a variation of the manifold metric can be compensated by a change of the gauge conditions and therefore leaves the partition function unchanged. As we can rewrite relation (2.19) in the form:

$$\delta_g Z_\psi = \frac{1}{i} Z_\psi \int d^n x |g|^{\frac{1}{2}} \langle T_{\mu\nu}(x) \rangle \delta g^{\mu\nu}, \quad |g| = \det(g_{\mu\nu}), \quad (2.20)$$

$T_{\mu\nu}$ being the energy-momentum tensor, this implies the vanishing of the v.e.v. of $T_{\mu\nu}$:

$$\langle T_{\mu\nu}(x) \rangle = \langle |g|^{\frac{-1}{2}} \left(\frac{\delta\psi}{\delta g^{\mu\nu}(x)}, S_\psi \right) \rangle = 0 \quad (2.21)$$

Notice that, according to the relation (2.11) $T_{\mu\nu}$ is in fact the BRST variation of $|g|^{\frac{-1}{2}} \frac{\delta\psi}{\delta g^{\mu\nu}}$. For theories where the Faddeev-Popov construction works this is analogous to Witten's original argument involving the property $T_{\mu\nu} = \{Q, \text{something}\}$. Nevertheless the result (2.21) is much more general in that it holds also for theories where the gauge-fixing term cannot be written as $\{Q, \dots\}$, e. g. theories with open gauge algebras (the BF theories we are going to discuss in sec. 4 are of this kind).

The only ingredient used here is metric-independence of the classical action $\mathcal{S}_{inv}(\phi)$ (the most natural realizations being theories of p-forms)⁽²⁾. This great generality is an advantage of B.V. approach.

3. Let us come now to the problem of the metric dependence of the functional measure. Some time ago Fujikawa [10] considered the problem of defining a partition function in a coordinate-independent way. More explicitly, suppose you have a theory with classical action $\mathcal{S}_{inv}$ defined on a Riemannian spacetime manifold, the action being diffeomorphism invariant (i.e. independent of the choice of the coordinates); Fujikawa showed that a definite metric dependence has to be assigned to the functional measure in order that the latter (and hence the partition function itself) be diffeomorphism-invariant. Even though it would seem that the use of a naive functional measure (which differs from Fujikawa's one only by a metric dependent overall factor) could work to evaluate the Green's functions $\langle \phi_1(x_1), \dots, \phi_n(x_n) \rangle$ of the theory, in fact it would lead to a number of inconsistencies in the definition of the energy momentum tensor and in the regularization procedure. Only by using the improved measure the definition $\langle T_{\mu\nu}(x) \rangle \equiv \frac{1}{i} |g|^{\frac{-1}{2}} \frac{\delta \ln(Z)}{\delta g^{\mu\nu}}$ of the v.e.v. of the energy momentum tensor is regularization independent and yields correct transformation properties to $\langle T_{\mu\nu}(x) \rangle$ under diffeomorphisms. Moreover, only in this way a number of local Ward Identities related to the conformal and diffeomorphism invariances of the theory acquire a well defined meaning (regularization independence and finiteness up to wave-function renormalization) at least for the explicit results obtained in the flat space-time limit, and a good high energy behaviour is found under the same circumstance.

Here we are going to approach the problem of the metric dependence of the functional measure from a rather different viewpoint. For definiteness we shall specialize to theories

of p-forms (i.e. antisymmetric tensor fields) which naturally lead to a metric independent $\mathcal{S}_{inv}(\phi)$. For these theories, apart from the important physical issues discussed above, we are compelled to introduce natural Gaussian measures through the requirement that the total (regularized) functional volume (denoted by Ω in the following) be metric independent. It is pleasant that this prescription leads in fact to Fujikawa's results as can be explicitly checked for the known examples.

