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NUMERICAL INVERSION OF A CLASS OF CHARACTERISTIC FUNCTIONS

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SUMMARY

In this paper a simple method is proposed to invert numerically a given characteristic function of an absolutely continuous distribution function. The method is based on Fourier series expansions. Exact error bounds are given and a few examples are discussed.

1. INTRODUCTION

In many applications we know the characteristic function

$$\psi(t) = \int_{-\infty}^{\infty} e^{itx} f(x) dx \quad (1.1)$$

of a probability density function $f(x)$, and want to determine the corresponding distribution function

$$F(x) = \int_{-\infty}^x f(\xi) d\xi \quad (1.2)$$

by numerical methods, using the fact that

$$F(x) = \frac{1}{2} - \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{e^{-itx}}{t} \psi(t) dt . \quad (1.3)$$

However, it turns out that the evaluation of (1.3) by numerical integration methods is often difficult and very time-consuming. Particularly if, for instance, $\psi(t)$ is highly oscillatory, or is slowly converging to zero for $|t| \rightarrow \infty$.

We therefore propose in this paper another method, applicable to a fairly wide class of probability density functions, by which we can avoid any integration process and for which we can easily find estimates for the error committed. This method is sometimes used in a more heuristic way [see, for example, Krylov (1969), Bendat (1966)] to invert certain Laplace transforms. It seems that in the literature nobody has so far applied this method, which is based on Fourier series, in a systematic and rigorous way for the numerical calculation of distribution functions.

Let us consider the following class of probability density functions:

Definition

We denote by C the set of probability density functions f with the following properties:

- a) f is continuous for all $x \in \mathbb{R}$;
- b) for every $f \in C$ there exist $c_- < 0$, $c_+ > 0$ and positive constants M_- , M_+ , such that

$$e^{-c_+x} f(x) \leq M_- \quad \text{for all } x \leq 0 ,$$

$$e^{-c_-x} f(x) \leq M_+ \quad \text{for all } x \geq 0 ;$$

- c) for every $f \in C$ there exists an x_0 such that f is monotonically non-increasing for $x > x_0$ and monotonically non-decreasing for $x < x_0$.

Then, for every $f \in C$, the Laplace transform

$$\phi(s) = \int_{-\infty}^{\infty} e^{-sx} f(x) dx \quad (1.4)$$

exists and represents a holomorphic function in the strip

$$c_- < \operatorname{Re} s < c_+ . \quad (1.5)$$

Therefore we shall always use $\phi(s)$ instead of $\psi(t)$ in the following.

We now make the following assumptions:

- $\alpha)$ we know $\phi(s)$ explicitly for all s in the strip (1.5);
- $\beta)$ we do not know explicitly the corresponding $f \in C$;
- $\gamma)$ we are given two small positive numbers ϵ_- and ϵ_+ .

We then consider the following problems:

- (A) Find two real quantities T_- and T_+ with $T_- < T_+$ such that

$$P(X \leq T_-) = \int_{-\infty}^{T_-} f(x) dx \leq \epsilon_- , \quad (1.6)$$

$$P(X \geq T_+) = \int_{T_+}^{\infty} f(x) dx \leq \epsilon_+ . \quad (1.7)$$

- (B) Find analytically a function $G(x)$ which can be computed "easily and rapidly" on a computer, and for which the inequality

$$-\epsilon_- \leq G(x) - F(x) \leq \epsilon_+ \quad (1.8)$$

is valid for all x with $T_- \leq x \leq T_+$, and where F is defined by (1.2).

In formulating these problems, which do not of course have a unique solution, we only want to give the reader an idea of the intention of the following considerations.

At first glance the class C of probability density functions seems to be a relatively special one. But in Section 5 of this paper we shall briefly discuss, and show in practical examples that the proposed method can also be applied for a more general class of probability density functions. But in order to avoid unnecessary theoretical complications in what is essentially a practical method we prefer to consider the particular class C of probability density functions,

especially as this class already covers most of the standard continuous distributions used in applied probability and statistics.

The proposed method will lead, in a natural way, to an approximating function $g(x)$ for $f(x)$ in the interval (T_-, T_+) . But, in general, we will not have as good an error estimate for the deviation of $g(x)$ from $f(x)$ as we will have for the deviation of $G(x)$ from $F(x)$. This will be discussed in Section 4.

2. DETERMINATION OF T_- AND T_+

For convenience, we may assume that x_0 , defined under (c) in the definition of C , is positive and not too large, so that we have

$$T_- \leq 0, \quad T_+ > 0.$$

This situation can always be attained by making a suitable translation.

