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SUBTRACTED GENERALIZED POLYLOGARITHMS

AND THE SINAC PROGRAM***)

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1. THE SUBTRACTED GENERALIZED POLYLOGARITHMS

In order to have a simple example of the definition of the subtracted generalized polylogarithms or subtracted Nielsen functions [1,2], we start with the expression of the Nielsen functions through their series expansion for real x , $|x| < 1$

$$S_{np}(x) = \sum_{s=0}^{\infty} \frac{(-1)^s S_{p+s}^{(p)}}{(p+s)!} \frac{x^{p+s}}{(p+s)^n}$$

where the $S_k^{(p)}$ are the Stirling numbers of the first kind, defined by

$$\log^j(1+x) = j! \sum_{k=j}^{\infty} S_k^{(j)} \frac{x^k}{k!}; \quad |x| < 1.$$

Then the m -subtracted $S_{1p}(x)$ is defined as

$$S_{1p}^{mm}(z) = \frac{1}{z^m} \left[\sum_{s=0}^{\infty} \frac{(-1)^s S_{p+s}^{(p)}}{(p+s)! (p+s)} z^{p+s} - \psi_{1p}^m(z) \right],$$

where

$$\psi_{1p}^m(z) = \sum_{i=0}^{m-p-1} \frac{(-1)^i S_{p+i}^{(p)} z^{p+i}}{(p+i)! (p+i)}.$$

It is easy to see that the first term of the S_{1p}^{mm} expansion is

$$\frac{1}{z^m} \left[\frac{(-1)^{m-p} S_m^{(p)} z^m}{m! m} \right] = (-1)^{m-p} \frac{S_m^{(p)}}{m! m},$$

so that

$$S_{1p}^{mm}(z) = \sum_{s=0}^{\infty} \frac{(-1)^{m-p+s} S_{m+s}^{(p)} z^s}{(m+s)! (m+s)}.$$

It seems at first sight that there is no *a priori* reason to duplicate the index m in the symbol of S_{1p}^{mm} . If we did so, this simply because the case investigated above is a degenerate case of S_{np}^{mk} where $k = m$ and $n = 1$. We define the general subtracted Nielsen function by

$$S_{np}^{mk}(z) = \sum_{s=0}^{\infty} \frac{(-1)^{m-p+s} S_{m+s}^{(p)} z^s}{(m+s)! (k+s)^n}.$$

This generalization is necessary because we need the integral

$$\int_0^1 u^{k-1} S_{1p}^{mm}(uz) du = -\frac{1}{m-k} \left[S_{1p}^{mm}(z) - S_{1p}^{mk}(z) \right].$$

Now, successive integrations can be performed and no other functions than $S_{np}^{mk}(z)$ appear. For instance, in performing

$$\int_0^1 y^{\ell-1} \int_0^1 u^{k-1} S_{1p}^{mm}(yuz) du dy,$$

the only functions appearing are $S_{1p}^{mm}(z)$, $S_{1p}^{mk}(z)$, $S_{1p}^{m\ell}(z)$.

In the general case, one has to deal with $1/[(k+s)(m+s)^n]$ on the first integration, which is still expressible in terms of S_{np}^{mm} and S_{np}^{mk} .

So far we have given the definition of S_{np}^{mk} in terms of a power series in the variable z . It is, however, possible to generalize the procedure for the analytic continuation given in [2] for $m = k = 0$, and find

$$\frac{1}{z} S_{np}^{mk}(z) = \frac{1}{\pi} \int_1^\infty \frac{\text{Im } S_{np}^{mk}(\zeta)}{\zeta(\zeta - z)} d\zeta.$$

The calculation of $\text{Im } S_{np}^{mk}$ is lengthy. For $m = k = 0$, it is given in [2].

We now have at our disposal all the tools necessary for the analytic computation of integrals of any repeated Feynman parameters $u, y, z, \dots$ sequence, through the four indices S_{np}^{mk} functions.

As practical examples we shall restrict ourselves to the simplified case $p = 1$ for which the operations of the S_{n1}^{mm} functions have been written down and are sufficient to complete by computer the integrals appearing in fourth order radiative corrections in Q.E.D., given in terms of Feynman parameters. It is important to notice at this point that for higher orders, the program has not been completed, nor the tables completed by the time of this writing. From the way it operates in the fourth order case and from the examples of the recently computed crossed slope [3] it is sufficient to demonstrate the power of such a program fundamentally based on the use of the subtracted Nielsen functions.

2. CASE $p = 1$

Since $S_s^{(1)} = (-1)^{s-1} (s-1)!$ We have the following definitions

$$S_n^{(1)}(x) = \frac{1}{x^n} [\log(1-x) + \Psi_{n-1}(x)], \quad \Psi_n(x) = \sum_{i=1}^n \frac{x^i}{i}.$$

These are called the "sublogs", or subtracted logarithms [in fact $S_{01}^{nn}(x)$].

In the same way

$$Sd_n(x) = \frac{1}{x^n} [Li_2(x) - \psi_{n-1}^{(2)}(x)]$$

$$\psi_n^{(2)}(x) = \sum_{i=1}^n \frac{x^i}{i^2}$$

called "subdilogs" [in the notation of the preceding section $S_{11}^{nn}(x)$]. We used the notation $Li_q(x) = S_{q-1,1}^{00}(x)$. One can go on for "subtrilog" and more generally "subpolylogs". According to the index of the polylogarithm v and the degree of subtraction n , these functions can be related to the general definition

$$\phi(z, n, v) = \sum_{k=0}^{\infty} \frac{z^k}{(k+n)^v} \quad *)$$

This way we find, according to the general analytic continuation of the S functions the following representation in the complex z plane

$$\phi(z, n, v) = \frac{1}{\Gamma(v)} \int_1^{\infty} \frac{dt}{t^n} \frac{(\log t)^{v-1}}{t-z}.$$

For instance, it is seen, as an example, that

$$\phi(z, 1, 2) = \frac{1}{z} Li_2(z).$$

Then, using the above representation,

$$\int_0^1 dz \phi(z, 1, 2) = - \int_0^1 du \log u \int_0^1 \frac{dz}{1-uz} = \int_0^1 \frac{du}{u} \log u \log(1-u) = \zeta(3) = Li_3(1).$$

which corresponds to the usual definition. This representation will be used in the following for the special case when the argument is $z \rightarrow (2z-1)/z$, which is the only non-trivial case occurring (see Appendix I).

We conclude this short section by mentioning that the function $U_n(a, w) = w^n/(1-wa)$ has the property

$$\int_0^1 dw U_n(wa) = \int_0^1 dw \frac{w^n}{1-wa} = -S\ell_{n+1}(a).$$

*) This function is often called Lerch's transcendent.