Consider a compactified n-dimensional Riemannian manifold M_n . The exterior algebra Λ^p of differential forms of degree p on M_n is naturally endowed with the scalar product

$$(\alpha_p, \beta_p) \equiv \int_{M_n} \alpha_p \wedge^* \beta_p = p! \int d^n x |g|^{\frac{1}{2}} (\alpha_p)_{\mu_1 \dots \mu_p} (\beta_p)^{\mu_1 \dots \mu_p}, \quad \alpha_p, \beta_p \in \Lambda^p \quad (3.1)$$

where the explicit integration over coordinates x in the RHS is of course to be understood chart by chart on M_n , and the Hodge operator $*$ in curved space in terms of components reads

$$(*\alpha_p)_{\mu_{p+1} \dots \mu_n} = \frac{|g|^{\frac{1}{2}}}{(n-p)!} \varepsilon_{\mu_1 \dots \mu_n} g^{\mu_1 \nu_1} \dots g^{\mu_p \nu_p} (\alpha_p)_{\nu_1 \dots \nu_p}. \quad (3.2)$$

In the sequel the symbol for the wedge product $\wedge$ will be dropped.

Define the Gaussian functional measure $\mathcal{D}\mu_{\omega_p}$ (corresponding to the p-form ω_p) to be an object such that

$$\int \mathcal{D}\mu_{\omega_p} e^{-\frac{1}{2} c(\omega_p, \omega_p)} = \Omega(c) \quad , \quad (3.3)$$

independently of the choice of the metric implicit in the scalar product $(\ , \)$ (c is some positive constant). As the theory is diffeomorphism-invariant we can choose local chartes of isothermal coordinates such that the metric can be expressed as

$$g_{\mu\nu}(x) = h_{\mu}(x) \delta_{\mu\nu} \quad (3.4)$$

and accordingly

$$\Omega(c) = \int \mathcal{D}\mu_{\omega_p} \exp \left[-\frac{1}{2} c \int d^n x \sum_{\{\mu_i\}} \frac{|g|^{\frac{1}{2}}(x)}{\prod_{i=1}^p h_{\mu_i}(x)} \omega_{\mu_1 \dots \mu_p}(x) \omega_{\mu_1 \dots \mu_p}(x) \right]. \quad (3.5)$$

Let

$$\bar{\omega}_{\mu_1 \dots \mu_p} \equiv \left[\frac{|g|^{\frac{1}{2}}}{\prod_{i=1}^p h_{\mu_i}} \right]^{\frac{1}{2}} \omega_{\mu_1 \dots \mu_p}; \quad (3.6)$$

the following formal definition of $\mathcal{D}\mu_{\omega_p}$ satisfies eq. (3.3):

$$\mathcal{D}\mu_{\omega_p} = \prod_x \prod_{\{\mu_i\}} d\bar{\omega}_{\mu_1 \dots \mu_p}(x). \quad (3.7)$$

We infer the simple rule: 1) a factor $|g|^{\frac{1}{4}}$ for each component $\omega_{\mu_1 \dots \mu_p}$; 2) a factor $h_{\mu_i}^{-\frac{1}{2}}$ for each index μ_i . As the number of components of ω_p is $\binom{n}{p}$ and each index appears in the measure $\binom{n-1}{p-1}$ times we find an overall factor $|g|^{\frac{1}{4} \binom{n}{p}} \prod_{\mu=1}^n h_{\mu}^{-\frac{1}{2} \binom{n-1}{p-1}}$. But $\prod_{\mu=1}^n h_{\mu} = |g|$, so we get

$$\mathcal{D}\mu_{\omega_p} = \prod_x \left(|g|^{\frac{1}{2}}(x) \right)^{\frac{1}{2} \binom{n}{p} - \binom{n-1}{p-1}} \prod_{\{\mu_i\}} d\omega_{\mu_1 \dots \mu_p}(x) \quad (3.8)$$

It is easy to recognize that the functional measure (3.8) is the proper one to evaluate the functional determinant of any positive definite linear operator A which is defined over Λ^p and is self-adjoint w.r.t. the scalar product $(\ , \)$ by the gaussian integral

$$\int \mathcal{D}\mu_{\omega_p} e^{-\frac{1}{2}(\omega_p, A \omega_p)} = (Det A)^{-\frac{1}{2}} \quad (3.9)$$