We now split $\phi(s)$ into a sum of two holomorphic functions:

$$\phi(s) = \phi_-(s) + \phi_+(s) \tag{2.1}$$

with

$$\phi_-(s) = \int_{-\infty}^0 e^{-sx} f(x) dx, \tag{2.2}$$

$$\phi_+(s) = \int_0^{\infty} e^{-sx} f(x) dx. \tag{2.3}$$

Because of (b) in the definition of C , $\phi_-(s)$ is holomorphic in $\operatorname{Re} s < c_+$ and $\phi_+(s)$ is holomorphic in $\operatorname{Re} s > c_-$. Therefore, it can be shown that $\phi_-(s)$ and $\phi_+(s)$ can be determined by Cauchy's integral theorem, knowing only $\phi(s)$.

Let α_- and α_+ be such that

$$c_- < \alpha_+ < 0, \tag{2.4}$$

$$c_+ > \alpha_- > 0. \tag{2.5}$$

Then, for any $T_- \leq 0$ and $T_+ > 0$, it follows that

$$\phi_-(\alpha_-) = \int_{-\infty}^0 e^{-\alpha_- x} f(x) dx \geq e^{-\alpha_- T_-} \int_{-\infty}^{T_-} f(x) dx = e^{-\alpha_- T_-} P(X \leq T_-),$$

$$\phi_+(\alpha_+) = \int_0^{\infty} e^{-\alpha_+ x} f(x) dx \geq e^{-\alpha_+ T_+} \int_{T_+}^{\infty} f(x) dx = e^{-\alpha_+ T_+} P(X \geq T_+),$$

exist, it follows from theorem 2 on page 247 in Doetsch's (1950) book that

$$\int_{-i\infty}^{i\infty} |\phi(s)|^2 ds$$

also exists. Therefore, since

$$a_k = \frac{\omega}{2\pi} \phi(ik\omega) \quad (3.3)$$

for all k , the sum

$$\sum_{k=-\infty}^{\infty} |a_k|^2 \quad (3.4)$$

exists, and according to the Fischer-Riesz theorem [e.g. Tricomi (1970)] the series

$$g(x) = \sum_{k=-\infty}^{\infty} a_k e^{ik\omega x} \quad (3.5)$$

represents in (T_-, T_+) a function which is at least quadratically integrable. We now show the following

Theorem 2

For $f \in C$ let a_k be given by (3.2). Then, for $g(x)$, defined by (3.5), we have

$$g(x) = f(x) + \sum_{k=1}^{\infty} [f(x + kT) + f(x - kT)] \quad (3.6)$$

in the interval $T_- \leq x \leq T_+$.

Proof

Since $a_k \rightarrow 0$ for $|k| \rightarrow \infty$ we have $S_n(x) \rightarrow g(x)$ in $T_- \leq x \leq T_+$ for $n \rightarrow \infty$, where

$$S_n(x) = \sum_{k=-n}^n a_k e^{ik\omega x} - \frac{1}{2}(a_n e^{in\omega x} - a_{-n} e^{-in\omega x}) \quad (3.7)$$

We insert (3.2) into (3.7), and after a simple calculation we obtain

$$S_n(x) = \frac{\omega}{\pi} \int_{-\infty}^{\infty} f(\xi) \left[\frac{1}{2} + \sum_{k=1}^n \cos k\omega(\xi - x) - \frac{1}{2} \cos n\omega(\xi - x) \right] d\xi \quad (3.8)$$

Using the well-known formula (Tricomi, p. 68)

$$\frac{1}{2} + \sum_{k=1}^n \cos kx - \frac{1}{2} \cos nx = \frac{\sin nx}{2 \tan \frac{x}{2}} \quad (3.9)$$

and introducing the substitution

$$\omega(\xi - x) = t ,$$

we obtain from (3.8)

$$S_n(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f\left(x + \frac{t}{\omega}\right) \frac{\sin nt}{\tan \frac{t}{2}} dt . \quad (3.10)$$

Since

$$\frac{\sin nt}{\tan \frac{t}{2}}$$

has the period 2π , (3.10) may be written in the form

$$\begin{aligned} S_n(x) = & \frac{1}{2\pi} \int_{-\pi}^{\pi} f\left(x + \frac{t}{\omega}\right) \frac{\sin nt}{\tan \frac{t}{2}} dt \\ & + \sum_{k=1}^{\infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} \left[f\left(x + kT + \frac{t}{\omega}\right) + f\left(x - kT + \frac{t}{\omega}\right) \right] \frac{\sin nt}{\tan \frac{t}{2}} dt . \end{aligned} \quad (3.11)$$

Since $f(x)$ is continuous and of bounded variation, it is easily seen that (Ticomi, p. 72)

$$\lim_{n \rightarrow \infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} f\left(y + \frac{t}{\omega}\right) \frac{\sin nt}{\tan \frac{t}{2}} dt = f(y) . \quad (3.12)$$

Using this fact in (9.11), we have only to show that it is permitted to take the limit for $n \rightarrow \infty$ under the summation sign.