3. THE SINAC PROGRAM

The SINAC program (Standard Integrals Analytic Calculation) fundamentally uses the CDC 6600 program SCHOONSHIP elaborated by Veltman [4]. This program turns out to be very suitable for the computer analytic calculation of parametric integrals in Q.E.D.

The standard integrals obtained by the classical use of ordinary Feynman parameters have been cast in a standard form which allows the classification of integrals in terms of five indices [5]. These integrals, in fourth order, are four-dimensional integrals and the integration variables are called here y, x, w and z. These integrals have the following general structure

$$I(k,m,n,r,s) = \int_0^1 dy \int_0^1 dx \int_0^1 dw \int_0^1 dz \frac{z^r w^s x^{2k-3+n-m} y^{k-2+n} \Delta^m}{\beta^n \Gamma^k}$$

with $\beta = 1 - xy(1 - wz + w^2 z^2)$; $\Delta = 1 - x[1 - zw(1 - w)]$; $\Gamma = w^2 x \beta + y \Delta^2$. The first three indices define classes of two-dimensional integrals characterized by three indices which, in the following, we shall call

$$SI(k,m,n) = \int_0^1 dy \int_0^1 dx \frac{x^{2k-3+n-m} y^{k-2+n} \Delta^m}{\beta^n \Gamma^k},$$

so that the integrals $I(k,m,n,r,s)$ are given by

$$I(k,m,n,r,s) = \int_0^1 dw \int_0^1 dz z^r w^s SI(k,m,n).$$

In $SI(k,m,n)$, the integrations are done in the order y first, then x. They are trivial because of the fact that, once the integration over y is performed and the limits inserted, the quadratic denominator in x factorizes (discriminant = perfect square). These two integrals can be done easily by computer using suitable derivatives with respect to A, B, and C of the generating function

$$J_0 = \int_0^1 \frac{dy}{(A + By)(1 - Cy)} = \frac{1}{AC + B} \log \frac{A + B}{A(1 - C)},$$

A + By representing Γ , and $1 - Cy$ representing β , $AC + B = \Delta^2$. For $n > 0$, the y integration is therefore performed very easily by the program. The case when $n \leq 0$, though more involved, can be treated by choosing the generating function

$$J_1 = \int_0^1 \frac{dy}{(A + By)(1 - Cy)^n}.$$

This generalization is not included in the program, but an algorithm for it is given in Appendix IIa. Appendix IIb generalizes the x integration trivially. If J is the so obtained value of the integral over y , then

$$SI = \int_0^1 dx x^{2k-3+n-m} \Delta^m J .$$

The following notations are introduced:

$$C = 2(k + n) - m - 3$$

$$M = \log\left(1 - w_1 \frac{1 - 2z}{1 - z}\right) - \log(1 - w_1) - \log\left(1 - w_1 \frac{-z}{1 - z}\right)$$

$$w_1 = 1 - w$$

$$F = w^2$$

$$G_1 = \frac{w}{1 - w} \frac{1 - z}{z}$$

$$H_1 = \frac{w}{1 - w}$$

$$E_1 = \frac{w}{1 - w} (1 - z)$$

$$1 - D = w(1 - w)z .$$

The appearance of the $S\ell_n$ functions makes the whole expression to have regularity properties which are not evident. Although, by expanding, divergences appear at $z = 1$, $z = 1/2$, $z = 0$, $w = 0$, $w = 1$, the formulation of the integrand in terms of the closed $S\ell_n$ functions guarantees its finiteness when all terms of $SI(k,m,n)$ are taken into account. This can be proved rigorously for all integrals, but a few exceptions. To do this, the $\log M$ term must be put in the $S\ell$ form according for instance to

$$\frac{1}{w_1^2} \log(1 - w_1 a) = \frac{a^2}{a^2 w_1^2} [\log(1 - w_1 a) + w_1 a - w_1 a] = a^2 S\ell_2(w_1 a) - \frac{a}{w_1} .$$

Then the w_1^{-n} terms, $n > 0$, which appear in doing so, cancel exactly the infinities coming from the $S\ell_n(G_1)$, at $w = 1$. A similar analysis can be done once the w integration is performed for the singularities at $z = 1$, $1/2$, 0 , with the following exceptions for $k \leq 3$.

1) $I(2,0,0,1,2)$, $I(3,0,0,1,2)$, ... show a singularity at $w = 0$ coming from a term w^{-1} , which is not cancelled. This is nothing, but the well-known infra-red divergence. A simple limiting procedure introducing a fictive photon mass λ and then extracting the pathological $\log \lambda$ term, which is always multiplied by a

rational, has been given by Weneser, Bersohn and Kroll [6]. It turns out to be very suitable for computation. We do not discuss it here in more detail. As a special paper will be devoted to a detailed study of this problem will be treated separately.

2) When $C > 4$ the integrand shows a divergence for $w = 1$ or $z = 1$. However, in all physical applications, these divergences are spurious, as the combination in which these integrals appear, permits the factorization of $1 - zw$ or $1 - w$ or $1 - z$ factors or products thereof that always make the integrals finite at $w = 1$, or $z = 1$. In the case we have under investigation ($k = 1, 2, 3$) for example, an integral such as $SI(1, 0, 3)z^r w^s$ will always appear together with $-SI(1, 0, 3)z^{r+1} w^{s+1}$ and $(1 - wz)z^r w^s SI(1, 0, 3)$. Taken together they yield finite results. Therefore, all integrals of the same type will never be computed alone, but with the appropriate factor in front which will be determined by the physical problem under investigation.

With these prescriptions and with the exceptions mentioned, all integrals are shown to be finite globally. However, when expanding and integrating the various terms of the SI integrands, infinite spurious terms will show up which finally cancel. They are of the type

$$\int_0^1 w_1^{-n} \log(1 - w) dw, \int_0^1 w_1^{-n} dw, \int_0^1 z^{-n} \log z dz, \int_0^1 (1 - z)^n \log(1 - z) dz$$

$$\int_0^1 z^{-n} dz, \int_0^1 (1 - z)^{-n} dz, \int_0^1 (2z - 1)^{-n} dz, \dots, \int_0^1 w^{-n} dw.$$

Therefore a regularization procedure is permitted, which sets these integrals equal to zero, but only when all finite terms have been extracted and their analytical values substituted.

The program of integration is now straightforward.

First F, G_1, H_1, E_1, D in terms of w and z are substituted in the general formula for $SI(k, m, n)$ (but not in M).