So far we have assumed that the components of ω_p have even Grassman parity. In the case of p-forms with odd Grassman parity $\chi_p, \bar{\chi}_p$ an analogous construction leads to the measure

$$\mathcal{D}\mu_{\chi_p \bar{\chi}_p} = \prod_x \left(|g|^{\frac{1}{2}}(x) \right)^{2 \binom{n-1}{p-1} - \binom{n}{p}} d\chi_{\mu_1 \dots \mu_p}(x) d\bar{\chi}_{\mu_1 \dots \mu_p}(x) \quad (3.10)$$

One can also check that the functional measures (3.8), (3.10) are diffeomorphism invariant, namely they coincide with the ones which have been constructed [9] using Fujikawa's prescription [10]. So the measure $\mathcal{D}\mu_{\alpha_p}$ to be used in the definition of the partition function differs from the naive one by the metric dependent factor

$$\prod_x \left(|g|^{\frac{1}{2}}(x) \right)^{\eta(\alpha_p) [\frac{1}{2} \binom{n}{p} - \binom{n-1}{p-1}]} \quad (3.11)$$

for each p-form field α_p , where $\eta(\alpha_p)$ is the Grassman parity of its components.

4. Now it is a trivial matter to prove the metric independence of the partition function for a number of theories based on differential forms, namely topological Yang-Mills, Chern-Simons and BF theories. In the following an upper index will denote the ghost number of a form, a lower one its form degree. Since the classical actions S_{inv} of these theories are explicitly metric independent the considerations of the section 2. apply and the only potential source of metric dependence is the functional measure. Let us explicitly check that the metric dependence of the measure is in fact trivial.

For topological Yang-Mills in 4 dimensions $S_{inv} = \int_{M_4} tr(F F)$ the field content of the gauge-fixed theory [12] is

$$A_1^0 \ \psi_1^1 \ \bar{\chi}_2^{-1} \ b_2^0 \ c_0^1 \ \phi_0^2 \ \bar{c}_0^{-1} \ b_0^0 \ \bar{\phi}_0^{-2} \ \eta_0^{-1}; \quad (4.1)$$

as each even field is followed by an odd companion with the same form degree, the overall contribution is trivial and the improved measure is in fact metric independent.

For the Chern-Simons $\mathcal{S}_{inv} = \int_{M_3} \text{tr}(AF - \frac{1}{3}A^3)$ the BRST transformations (2.8),(2.11) coincide with the usual one

$$sA = Dc \quad sc = -c^2 \quad s\bar{c} = \lambda \quad s\lambda = 0 \quad (4.2)$$

(the field content is the same as in Yang-Mills theories: $c, \bar{c}$ are the usual Faddeev-Popov ghosts and λ is the Stueckelberg auxiliary field); the contributions of all fields to $|g|^{\frac{1}{2}w}$ are

$$w_A = \frac{1}{2}, \quad w_c = w_{\bar{c}} = -\frac{1}{2}, \quad w_\lambda = \frac{1}{2}$$

so $w = 0$ and again the measure is metric independent.

Consider now the abelian BF system with classical action

$$\mathcal{S}_{inv} = \int_{M_n} B_{n-2} dA, \quad n > 1 \quad (4.3)$$

with B_{n-2} a $(n-2)$ -form and A a 1-form on the n -dim. manifold M_n . The theory is $(n-3)$ -stage reducible and one has to introduce ghosts of ghosts (for $n > 3$) in order to perform its quantization. This can be done in the B.V. formalism [5], or in other equivalent ways [13], [14], [5]. As it is well known, the set of fields needed in the quantization can be arranged in two pyramids as shown in Fig. 1.

The fields of the gauge-fixing sector are the antighosts $\bar{C}'_q, \bar{\omega}_0^{-1}$ and the Stueckelberg fields Π'_q, π_0^0 . To each $\bar{C}'_q$ there corresponds Π_q^{p+1} (along the same vertical, see Fig. 1), which has opposite Grassman parity, so that their total contribution to w is zero; the same occurs with $\bar{\omega}_0^{-1}, \pi_0^0$. This is a general fact: as to each antighost there corresponds a Stueckelberg field (its BRST transform) the gauge-fixing sector cannot contribute to w . The contribution to w of the remaining part of the fields (the so-called "geometrical sector", or "principal branch", in the language of [2]) $\{\omega_0^1, A_1^0, B_{n-2-k}^k\}$ ($k = 0, 1, \dots, n-2$) can be explicitly calculated by the rule (3.11). We get

$$w = -\frac{1}{2}\binom{n}{0} + \frac{1}{2}\binom{n}{1} - \binom{n-1}{0} + \sum_{k=2}^n (-1)^k \left[\frac{1}{2}\binom{n}{n-k} - \binom{n-1}{n-k-1} \right] = 0. \quad (4.4)$$