According to a theorem, given, for example, by Ostrowski (1968) this is permissible if the sum in (3.11) converges uniformly with respect to n .

To show that this is the case, we use the following well-known mean-value theorem:

Let $f_1(t)$ and $f_2(t)$ be real continuous functions in $a \leq t \leq b$. In addition, let $f_1(t)$ be non-negative and non-increasing. Then there exists a value ξ_1 with $a < \xi_1 < b$ such that

$$\int_a^b f_1(t) f_2(t) dt = f_1(a) \int_a^{\xi_1} f_2(t) dt . \quad (3.13)$$

If $f_1(t)$ is non-decreasing, there exists a value ξ_2 with $a < \xi_2 < b$ such that

$$\int_a^b f_1(t) f_2(t) dt = f_2(b) \int_{\xi_2}^b f_2(t) dt . \quad (3.14)$$

Therefore, taking

$$f_1(t) = f\left(y + \frac{t}{\omega}\right) ,$$

$$f_2(t) = \frac{1}{2\pi} \frac{\sin nt}{\tan \frac{t}{2}} ,$$

$$a = -b = -\pi ,$$

and observing that $f(x)$ has property (c) of the definition of C, we can easily show that

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} f\left(y + \frac{t}{\omega}\right) \frac{\sin nt}{\tan \frac{t}{2}} dt = A_n(y) , \quad (3.15)$$

where

$$A_n(y) = \begin{cases} f\left(y - \frac{\pi}{\omega}\right) \frac{1}{2\pi} \int_{-\pi}^{\xi_1} \frac{\sin nt}{\tan \frac{t}{2}} dt & \text{for } y - \frac{\pi}{\omega} \geq x_0 , \\ f\left(y + \frac{\pi}{\omega}\right) \frac{1}{2\pi} \int_{\xi_2}^{\pi} \frac{\sin nt}{\tan \frac{t}{2}} dt & \text{for } y + \frac{\pi}{\omega} \leq x_0 . \end{cases} \quad (3.16)$$

According to Tricomi, p. 72, it can be proved that the integrals in (3.16) are bounded for all n . Therefore, it follows immediately that the sum (3.11) can be majorized by the convergent sum

$$A \sum_{k=1}^{\infty} \left[f\left(x + kT - \frac{\pi}{\omega}\right) + f\left(x - kT + \frac{\pi}{\omega}\right) \right] ,$$

where A is a suitable positive constant. Theorem 2 is thus proved.

Before we discuss (3.6) as an approximation to $f(x)$ in (T_-, T_+) , we consider the function

$$G(x) = \int_{T_-}^x g(\xi) d\xi , \quad T_- \leq x \leq T_+ , \quad (3.17)$$

as an approximation to $F(x)$, defined by (1.2). We prove the following

Theorem 3

Let $G(x)$ be defined by (3.17), where $g(x)$ is given by (3.5). Then, for given ϵ_- , ϵ_+ and for T_- , T_+ such that (1.6) and (1.7) are satisfied, the inequality

$$-\epsilon_- \leq G(x) - F(x) \leq \epsilon_+ \quad (3.18)$$

is valid for all x with $T_- \leq x \leq T_+$, where $F(x)$ is defined by (1.2).

Proof

Using (3.6) and (3.17) we obtain in an elementary way

$$\begin{aligned} G(x) &= \int_{T_-}^x g(\xi) d\xi + \sum_{k=1}^{\infty} \int_{T_-}^x [f(\xi + kT) + f(\xi - kT)] d\xi \\ &= F(x) - \int_{-\infty}^{T_-} f(\xi) d\xi + \sum_{k=1}^{\infty} \int_{T_-}^x f(\xi - kT) d\xi + \sum_{k=1}^{\infty} \int_{T_-}^x f(\xi + kT) d\xi \\ &= F(x) - \sum_{k=1}^{\infty} \int_x^{T_+} f(\xi - kT) d\xi + \sum_{k=1}^{\infty} \int_{T_-}^x f(\xi + kT) d\xi . \end{aligned} \quad (3.19)$$