Second, the $S\ell_n$ are expanded, and the $\log(-G_1) \dots$ written as

$$\log \left(1 + \frac{w}{1 - w} \frac{1 - z}{z} \right) = \log \left(1 - w \frac{2z - 1}{z} \right) - \log(1 - w) \text{ etc.}$$

Then a `RATIO` command expands the expression in terms of w and w_1 powers $[(w^2/w_1^3) = (1 - 2w_1 + w_1^2)/w_1^3 \text{ as example}]$.

At this point, we remark that the combination

$$M = \log \left(1 - w_1 \frac{1 - 2z}{1 - z} \right) - \log (1 - w_1) - \log \left(1 - w_1 \frac{-z}{1 - z} \right)$$

appears several times in all integrals. The program calculates easily $\int_0^1 \int_0^1 z^n w_1 M dz dw$ by partial integration in a way quite similar to the one given in Appendix III. The case $\int_0^1 \int_0^1 (M/zw_1) dz dw$ is an exception and the value of this integral is given explicitly as a substitution. It introduces however special transcendental constants like $\zeta(3)$, $\pi^2 \log 2$.

$$\int_0^1 dw_1 \int_0^1 dz z w_1 M = \frac{19\pi^2}{192} - \frac{11}{16} \quad *)$$

$$\int_0^1 dw_1 \int_0^1 dz \frac{M}{z w_1} = \frac{\pi^2}{2} \log 2 - \frac{5}{4} \zeta(3)$$

$$\int_0^1 dw_1 \int_0^1 dz \frac{M}{z^3 w_1^2} = \frac{\pi^2}{6} + \frac{3}{4} \quad *)$$

.

etc.

Here the two integrals marked by an asterisk are examples of automatically performed partial integration.

Then the integration is performed in all remaining terms (other than M terms). The formula

$$\int_0^1 dw w^{m-1} S\ell_n(wz) = -\frac{1}{n-m} [S\ell_n(z) - S\ell_n(z)]$$

is used, generating for example

$$\int_0^1 dw w^{-3} \log (1 - wa) = -\frac{1}{2} \left[(1 - a^2) \log (1 - a) + a + \frac{a^2}{2} \right]$$

after regularization with respect to w. Also

$$\int_0^1 dw w \log (1 - wa) = \frac{1}{2} \left(1 - \frac{1}{a^2} \right) \log (1 - a) - \frac{1}{4} - \frac{1}{2a}$$

and

$$\int_0^1 \frac{dw}{w} \log (1 - wa) = -\text{Li}_2(a)$$

and so on and so forth.

One has thus a set of such integral formulae over w which are substituted after the `RATIO` command for w and w_1 has been executed.

The quantity a which appears in the above integration formulae over w arise from the log terms in the expansions of $S\ell_n(-G_1)$, etc. For example one has written (before the `RATIO` command)

$$\log(1 + G_1) = \log\left(1 + \frac{w}{1-w} \frac{1-z}{z}\right) = \log\left(1 - w \frac{2z-1}{z}\right) - \log(1-w).$$

In this case, we have $a_3 = (2z-1)/z$ and $a_5 = 1$.

There are five such quantities a , which are functions of z :

$$\begin{aligned} a_1 &= \frac{1-2z}{1-z}; & a_2 &= -\frac{z}{1-z}; \\ a_3 &= \frac{2z-1}{z}; & a_4 &= z; & a_5 &= 1. \end{aligned}$$

This ends the w integration with the use of a set of trivial integrals, the last being

$$\int_0^1 dw = 1.$$

Then one can use the regularization formulae over w and w_1

$$\int_0^1 w_1^{-n} \log(1-w) dw = 0, \quad \int_0^1 w^{-n} \log w dw = 0; \quad \int_0^1 w_1^{-n} dw = 0, \quad \int_0^1 w^{-n} dw = 0.$$

The final step is the z integration. New `RATIO` commands clear up expressions like, say

$$\frac{z}{(1-z)^2} \frac{1}{2z-1} = \left[\frac{1}{(1-z)^2} - \frac{1}{1-z} \right] \frac{1}{2z-1} = \frac{1}{(z-1)^2} + \frac{1}{1-z} + \frac{2}{2z-1}$$

or

$$\frac{z}{(2z-1)^2} = \frac{1}{2} \left[\frac{1}{(2z-1)^2} + \frac{1}{2z-1} \right].$$

As factors of such rationals in z appear

$$\text{Li}_2\left(\frac{2z-1}{z}\right), \quad \text{Li}_2(z), \quad \log \frac{1-z}{z}, \quad \log(1-z),$$

and constants. Here again the program proceeds by partial integration with respect to z and uses three functional relations satisfied by the Li_2 functions. A non trivial example on how such partial integrations proceed is given in Appendix III. Only the integrations introducing special transcendental constants are given explicitly

$$\int_0^1 \frac{dz}{1-z} \operatorname{Li}_2\left(\frac{2z-1}{z}\right) = -\frac{\pi^2}{2} \log 2 - \frac{3}{4} \zeta(3)$$

$$\int_0^1 \frac{dz}{2z-1} \operatorname{Li}_2\left(\frac{2z-1}{z}\right) = \frac{7}{8} \zeta(3) + \frac{\pi^2}{4} \log 2$$

$$\int_0^1 \frac{dz}{z} \operatorname{Li}_2(z) = \zeta(3)$$

$$\int_0^1 \frac{dz}{1-z} \operatorname{Li}_2(z) = -2 \zeta(3)$$

after regularization.

Then

$$\int_0^1 \operatorname{Sl}_1(z) dz = -\frac{\pi^2}{6}$$

$$\int_0^1 \operatorname{Sl}_1(2z-1) dz = -\frac{\pi^2}{8}$$

$$\int_0^1 \operatorname{Sl}_3(2z-1) dz = -\frac{1}{2}$$

⋮

With the exception of the first two, all remaining integrals follow automatically and are computed by the program according to

$$\int_0^1 \operatorname{Sl}_n(z) dz = -\frac{1}{n-1} \Psi_{n-1}(1) = \int_0^1 \operatorname{Sl}_n(1-z) dz$$

$$\begin{aligned} \int_0^1 \operatorname{Sl}_n(2z-1) dz = & -\frac{1}{2} \frac{1}{n-1} \left[\Psi_{n-1}(1) + (-1)^n \Psi_{n-1}(-1) \right] \\ & - \frac{1}{2} \frac{1}{n-1} \left[(-1)^n + 1 \right] \log 2 . \end{aligned}$$

The last formulae are equivalent to

$$\int_0^1 \frac{dz}{z} \log (1 - z) = - \frac{\pi^2}{6} = \int_0^1 \frac{dz}{1 - z} \log z$$

$$\int_0^1 \frac{dz}{2z - 1} \log 2(1 - z) = - \frac{\pi^2}{8}$$

$$\int_0^1 \frac{dz}{1 - 2z} \log 2z = - \frac{\pi^2}{8}$$

$$\int_0^1 \frac{dz}{(2z - 1)^3} \log \frac{1 - z}{z} = -1$$

etc.

after regularization.