The first three terms arise from ω_0^1, A_1^0 , the sum from the $B_{n-k}^{k-2}'s$, and it is easy to see that they cancel each-other identically. This result can be checked also by noting that the ω_0^1, A_1^0 contributions to w are the same as those of some new forms B_n^{-2}, B_{n-1}^{-1} respectively,

so the total contribution to w is that of a complete sequence of p-forms $B_p(p = 0, 1, \dots, n)$ with alternating Grassman parity;

$$w = \sum_{k=0}^n (-1)^k \left[\frac{1}{2} \binom{n}{n-k} \right] - \sum_{k=0}^n (-1)^k \left[\binom{n-1}{n-k-1} \right], \quad (4.5)$$

each sum vanishing by virtue of Leibniz formula.

The obtention of the quantum action is straightforward. For definiteness, let us specialize to 4 dimensions and take gauge conditions of Lorentz type:

$$\psi \equiv \int \text{tr} [\bar{C}_1^{-1} d^* B_2^0 + \bar{C}_0^{-2} d^* B_1^1 + \bar{C}_0^0 d^* \bar{C}_1^{-1} + \omega_0^{-1} d^* A] \quad (4.6)$$

It is convenient to introduce a new set $\{\chi\}$ of independent sources of the BRST transforms, in such a way that the source sector of the action is expressed as wedge products of forms (without any Hodge operation) and is therefore explicitly metric independent⁽³⁾. The sources $(\chi_2^{-1}, \chi_3^{-2}, \chi_3^{-1}, \chi_3^0, \chi_4^1, \chi_4^0, \chi_4^{-1})$ correspond to the fields $(B_2^0, B_1^1, A, \bar{C}_1^{-1}, \bar{C}_0^{-2}, \bar{\omega}_0^{-1}, \bar{C}_0^0)$ respectively.

The corresponding quantum action (eq. (2.9)) is given by

$$\begin{aligned} S_\psi = \int [& B_2^0 dA - (*d\bar{C}_1^{-1})dB_1^1 + (*d\bar{C}_0^{-2})dB_0^2 - (*d\bar{\omega}_0^{-1})d\omega_0^1 + (d^*B_2^0 + \\ & *d\bar{C}_0^0)\Pi_1^0 + (d^*B_1^1)\Pi_0^{-1} + (d^*\bar{C}_1^{-1})\Pi_0^1 + (d^*A)\pi_0^0 + \chi_2^{-1}dB_1^1 + \\ & \chi_3^{-2}dB_0^2 + \chi_3^{-1}d\omega_0^1 + \chi_3^0\Pi_1^0 + \chi_4^1\Pi_0^{-1} + \chi_4^0\pi_0^0 + \chi_4^{-1}\Pi_0^1] \end{aligned} \quad (4.7)$$

The BRST transformations are readily obtained from eqs. (2.11) and are given by (note that in this case $s = s^{(\psi)}$):

$$\begin{aligned} s^{(\psi)} B_2^0 &= dB_1^1 & s^{(\psi)} B_1^1 &= dB_0^2 & s^{(\psi)} B_0^2 &= 0 \\ s^{(\psi)} A &= d\omega_0^1 & s^{(\psi)} \omega_0^1 &= 0 \\ s^{(\psi)} \bar{C} &= \Pi & s^{(\psi)} \Pi &= 0, & \bar{C} &= (\bar{C}_0^{-2}, \bar{C}_1^{-1}, \bar{C}_0^0, \bar{\omega}_0^{-1}), & \Pi &= (\Pi_0^{-1}, \Pi_1^0, \Pi_0^1, \pi_0^0). \end{aligned} \quad (4.8)$$