Since

$$\begin{aligned} \sum_{k=1}^{\infty} \int_x^{T_+} f(\xi - kT) d\xi &\leq \sum_{k=1}^{\infty} \int_{T_-}^{T_+} f(\xi - kT) d\xi = \int_{-\infty}^{T_-} f(\xi) d\xi \leq \epsilon_- , \\ \sum_{k=1}^{\infty} \int_{T_-}^x f(\xi + kT) d\xi &\leq \sum_{k=1}^{\infty} \int_{T_-}^{T_+} f(\xi + kT) d\xi = \int_{T_+}^{\infty} f(\xi) d\xi \leq \epsilon_+ \end{aligned}$$

for $T_- \leq x \leq T_+$, it is easily seen that (3.18) is true.

We now see that theorem 2 together with (3.17) and theorem 3 solve the problem (B) if the computation of $G(x)$ is considered to be "easy and rapid".

For practical purposes we write $g(x)$ and $G(x)$ in an explicit real form.

From (3.3) and the fact that $\phi(0) = 1$ we have $a_0 = \omega/2\pi$.

We set

$$A_k = \operatorname{Re} \phi(ik\omega) \quad (3.20)$$

$$B_k = -\operatorname{Im} \phi(ik\omega) \quad (3.21)$$

and obtain from (3.5) and (3.17)

$$g(x) = \frac{\omega}{\pi} \left\{ \frac{1}{2} + \sum_{k=1}^{\infty} (A_k \cos k\omega x + B_k \sin k\omega x) \right\} \quad (3.22)$$

$$G(x) = \frac{1}{\pi} \left\{ A + \frac{\omega x}{2} + \sum_{k=1}^{\infty} \left(\frac{A_k}{k} \sin k\omega x - \frac{B_k}{k} \cos k\omega x \right) \right\}, \quad (3.23)$$

with

$$A = -\frac{\omega T_-}{2} - \sum_{k=1}^{\infty} \left(\frac{A_k}{k} \sin k\omega T_- - \frac{B_k}{k} \cos k\omega T_- \right). \quad (3.24)$$

4. THE FUNCTION $g(x)$

From theorem 2 we know that the function $g(x)$, originally defined by (3.5), is identical with the right-hand side of (3.6). Starting from this formula we want to find an estimate for the difference $g(x) - f(x)$ in the interval (T_-, T_+) .

Because of property (c) in the definition of C and because of the fact that we may always assume that $T_- < x_0 < T_+$ is valid, it is easily seen that

$$\sum_{k=1}^{\infty} f(x + kT) \leq \frac{1}{x - T_-} \int_{T_+}^{x+T} f(\xi) d\xi + \frac{1}{T} \int_{x+T}^{\infty} f(\xi) d\xi, \quad (4.1)$$

$$\sum_{k=1}^{\infty} f(x - kT) \leq \frac{1}{T_+ - x} \int_{x-T}^{T_-} f(\xi) d\xi + \frac{1}{T} \int_{-\infty}^{x-T} f(\xi) d\xi. \quad (4.2)$$

for $T_- < x < T_+$.

Assuming that (1.6) and (1.7) are true and using (3.6), (4.1) and (4.2) we easily obtain

$$0 \leq g(x) - f(x) \leq \frac{1}{x - T_-} \int_{T_+}^{x+T} f(\xi) d\xi + \frac{1}{T_+ - x} \int_{x-T}^{T_-} f(\xi) d\xi + \frac{\epsilon_- + \epsilon_+}{T}. \quad (4.3)$$

From (4.3) two different inequalities may be derived. The first one uses the fact that the integrals in (4.3) can be estimated by ϵ_+ and ϵ_- , respectively, leading to

$$0 \leq g(x) - f(x) \leq \frac{\epsilon_+}{x - T_-} + \frac{\epsilon_-}{T_+ - x} + \frac{\epsilon_- + \epsilon_+}{T} \quad (4.4)$$

for $T_- < x < T_+$.

This estimate of the deviation of $g(x)$ from $f(x)$ is very bad in the neighbourhood of the end points of the interval. It has, however, the advantage that no further knowledge about $f(x)$ is necessary.

Let us now assume that we know the order of magnitude of $f(T_-)$ and $f(T_+)$. We then can use the following inequality

$$0 \leq g(x) - f(x) \leq f(T_-) + f(T_+) + \frac{\varepsilon_- + \varepsilon_+}{T} \quad (4.5)$$

for $T_- \leq x \leq T_+$, which is easily derived from (4.3).

5. GENERALIZATIONS AND EXAMPLES

We assume first of all that $f(x)$ is not necessarily continuous, but that $F(x)$ is absolutely continuous. It can then be shown that the representation (3.22) of $G(x)$ is still valid and that theorem 3 remains true.