Elementary integrals which the program performs in all generality, give for example:

$$\int_0^1 dz \, z \log z = - \frac{1}{4}$$

$$\int_0^1 dz \, z \log (1 - z) = - \frac{3}{4}$$

$$\int_0^1 dz \log z = -1$$

•
•
•

$$\int_0^1 z \, dz = \frac{1}{2}$$

$$\int_0^1 dz = 1 \, .$$

Then finally, and only then, the regularization is performed by letting

$$\int_0^1 dz \frac{\log z}{z^n} = 0$$

$$\int_0^1 \frac{dz}{(2z - 1)^n} = 0$$

$$\int_0^1 dz \frac{\log (1 - z)}{(1 - z)^n} = 0$$

$$\int_0^1 \frac{dz}{z^n} = 0$$

$$\int_0^1 \frac{dz}{(1 - z)^n} = 0$$

n integer ≥ 0 .

The only elementary integrals that are performed after the regularization has been done, are

$$\int_0^1 z^n dz = \frac{1}{(n + 1)}, \quad n \geq 0.$$

The output will then give the value of the desired integral analytically. Thus, for computing, say, $I(2,1,2,3,5)$, one simply punches a card with

Z**3*W**5*SI(2,1,2),

adds the deck, and the analytical value of the integral comes out. It must be finally mentioned that it is of course also possible to integrate simultaneously a sum of polynomials in z and w , times various integrals, like

$$\sum_{k,m,n} P_{k,m,n}(z,w) SI(k,m,n)$$

The listing given in the example is restricted to the cases $k = 1, 2, 3$ which are the only occurring in the Lamb-shift calculation. Generalizations to higher k 's can be performed according to the Appendices I, IIa, IIb and II. Moreover this listing is the one based on SCHOONSCHIP. Of course the program has a greater generality and can be implemented into any algebraic manipulation program. A translation within REDUCE is on the way [7].

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Example

To compute

$$\begin{aligned} \int_0^1 dy \int_0^1 dx \int_0^1 dw \int_0^1 dz (1 - zw)^4 \frac{z^4 w^5 x^3 y^5 \Delta^4}{\beta^4 \Gamma^3} = \\ = - \frac{27149}{1008} - \frac{249}{56} \pi^2 \log 2 + \frac{3057}{112} \zeta(3) + \frac{66847}{26880} \pi^2 . \end{aligned}$$

SCHOONSHIP	TEXT
Z1	$-(1 - z)$
W1	$-(1 - w) = -w_1$
Z2	$z - \frac{1}{2}$
SD1	$(1 - D)^{-1}$
HMG	$(H_1 - G_1)^{-1}$
SPE	$(1 + E_1)^{-1}$
SL2G	$S\ell_2(-G_1)$
SL2E	$S\ell_2(-E_1)$
SL2H	$S\ell_2(-H_1)$
SL1G	$S\ell_1(-G_1)$
SL1H	$S\ell_1(-H_1)$
SL1E	$S\ell_1(-E_1)$
SG	$-G_1^{-1}$
SE	$-E_1^{-1}$
SH	$-H_1^{-1}$
SSG	$-G_1$
SSE	$-E_1$
SSH	$-H_1$
PAR	M
SLW3	$\log \left(1 - w \frac{2z - 1}{z} \right)$
SLW4	$\log (1 - wz)$
SLW5	$\log (1 - w)$
R	$1 - wz$
SLI3	$Li_2 \left(\frac{2z - 1}{z} \right)$
SLI4	$Li_2(z)$
SL3Z	$S\ell_3(z)$
SL2Z	$S\ell_2(z)$
SL1Z	$S\ell_1(z)$
SL3	$\log \frac{1 - z}{z}$
SL4	$\log (1 - z)$
ZETA3	$\zeta(3)$
P2=P1**2	π^2

REFERENCES

- [1] N. Nielsen, Nova Acta Leopoldina Halle 90 (1909) 125.
- [2] K.S. Kölbig, J.A. Mignaco and E. Remiddi, BIT 10, (1970) 38.
- [3] A. Petermann, Phys. Letters 35 B, (1971) 325.
- [4] M. Veltman, SCHOONSHIP, CERN Internal Report (1967).
- [5] R. Karplus and N.M. Kroll, Phys. Rev. 77, (1960) 536.
- [6] J. Weneser, R. Bersohn and N.M. Kroll, Phys. Rev. 91, (1953) 1257.
- [7] J. Fox and A. Hearn (private communication).

The function

$$\phi\left(\frac{2z-1}{z}, n, 2\right) = \text{Sd}_n\left(\frac{2z-1}{z}\right)$$

is computed with the help of the function

$$\begin{aligned} f_2(u) &= \int_0^u \text{Sl}_2\left(\frac{2z-1}{z}\right) dz = \frac{1}{8} \left[\log 2(1-u) + \log 2u - \text{Sl}_1(2u-1) \right. \\ &\quad \left. - \text{Sl}_1(1-2u) - 2 \log 2 \right. \\ &\quad \left. - \frac{1}{4} \left[\text{Li}_2(2u-1) - \text{Li}_2(1-2u) = \frac{\pi^2}{6} \right] \right. \\ &\quad \left. + \left[\frac{u}{2} - \frac{1}{4} \{ (1-u) \log(1-u) + u \log u \} \right] \right]. \end{aligned}$$

Therefore, if we have to compute:

$$\phi\left(\frac{2z-1}{z}, 2, 2\right)$$

for example, we have

$$\phi\left(\frac{2z-1}{z}, 2, 2\right) = - \int_0^1 \frac{du \, u \, \log u}{1-u} \frac{2z-1}{z} = \int_0^1 du \, \log u \, \text{Sl}_2\left(\frac{2u-1}{u}\right).$$

The integration is performed on integrating by parts, giving

$$\int_0^1 du \, \text{Sl}_2\left(\frac{2u-1}{u}\right) \log u = f_2(u) \log u \Big|_{u=0}^1 - \int_0^1 f_2(u) \frac{du}{u}.$$

There is an algorithm in the program which automatically performs all necessary integrations. For instance, first $f_2(u) \log u \Big|_0^1$ is set equal to zero.