Upon integration over the Π 's, $\bar{C}_0^{-2}$, $\bar{C}_1^{-1}$ and A the partition function can be put into a form which will prove to be useful for the evaluation of some v.e.v.'s:

$$Z(\chi = 0) = \int \mathcal{D}B_0^2 \mathcal{D}B_1^1 \mathcal{D}B_2^0 \mathcal{D}\bar{C}_0^0 \mathcal{D}\pi_0^0 \mathcal{D}\bar{\omega}_0^{-1} \mathcal{D}\omega_0^1 e^{-S_{eff}(B, \omega, \bar{\omega})}$$

$$\delta(dB_2^0) \delta(*d\pi_0^0) \delta(d^*dB_1^1) \delta(d^*dB_2^0) \delta(d^*B_2^0) \delta(*d\bar{C}_0^0) \delta(d^*B_1^1) (det' \Delta_0)^{\frac{1}{2}} \quad (4.9)$$

where S_{eff} is the effective action depending only on the fields $B's, \omega, \bar{\omega}$, and $det' \Delta_0$ denotes the determinant of the laplacian Δ_0 defined on the 0-forms.

It is now easy to construct a set of quantities Ω_k with metric independent v.e.v. $\langle \Omega_k \rangle$, usually called observables. These $\langle \Omega_k \rangle$'s constitute a set of nontrivial correlation functions (indeed we shall prove that they involve only the zero modes of the relevant fields).

Paralleling Witten's construction of T.Y.M. we define Ω_k by $\Omega_k \equiv \int_{\gamma_k} \Delta_k$ where γ_k is a k-homology cycle of the manifold M and Δ_k are some polynomials. They are given by

$$\Delta_0 \equiv (B_0^2)^2; \quad \Delta_1 \equiv 2B_0^2 B_1^1; \quad \Delta_2 \equiv 2B_0^2 B_2^0 + (B_1^1)^2; \quad \Delta_3 \equiv 2B_1^1 B_2^0; \quad \Delta_4 \equiv (B_2^0)^2 \quad (4.10)$$

and obey:

$$\begin{aligned} s^{(\psi)} \Delta_4 - d\Delta_3 &= -2B_1^1 dB_2^0; & s^{(\psi)} \Delta_3 - d\Delta_2 &= -2B_0^2 dB_2^0; \\ s^{(\psi)} \Delta_2 - d\Delta_1 &= 0; & s^{(\psi)} \Delta_1 - d\Delta_0 &= 0; & s^{(\psi)} \Delta_0 &= 0 \end{aligned} \quad (4.11)$$

Inserting the above expressions into the path integral and performing an integration over A one gets $\delta(dB_2) \delta(*d\pi_0^0)$; therefore

$$\langle s^{(\psi)} \Delta_k - d\Delta_{k-1} \rangle = 0 \quad (4.12)$$

(since dB_2^0 is set equal to zero by the δ -functions shown in eq. (4.9)), i.e. Δ_k are " weak cocycles " in the sense that the descent equations are satisfied after taking the v.e.v. So defining $\Omega_k = \int_{\gamma_k} \Delta_k$ one finds

$$\delta_g \langle \Omega_k \rangle = \langle \int_{\gamma_k} \Delta_k (\delta_g \psi, S_\psi) \rangle = \langle \int_{\gamma_k} (\Delta_k \delta_g \psi, S_\psi) - \delta_g \psi (\Delta_k, S_\psi) \rangle; \quad (4.13)$$

the first term vanishes by eq. (2.18) (notice that Δ_k does not depend on the antifields). The second term can be easily shown to be proportional to $\langle \delta_g \psi \int_{\gamma_k} d\Delta_{k-1} \rangle$. For $k=0,1,2$ this follows trivially from eqs. (4.11). In the case of $k=3,4$ one has an additional term of the form $\langle \delta_g \psi \int_{\gamma_{2+j}} B_j^{2-j} dB_2^0 \rangle$ ($j=0,1$ respectively). These terms in fact vanish by the following argument. As $\delta_g \psi$ is linear in A decompose it into the sum of two terms, $\delta_g \psi_0$ independent of A and $\delta_g \psi_1$ which depends on A . Integration over A in the piece involving