This means that we can, for instance, also handle distributions where the probability density function has a square root singularity at $x = 0$, such as the $\chi^2(1)$ distribution with one degree of freedom.

Let us take, for example, the Laplace transform of the density of the $\chi^2(1)$ distribution which is given by

$$\phi(s) = \frac{1}{\sqrt{1+2s}} \quad (5.1)$$

Here the quantity T_- is given by $T_- = 0$.

To determine T_+ we use (2.14) with $c_- = -\frac{1}{2}$. It turns out that

$$T_+ = z \quad (5.2)$$

where z is the root of the equation

$$e^{1-z} - \varepsilon^2 z = 0 \quad (5.3)$$

for $z > 1$ and $\varepsilon = \varepsilon_+$.

For example, the choice of $\varepsilon = 0.005$ leads to $T_+ = 9.36$. A comparison with corresponding tables shows that this is a reasonable value for T_+ .

We now consider an example where condition (b) in the definition of C is not completely satisfied, namely the Landau distribution [Landau (1944)] which has

$$\phi(s) = e^{s \ln s} \quad (5.4)$$

Here $c_- = 0$, and c_+ can be any positive number. Therefore, only T_- can be determined by the method of theorem 1. Formula (2.13) easily yields

$$T_- = -1 - \ln \ln \frac{1}{\varepsilon_-} \quad (5.5)$$

From Börsch-Supan's (1961) table for the probability density function $f(x)$ of Landau's distribution, we find that, for example, $f(-4) = 0.339 \times 10^{-8}$. Setting $T_- = -4$ in (5.5), we obtain $\varepsilon_- \approx 10^{-20}$, which shows that the order of magnitude of T_- is very well determined by (5.5).

To find T_+ we have to use other methods. In general, one would try to use the Chebyshev inequality. Unfortunately, it cannot be applied in this case because the mean value of Landau's distribution does not exist. However, Börsch-Supan has shown that for

$$x = w + \ln w + \gamma - \frac{3}{2}, \quad (5.6)$$

$$w_1 = x \left(1 - \frac{\ln x + \gamma - 3/2}{x + 1} \right) \quad (5.7)$$

$$\gamma = 0.5772 \dots, \quad (\text{Euler's constant}) \quad (5.8)$$

and $w_1 \geq 2.52$ we have

$$f(x) = \frac{1}{w^2 + \pi^2} + R(w), \quad (5.9)$$

$$f(x) \leq \frac{1}{w_1^2 + \pi^2} + R_1(w_1), \quad (5.10)$$

with

$$|R(w)| \leq \frac{2\pi^2 - 15}{4w^4} \exp\left(\frac{0.926}{w}\right) + \frac{17\pi^2}{6w^5} = R_1(w). \quad (5.11)$$

Using these formulae it can be shown that, for example, for $T_+ \geq 200$ we have

$$P(X \geq T_+) \leq \frac{1.043}{T_+} + \frac{1}{T_+^3} + \frac{8}{T_+^4}. \quad (5.12)$$

Taking again $\varepsilon_+ = 0.005$, we have $T_+ = 209$.

The author [Schorr (to appear)] has used this method for the generation of Landau- and Vavilov-distributed random numbers, where the Vavilov distribution [Vavilov (1957)] is a generalization of the Landau distribution having the ϕ function

$$\phi(s) = \exp \left[\kappa \{ 1 + \beta^2 \gamma + (s + \beta^2) (\ln s - \text{Ei}(-s)) - e^{-s} \} \right], \quad (5.13)$$

where γ is Euler's constant, κ and β^2 are free parameters with $\kappa > 0$ and $0 \leq \beta^2 \leq 1$, and $\text{Ei}(-s)$ is the exponential integral

$$\text{Ei}(-s) = \gamma + \ln s + \int_0^1 \frac{e^{-sx} - 1}{x} dx . \quad (5.14)$$

Taking the particular values $T_- = -5$, $T_+ = 233.5$ and $N = 200$ terms in the Fourier expansion of $g(x)$ for the Landau distribution, we find that for $-3.5 \leq x \leq 100$ the deviation $g(x) - f(x)$ is smaller than 0.0005.

Finally we consider condition (c) of the definition of C. It is easily seen from the proofs of our theorems that the method can still be employed in many cases in which $f(x)$ has several maxima and minima. All that is required is that $f(x)$ should be monotonic outside some interval of length T , where T is not so large as to prevent the Fourier series from converging with reasonable rapidity.

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