Next $\int_0^1 f_2(u) \, du/u$ gives rise to the three terms

$$-\frac{\pi^2}{48}, \quad \frac{7}{16} \zeta(3) + \frac{\pi^2}{8} \log 2, \quad -\frac{1}{2} - \frac{\pi^2}{24},$$

giving the desired result

$$\text{SD2} = \frac{7}{16} \zeta(3) + \frac{\pi^2}{8} \log 2 - \frac{\pi^2}{16} - \frac{1}{2}.$$

In the same way,

$$\phi\left(\frac{2z-1}{z}, 3, 2\right) \text{ etc.},$$

are computed, the first requiring computation of

$$\text{SD3}\left(\frac{2z-1}{z}\right) = \frac{21}{64} \zeta(3) + \frac{3}{32} \pi^2 \log 2 - \frac{7\pi^2}{96} - \frac{3}{16} .$$

[SD4 would require $\int_0^1 f_2(u)(2u \log u + u) du$ and so on], all operations which are in the domain of applicability of the program.

APPENDIX IIa

If $n' = -n$, $n \leq 0$, $n' \geq 0$, the integral to compute is

$$\int_0^1 dy \frac{(1 - Cy)^{n'} y^{k-n'-2}}{(A + B)^k} =$$

$$= \sum_{j=0}^{n'} \frac{C^j (-1)^j}{(k-1)!} \binom{n'}{j} \sum_{\ell=0}^{n'-j} \frac{(n' - j - \ell)! (k - n' + j + \ell - 2)!}{A^{n'-j-\ell+1} (A + B)^{k-n'+j+\ell-1}} \binom{n'-j}{\ell}.$$

The rules are the same as for $n > 0$, i.e.

- 1) A is replaced by $w^2 u / (1 - D)$.
- 2) x by $u / (1 - D)$.
- 3) $(A + B)$ by $(1 + G_1 u) (1 + H_1 u)$.
- 4) C by $(1 - wz + w^2 z^2) u / (1 - D) \equiv Eu / (1 - D)$.
- 5) dx by $du / (1 - D)$.

The generalization of the x integration (u integration after the change of variable above) is done in Appendix IIb.

APPENDIX IIb

Concerning the x integration, the following generalizations have to be made:

- a) Each time the integral

$$\int_0^1 \frac{du u^n}{(1 + E_1 u)^\alpha}$$

occurs, replace it by the values (A.1) or (A.2) (see below) according to whether $\alpha - n \leq 1$ or not. (A.1) and (A.2) are written for G_1 ; if E_1 or H_1 occurs the change is evident.)

- b) Each time an integral of the form

$$\int_0^1 \frac{u^k du}{(1 + G_1 u)^\beta (1 + H_1 u)^\beta}$$

occurs, write it as

$$\frac{1}{G_1^\beta} \frac{1}{H_1^\beta} \frac{1}{\left(u + \frac{1}{G_1}\right)^\beta} \frac{1}{\left(u + \frac{1}{H_1}\right)^\beta}.$$

Then calling $UG = u + 1/G_1$ and $UH = u + 1/H_1$, one makes the following sequence of substitutions

```
ID  RATIO,UH,UG,C
ID  C=SHMG*SG*SH
ID  MULTI,C**(-1)=SSG*SSH*HMG
ID  SHMG=HMG**(-1) ,
```

c) Then the program applies to the formulae (A.1) or (A.2) for

$$\int_0^1 \frac{du u^n}{(1 + H_1 u)^\alpha}, \quad \int_0^1 \frac{du n^n}{(1 + G_1 u)^\alpha}$$

according to whether $\alpha - n \leq 1$ or not. The formulae (A.1) and (A.2), written for G_1 , are

$$\begin{aligned} \int_0^1 \frac{u^n du}{(1 + G_1 u)^\alpha} &= - \sum_{q=0}^{\alpha-2} \frac{n!}{(\alpha-1)!} \frac{(\alpha-2-q)!}{(n-q)!} \frac{1}{G_1^{q+1}} \frac{1}{(1 + G_1)^{\alpha-q-1}} \\ &- \frac{n!}{(n-\alpha+1)!} \frac{1}{(\alpha-1)!} \frac{1}{G_1^{\alpha-1}} S\ell_{n-\alpha+2}(-G_1), \end{aligned} \quad (A.1)$$

(for $\alpha - n \leq 1$)

$$\begin{aligned} \int_0^1 \frac{u^n du}{(1 + G_1 u)^\alpha} &= - \frac{1}{\alpha-1} \frac{1}{G_1} \frac{1}{(1 + G_1)^{\alpha-1}} \\ &- \sum_{m=0}^{n-2} \frac{n(n-1) \dots (n-m)}{(\alpha-1)(\alpha-2) \dots (\alpha-m-2)} \frac{1}{G_1^{m+2}} \frac{1}{(1 + G_1)^{\alpha-m-2}} \\ &- \frac{n!}{(\alpha-1)!} \frac{(\alpha-n-2)!}{G_1^{n+1}} \left[\frac{1}{(1 + G_1)^{\alpha-n-1}} - 1 \right], \end{aligned} \quad (A.2)$$

(for $\alpha - n \geq 2$).

Note that all terms with $m < 0$ in the sum are zero. This way, the x integration is performed in the most general case.

The procedure used in the program to integrate the regularized value of, say,

$$\int_0^1 \frac{dz}{(1-z)^n} \text{Li}_2\left(\frac{2z-1}{z}\right)$$

proceeds as follows. This is first integrated by parts according to

$$\begin{aligned} \int_0^1 \frac{dz}{(1-z)^n} \text{Li}_2\left(\frac{2z-1}{z}\right) &= \frac{1}{n-1} \frac{1}{(1-z)^{n-1}} \text{Li}_2\left(\frac{2z-1}{z}\right) \Big|_0^{1-\eta} \\ &+ \frac{1}{n-1} \int_0^1 \frac{dz}{(1-z)^{n-1}} \frac{\log \frac{1-z}{z}}{(2z-1)z}. \end{aligned}$$

The second term on the right hand side is evaluated easily, being of the type of the integrals involving lower transcendentals. The first term however must be handled with care, as it is divergent at both limits. We look first at the value for $z = 0$ and write it as

$$\begin{aligned} \frac{1}{n-1} \frac{1}{(1-z)^{n-1}} \text{Li}_2\left(1 - \frac{1-z}{z}\right) \Big|_{z=0} &= \frac{1}{n-1} \frac{1}{(1-z)^{n-1}} \left[-\text{Li}_2\left(\frac{1-z}{z}\right) \right. \\ &+ \left. \frac{\pi^2}{6} - \log \frac{2z-1}{z} \log \frac{1-z}{z} \right]_{z=0} = \\ &= \frac{1}{n-1} \frac{1}{(1-z)^n} \left[\text{Li}_2\left(\frac{z}{1-z}\right) - \frac{\pi^2}{6} + \right. \\ &+ \left. \log(2z-1) \log z + \log(1-z) \log z - \frac{1}{2} \log^2 z \right] \\ &= -\frac{1}{n-1} \frac{\pi^2}{6} - \frac{1}{2(n-1)} \log^2 z \Big|_{z=\epsilon \rightarrow 0}. \end{aligned}$$