$\delta_g \psi_0$ produces a $\delta(dB_2^0)$ (see eq. (4.9)), so its contribution is zero; the term containing $\delta_g \psi_1$ can be shown to be zero by rewriting it in the form $-\langle \delta_g \psi_1 \int_{\gamma_2+j} (dB_j^{2-j}) B_2^0 \rangle$ and integrating over $\bar{C}_1^{-1}, \bar{C}_0^{-2}$ (these integrations yield $\delta(dB_j^{2-j})$ for $j=0,1$ respectively). Summing up:

$$\delta_g \langle \Omega_k \rangle = 0 \quad (4.14)$$

implying the metric independence of the $\langle \Omega_k \rangle$'s. An explicit evaluation of the observables using the expression (4.9) for the partition function gives

$$\begin{aligned} \langle \Omega_0 \rangle &= \langle \int_{\gamma_0} h_0 h_0 \rangle, \quad \langle \Omega_1 \rangle = 2 \langle \int_{\gamma_1} h_0 h_1 \rangle, \\ \langle \Omega_2 \rangle &= \langle \int_{\gamma_2} 2h_0 h_2 + (h_1)^2 \rangle, \quad \langle \Omega_3 \rangle = 2 \langle \int_{\gamma_3} h_1 h_2 \rangle, \quad \langle \Omega_4 \rangle = \langle \int_{\gamma_4} h_2 h_2 \rangle \end{aligned} \quad (4.15)$$

where h_p 's are the harmonic components of the corresponding fields B_p^{2-p} .

In fact the abelian BF model, in the absence of zero modes, is known to represent the Ray-Singer torsion $T(M)$ of M . Nevertheless it seems that the abelian BF theory is related to the cohomology of M . It should be interesting to perform a full gauge-fixing of the zero modes and determine to what extent the observables specify the topology of M .

Let us discuss now the non abelian BF theory (for details see Refs. [15], [16]). It is constructed out of Lie algebra-valued forms B_{n-2}^0 and a Yang-Mills connection A . The invariant action is given by

$$S_{inv} = \int_{M_n} \text{tr}(B_{n-2}^0 F) \quad (4.16)$$

where F is the usual Yang-Mills curvature. The set of fields needed in the quantization is the same as in the previous abelian case, implying that $w = 0$ as well. The proof of metric independence of the partition function of all previously mentioned theories is so completed. A discussion of the observables of this theory will be given elsewhere.

Conclusions

5. We furnished quite a general proof that the partition function of a TQFT (of either Witten or Schwarz type, see Ref. [17]) is metric independent. Our proof includes theories for which the basic BRST algebra is open or reducible, since it relies on the general Batalin-Vilkovisky formalism. The independence of the functional measure on the metric has been fully discussed and displayed for a number of examples including T.Y.M.

and Chern-Simons Theory (closed irreducible BRST algebra), abelian BF theory (closed reducible BRST algebra), nonabelian BF theory (open, reducible BRST algebra). The main ingredient is just the metric independence of the classical action S_{inv} . A final issue is the construction of observables for the abelian BF theory, which are proved to be related to bilinear functionals of harmonic forms (zero modes). An analogous result for the non abelian case is possible and omitted here for the sake of brevity. It remains to perform a suitable BRST gauge-fixing of the zero modes and establish to what extent this set of observables fully specify the topology of the base manifold M . An analogous construction for the non abelian case is also desirable.

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Notes

- (1) Maybe it is worth stressing that this change of variables is not a BRST transformation as the one used to prove the Ward Identities (2.13), since the anticommuting parameter μ has been replaced by the functional $\delta\psi(\Phi)$; moreover, both depend on $\bar{\Phi}^*$ and the gauge-fixing ψ .
- (2) The present argument applies trivially to theories where $S_{inv}(\phi)$ is metric dependent but of the form $\{Q, \Sigma\}$, since the v.e.v. of a BRST-exact object is zero.
- (3) They are obviously related to $\bar{\Phi}^*$ by Hodge duality.

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