The last term $\left[2(n-1)\right]^{-1} \log^2 z \Big|_{z=\epsilon \rightarrow 0}$ can be discarded because, according to our regularization procedure,

$$\text{Reg} \int_{\epsilon \rightarrow 0}^1 \frac{\log z}{z} dz = 0.$$

This integral being also equal to $\frac{1}{2} \log^2 1 - \frac{1}{2} \log^2(\epsilon)$ it follows that the regularization requires that $\log^2 \epsilon \Big|_{\epsilon \rightarrow 0} = \log^2 1 = 0$. For the upper limit $1-\eta$, with $\eta \rightarrow 0$, the value is written as

$$\begin{aligned} & \frac{1}{n-1} \frac{1}{\eta^{n-1}} \left[-\text{Li}_2\left(\frac{1-z}{z}\right) + \frac{\pi^2}{6} - \log\left(1 - \frac{1-z}{z}\right) \log \frac{1-z}{z} \right] \\ &= \frac{1}{n-1} \frac{1}{\eta^{n-1}} \left[-\text{Li}_2\left(\frac{\eta}{1-\eta} + \frac{\pi^2}{6}\right) \right] - [\log(1-\eta) - \log(1-2\eta)] \times \\ & \quad \times [\log(1-\eta) - \log \eta] . \end{aligned}$$

One then expands:

$$\text{Li}_2\left(\frac{\eta}{1-\eta}\right) = \frac{\eta}{1-\eta} + \frac{1}{4} \frac{\eta^2}{(1-\eta)^2} + \dots + \frac{1}{n^2} \frac{\eta^n}{(1-\eta)^n} + \dots .$$

Applying

$$(1-\eta)^{-n} = 1 + n\eta + \frac{n(n+1)}{2!} \eta^2 + \dots + \frac{(n+k-1)!}{(n-1)! k!} \eta^k + \dots$$

to the various terms in the Li_2 expansion, one gets for $\text{Li}_2[\eta/(1-\eta)]$ a power series in η for very small values of η . Then one writes

$$\log(1-2\eta) = 2\eta + \frac{4\eta^2}{2} + \frac{8\eta^3}{3} + \dots$$

$$\log(1-\eta) = \eta + \frac{\eta^2}{2} + \dots .$$

It follows that

$$\begin{aligned} & -[\log(1-2\eta) - \log(1-\eta)] [\log \eta - \log(1-\eta)] = \\ &= \left(\eta + \frac{3}{2} \eta^2 + \frac{7}{3} \eta^3 + \dots \right) \left(\eta + \frac{\eta^2}{2} + \frac{\eta^3}{3} + \dots \right) \end{aligned}$$

+ terms of the form $\eta^m \log \eta$. Adding the Li_2 series and the log series, the result of the limit $1-\eta$ ($\eta \rightarrow 0$) is thus of the form

$$\begin{aligned} & \frac{1}{\eta^{n-1}} \left[a_1 \eta + a_2 \eta^2 + \dots + a_k \eta^k + \dots + b_1 \eta \log \eta + b_2 \eta^2 \log \eta \right. \\ & \quad \left. + \dots + b_k \eta^k \log \eta + \dots \right] . \end{aligned}$$

For terms with $k > n-1$, the result is $\eta^{k-n+1} \rightarrow 0$ when $\eta \rightarrow 0$. For terms in which the exponent is smaller or equal to $n-1$ one sees that they are of the form $1/\eta^r$, $r \geq 0$. Our regularization procedure prescribes that

$$\text{Reg}_{\eta \rightarrow 0} \int_0^{1-\eta} \frac{dz}{(1-z)^{r+1}} = 0 ,$$

but it is also equal to

$$\frac{1}{r} \frac{1}{(1-z)^r} \bigg|_0^{1-\eta} = \frac{1}{r} \left(\frac{1}{\eta^r} - 1 \right) ,$$

so that whatever r is, in order to satisfy the regularization, $1/\eta^r$ must be put equal to 1.

Finally, terms of the form

$$\frac{1}{\eta^{m-1}} \log \eta , \quad m \geq 1 ,$$

are to be set equal to zero. Indeed, the regularization prescribes

$$\text{Reg}_{\eta \rightarrow 0} \int_0^{1-\eta} \frac{\log(1-z)}{(1-z)^m} = 0 = \frac{1}{m-1} \frac{1}{(1-z)^{m-1}} \log(1-z) \bigg|_0^{1-\eta} + \frac{1}{m-1} \text{Reg}_{\eta \rightarrow 0} \int_0^{1-\eta} \frac{dz}{(1-z)^m} .$$

The last integral is zero by our prescription, so that

$$\frac{1}{m-1} \frac{1}{\eta^{m-1}} \log \eta - \frac{1}{m-1} \log 1 = \frac{1}{m-1} \frac{1}{\eta^{m-1}} \log \equiv 0 .$$

TIME

.02 SECONDS

```

PRINT NLIST
PRINT NSTATISTICS
C
INDEX R,RQ
SYMBOLS SD1,LOG2,LWHO
FUNCTIONS SLZ,Q07,MA,SLH,SLE,PER,PSI,SI,Q
F XX,F4,F1,F2,F3,X1,X2,X3,U1,U2,U3 ,Q0,Q1,Q2, S1,S2,S3,FL3,XP,SLG
C
C PO*(X,N)=X**N
X PS(X,N)=DS(K,1,N,(POW(X,K)*DD(K)))
C
Z SINAC=
R**4*Z**4*W**5*SI(3,4,4)
C
C START OF Y INTEGRATION
C
L 1 ID SI(3,0,-1)=SI(3,1,0)-SSE*SD1*(-1)*SI(3,0,0)
+1./2.*W**(-4)*SD1*HMG*(SSH*SLH(1)-SSG*SLG(1))
L 3 ID SI(3,0,0)=1./2.*W**(-2)*SD1**3*(SPG+SPH)*HMG**2
+WW**(-2)*SD1**3*HMG**3*(SSG*SLG(1)-SSH*SLH(1))
L 5 ID SI(3,1,0)=1./2.*W**(-2)*SD1**2*HMG**2*(SSG*SPG+SSG*SLG(1))
+1./2.*W**(-2)*SD1**2*HMG**2*(SSH*SPH+SSH*SLH(1))
+WW**(-2)*HMG**3*SSH*SSG*(SLG(1)-SLH(1))*SD1**2
L 7 ID SI(K+,M+,N+)=SD1*POW(SD1,2.*K+N-M-3)*POW(U,2.*K+N-M-3)
*SO(K,X,N)
L 9 ID SO(K+,X,N+)=SI(K,N)+S2(K,N)+S3(K,N)
L10 ID SI(K+,N+)=DS(R,1,K-1,(POW(A,N-1)*POW(ACB,R+1-N-K)*POW(AR,-R)
*DD(R)*POW(-1,N-1-1)*DB(N+K-K-2,N-1)))
L13 ID S2(K+,N+)=DS(RQ,1,N-1,(POW(A,N-RQ-1)*POW(ACB,RQ+1-N-K)*POW(C1,-RQ)
*DD(RQ)*POW(-1,N-1-RQ)*DB(N+K-RQ-2,K-1)))
L16 ID S3(K+,N+)=POW(-1,N-1)*POW(ACB,1-N-K)*POW(A,N-1)*DR(N+K-2,N-1)
*LWHO
C
C END OF Y INTEGRATION
C

```

```

C
C
C
L18 ID MULTI,ACB**(-1)=1.
L19 ID A=SD1*U**W**2
L20 ID MULTI,AR**(-1)=C
L21 ID C=SPGU*SPHU
L22 ID MULTI,C1**(-1)=SPEU
L23 ID U**N1+SPEU**2=SLE(N1+1)
L25 ID U**N1+SPGU**2*SPHU**2=(N1-2)*HMG**2*(SLG(N1-1)+SLH(N1-1))
      +HMG**2*(SPG+SPH)+2.*HMG**3*(SLG(N1-2)-SLH(N1-2))
L26 ID U**N1+SPGU*SPHU=-HMG*(SLG(N1)-SLH(N1))
      * YEP
      ID U**N1+SPEU**2=SE*SPE+N1*SE*SLE(N1)
      ID U**N1+SPGU**3=1./2.*SPE**2*DK(N1,1)
      ID U**N1+SPGU**2*(N1-2)*SPE**2*(N1-2)-N1/2.*SE**2*SPE*DT(N1-2)
      +1./2.*SE**2*SLE(N1-1)*DT(N1-2)
      ID -N1*(N1-1)/2.*SE**2*SLE(N1+1)*PER(N1+1)
      ID U**N1+LWHO=DD(N1+1)*SLH(1)
      ID SLH(0)=SSH*SLH(1)
      ID SLG(0)=SSG*SLG(1)
      ID LWHO=PER(1)
      ID SPFU**3=1./2.*SE*(SPE**2-1.)
      ID SPEU**2=SPE
      ID SPEU=-SLE(1)
      AL SPGU*SPHU=(-SSG*SLG(1)+SSH*SLH(1))*HMG
      C
      C
END OF X INTEGRATION

```



```

L 1 ID W**(-4)*SLW3=-1./3.*(SL3-A3**3*SL3+A3+1./2.*A3**2+1./3.*A3**3)
L 3 ID W**(-4)*SLW4=-1./3.*(SL4-Z**3*SL4+Z+1./2.*Z**2+1./3.*Z**3)
L 5 ID W**(-4)*SLW5=-11./18.
L 6 ID W**(-3)*SLW3=Z1/Z*SL3+1./2.*Z1**2/Z**2*SL3-Z2/Z-Z2**2/Z**2
L 6 AL W**(-3)*SLW4=-1./2.*SL4+1./2.*Z**2*SL4-1./2.*Z-1./4.*Z**2
L 6 AL W**(-3)*SLW5=-3./4.
L 7 ID W**(-2)*SLW3=Z1/Z*SL3-2.*Z2/Z
L 7 AL W**(-2)*SLW4=Z1*SL4-Z
L 7 AL W**(-2)*SLW5=-1.
L 8 ID W**(-1)*SLW3=-SLI3
L 9 ID W**(-1)*SLW4=-SLI4
L 10 ID W**(-1)*SLW5=-P2/6.
L 11 ID W**(-1)*R**(-2)*Z**(-1)*Z1**(-1)+N*Z**(-1)*SLZ(N)
L 12 ID R**(-2)*Z1**(-1)
L 12 AL W**(-1)*R**(-1)*W**(-1)+Z/R
L 12 AL W1**(-1)*R**(-1)*W1**(-1)*Z1**(-1)-Z/Z1*R**(-1)
L 13 ID W**N+R**(-1)*SLZ(N+1)
L 13 AL R**(-1)*SLI2
L 15 ID SLZ(N+1)=SZ**N*(SL4+PS(Z,N=1))
L 20 ID SZ=Z**(-1)
L 21 ID W**(-2)*R3**(-1)*Z2/Z*SL3
L 21 AL W**(-1)*R3**(-1)*SL3
L 22 ID W1**(-4)*R3**(-1)*Z2**3*Z1**(-4)*SL3
L 23 ID W1**(-3)*R3**(-1)*Z2**2*Z1**(-3)*SL3
L 23 AL W1**(-2)*R3**(-1)*Z2**2*Z1**(-2)*SL3
L 23 AL W1**(-1)*R3**(-1)*Z/Z1*SL3
L 24 ID W**2*R3**(-1)*SL3A3
L 24 AL W**R3**(-1)*SL2A3
L 24 AL R3**(-1)*SL1A3
L 25 ID W**2*SLW3=1./3.*Y3
L 26 ID Y3=(1.-SA3**3)*SL3-SA3**2-1./2.*SA3-1./3.
L 28 ID W**2*SLW4=1./3.*Y4
L 29 ID Y4=(1.-Z**(-3))*SL4-Z**(-2)-1./2.*Z**(-1)-1./3.
L 29 AL W**2*SLW5=-11./18.
L 31 ID W*SLW3=(-1./32.*Z2**(-2)-1./8.*Z2**(-1)+3./8.)*SL3
      -1./2.-1./8.*Z2**(-1)
      W*SLW4=1./2.*(SL4-Z**(-2)*SL4-1./2.-1./Z)
      W*SLW5=-3./4.
L 33 ID SLW3=SL3-1./2.*Z/Z2*SL3-1.
L 35 AL SLW4=Z1/Z*SL4-1.
L 35 AL SLW5=-1.
C
L 36 ID MULTI,W**(-1)=0
L 36 AL MULTI,W1**(-1)=0
L 37 ID W**N+DD(N+1)
C
C
C
END OF W INTEGRATION

```

```

*
L 1 ID SL3A3=9A3**3*(SL3+A3+1./2.*A3**2)
L 3 ID SL2A3=9A3**2*(SL3+A3)
L 5 ID SL1A3=9A3*SL3
L 6 ID SA3=1./2.*Z/Z2
L 6 AL A3=2.*Z2/Z
L 7 ID SL3Z=Z**(-3)*SL4+Z**(-2)+1./2.*Z**(-1)
L 7 AL SL2Z=Z**(-2)*SL4+Z**(-1)
L 7 AL SL1Z=SL4/Z
L 8 ID RATIO,Z1,Z,1.
L 9 ID RATIO,Z2,Z1,A
L10 ID A=1./2.
L11 ID MULTI,A**(-1)=2.
L12 ID RATIO,Z2,Z,B
L13 AL B=1./2.
L13 AL MULTI,B**(-1)=2.
L14 ID Z1=Z=1.
L15 ID Z**N+SLI3=F1(N)
L16 ID F1(N+)=DD( N+1 ) *P2/6.+1./2.*DD( N+1 ) *Z**N*Z2**(-1)*SL33
L18 ID RATIO,Z2,Z,B
L19 ID B=1./2.
L20 ID MULTI,B**(-1)=2.
L21 ID Z2**(-1)*SL33=P2/2.
L22 ID SL33=SL44=SS
L23 ID Z2=Z=1./2.
L24 ID Z**J+*SS=DD(J+1)*DD(J+1)
L26 ID Z=1.=U
L27 ID U**L+*SL44=DD(L+1)*DD(L+1)
L29 ID SS=1.
L30 ID SL44=1.
L31 ID U=1.=Z
*
L 1 ID YEP
L 2 ID Z1**(-1)*SLI3=3./4.*ZETA3+P2/2.*LOG2
L 4 ID Z1**N+*SLI3=FL3(-N=1,-N=1,-N=1)
L 4 ID FL3(K+,N+,M+)= DD(M)*SH**M*XP(K,N)= DD(M)*P2/3.*(-1)**M
      -1./2.* DD(M)*SZ1**M*Z2**(-1)*SL3*Z**(-1)
L 6 ID SH**N+=S**N
L 7 ID S=H**(-1)
L 8 ID SZ1=Z1**(-1)
L 9 ID RATIO,Z1,Z,1.

```

```

L10 ID      RATIO,Z2,Z,R
L11 ID      B=1./2.
L11 AL      MULTI,R**(-1)=2.
L12 ID      RATIO,Z2,Z1,A
L13 ID      A=-1./2.
L13 AL      MULTI,A**(-1)=-2.
L13 AL      Z2=Z-1./2.
L13 AL      Z1=Z-1.
L13 *      YEP
L1 ID      Z1**(-1)*SL3=-P2/6.
L1 AL      Z2**(-1)*SL3=-P2/2.
L1 AL      Z**(-1)*SL3=-P2/6.
L2 ID      Z1**(-1)*SL3=F3(-N)
L3 ID      F3(N+)=(-1)**N*Q0(N-1)* DD( N-1 )
L3 *      YEP
L1 ID      Q0(N+)=DS(K,1,N,(DD(K)))
L1 *      YEP
L1 ID      XP(K+,N+)=Q(K)*Q1(K)-F2(K,N)
L2 ID 15 F2(K+,J+)=F2(K,J-1)*DT(J=5./2.)+F1(J,K)
L2 *      YEP
L1 ID 15 F1(N+,K+)=F1(N,K-1)*DT(K=3./2.)
      +DT(K)*DB(N+K-1,N-1)*H**N*H**K*DD(N)* DD(N)
L1 *      YEP
L1 ID      Q(N+)=Q2(N)-Q1(N)
L2 ID      Q2(N+)=DS(K,1,N,(DD(K)*POW(HH,K)))
L2 *      YEP
L1 ID      Q1(N+)=DS(K,1,N,(DD(K)*POW(H,K)))
L4 ID      HH=2.*H
L4 *      YEP
L1 ID      MULTI,H**(-1)=1.
L2 ID      H=0
L2 AL      Z2**(-3)*SLI3=12.
L2 AL      Z2**(-2)*SLI3=-2./3.*P2
L3 ID      Z2**(-1)*SLI3=7./4.*ZETA3+P2/2.*LOG2
L3 AL      SLI3=-P2/12.
L3 *      YEP
L1 ID      Z1**(-1)*SLI4=2.*ZETA3
L2 ID      Z1**(-1)*SLI4=F4(-N-1)
L4 ID      F4(J+)=DD(J)*SH**J*XX(J)
L5 ID      SH**J+=S**N

```

```

L 6 S=H*(-1)
L 7 XX(N+)=DS(K,1,N,(DD(K)*DD(K))*POW(H,K))
L10 MULTI,H*(-1)=1.
L11 H=0
L11 Z**(-4)*SLI4=71./108.-P2/18.
L12 Z**(-1)*SLI4=ZETA3
L13 Z**(N+)*SLI4=DD(-N-1)*P2/6.+DD(-N-1)*POW(H,N+1)
      *DS(K,1,-N,(POW(H,K)*DD(K)*DD(K))+DD(-N-1)*DD(-N-1)*Q0(-N-1)*Q0(-N-1))
L16 MULTI,H*(-1)=1.
L17 H=0
      *
      YEP
L 1 Z1*(-1)*SL3=P2/6.
L 2 Z1*(N+)*SLI3=(-1)**N*Q0(-1-N)*DD(-1-N)
L 4 Q0(N+)=DS(K,1,N,(DD(K)))
L 6 Z2*(-1)*SL3=P2/2.
L 7 Z2*(N+)*SL3=POW(2,-N-1)*DD(-N-1)*(1.-POW(-1,-N))
      *(Q0(-N-1)+POW(-1,-N)*Q07(-N-1))
L10 Q07(N+)=DS(K,1,N,(DD(K)*POW(-1,K)))
L13 Q0(N+)=DS(K,1,N,(DD(K)))
L15 Z*(-1)*SL4=P2/6.
L15 Z2=Z-1./2.
L16 Z*(N+)*SL3=Z**N*SL+DD(N+1)*DD(N+1)
L18 Z*(N+)*SL4=Q0(-1-N)*DD(-1-N)*DT(-N-2)+Z**N*SL*DT(N)
      *
      YEP
L 1 Q0(N+)=DS(K,1,N,(DD(K)))
L 3 Z=1.-U
L 4 U*(N+)*SL=-DD(N+1)*DD(N+1)
L 6 U*(N+)=DD(N+1)
L 6 SL=-1.
L 8 SL4=1.
L 9 SL3=0
L 9 P2=PI**2
L 9 MULTI,Z2*(-1)=0
L 9 MULTI,Z1*(-1)=0
L 9 MULTI,Z*(-1)=0
L10 Z*(N+)=DD(N+1)
C
C END OF Z INTEGRATION
C
C SEE OUTPUT BELOW
C
C PRINT OUTPUT
N R
* YEP

```

SIAC

$$= -27149./1008., -249./56.*LOG2*PI**2 + 3057./112.*ZETA3 + 66847./26880.*PI**2 + 0.,$$