PHOTON–PHOTON SCATTERING WITH SYNCHROTRON RADIATION

(Submitted to Physical Review)

*) Institute of Theoretical Science, University of Oregon, Eugene, Oregon, USA, and Institute for Theoretical Physics, University of Helsinki, Helsinki, Finland.
PHOTON–PHOTON SCATTERING WITH SYNCHROTRON RADIATION

Paul L. Csonka*)
Institute of Theoretical Science
University of Oregon, Eugene, Oregon, USA
and
Institute for Theoretical Physics
University of Helsinki, Helsinki, Finland

K.S. Kölblig
CERN, Geneva, Switzerland

ABSTRACT

The number and (energy as well as polarization) spectrum of photons scattered in the forward direction in the reaction centre-of-mass frame are calculated when two beams of synchrotron radiation are scattered on each other. The synchrotron radiation is produced by electrons or positrons. Its properties depend on the following parameters: energy and local radius of orbit curvature of the emitting particles, intensity of the emitting current, and angle of emission of the synchrotron radiation. The dependence of the calculated numbers on these parameters is exhibited.

Geneva – August 1973

*) Alfred P. Sloan Fellow.
1. BACKGROUND

Quantum electrodynamics predicts the (elastic as well as inelastic) scattering of two photons on each other. This was first noted by Halpern. Shortly thereafter, Heisenberg and Euler performed careful calculations to evaluate this effect, concentrating their attention on low-energy scattering. Akhiezer studied photon-photon scattering at high energies. Several years later, Karplus and Neuman gave a general discussion of all the photon-photon scattering amplitudes in quantum electrodynamics up to fourth order, and evaluated them for certain selected values of the scattering parameters. The subject was well reviewed and simplified and the discussion was made more complete by Jauch and Rohrlich. Subsequently Sannikov, De Tollis and Violini applied the techniques of dispersion relations and succeeded in considerably simplifying the treatment of this, nevertheless, still complicated theoretical problem.

The calculated photon-photon cross-sections are very small, and are beyond the range of traditional experimental techniques. For photons in the visible spectrum, the elastic cross-section is about $10^{-6.5}$ cm$^2$. Although the first calculations were performed several decades ago, elastic scattering of real photons on each other in vacuum was never observed. During the last few years, experimental technology started on a new "spiral" of development which is still in progress. This progress is tied to the availability of new accelerators, storage rings and the accompanying detection devices. The question arises as to whether the new technology can be used to study photon-photon scattering at either low or at high energies. In earlier works by one of the authors various possibilities were surveyed, the obtainable counting numbers were estimated, and it was concluded that several new approaches may become feasible in the foreseeable future. It was found that one of the most promising techniques would consist of using storage rings to produce two beams of synchrotron radiation and scattering these two beams on each other.

At the present time, several new electron storage rings are being planned or are under construction. The time seems appropriate to make a more detailed study of the possibility of observing photon-photon scattering by colliding two beams of synchrotron radiation.

2. PURPOSE OF THE PRESENT PAPER

Our purpose is to calculate the properties of elastically scattered photons which are produced in the collision of two beams of synchrotron radiation.

The two beams of synchrotron radiation are produced by electron or positron beams circulating in one or more rings (see Fig. 1). One can use two beams in
one ring, or two rings with one beam in each. Either or both beams can be electron or positron beams. In the following, unless explicitly stated otherwise, "electron" means either an electron or a positron. The energy spectrum and polarization of the radiation can be varied by changing the energy of the circulating particles, by varying the angle of emission of the radiation used, and by adjusting the local radius of curvature of the electron orbits at the points where the synchrotron radiation is emitted. The influence of these parameters on the properties of the scattered photons is studied in this paper.

Of course, the local radius of curvature may differ from the average radius of curvature. One can produce a "kink" or "hump" in the circulating particle orbit. This will locally increase the amount of synchrotron radiation emitted while the particles travel through the kink or hump, but will change only by a small amount the average (over one revolution) energy loss due to synchrotron radiation per circulating particle. The storage ring(s) can be used in the customary way for the usual experiments, and in addition, at the same time, the circulating particles can be employed to produce the two beams of synchrotron radiation for the purposes of the photon-photon scattering experiments discussed here.

In the type of experiment outlined in Fig. 1, the average reaction centre-of-mass frame is the laboratory rest frame, and the calculated differential cross-section as a function of angle does not have any sharp peaks in this frame. One can measure the spectrum of photons leaving the interaction region along any particular line. The elimination of the background is easiest if one measures photons leaving with a momentum perpendicular to the plane (x, y) in which both electron beams circulate. Detectors can be placed far away from this plane, and the relatively high energy photons detected without strong background interference. At least from this point of view, the more interesting quantities we could have calculated are the polarization and energy spectrum and the total number of photons leaving the interaction region with a momentum perpendicular to the (x, y) plane. Instead, we chose to calculate at first the properties of those photons which are scattered in the forward direction in the reaction centre-of-mass frame. The calculation of these is somewhat simpler and it seemed reasonable to perform this calculation first. The results are exact for forward scattering and are approximately valid near the forward direction. They are reported in the present paper and are summarized in the figures. At a later time we intend to study the spectrum and number of photons scattered perpendicularly to the (x, y) plane.

In the Appendix we list what we believe are misprints or arithmetic errors in the published literature concerning photon-photon scattering.
3. **CALCULATIONS**

At each point B where synchrotron radiation is emitted, we define a local Cartesian coordinate frame as explained in Fig. 2. The direction of any vector \( \vec{k} \) originating at B can be characterized in this coordinate system by two angles \( \phi \) and \( \psi \).

With each photon of momentum \( \vec{k} \), emitted at B, we associate a Cartesian coordinate system whose axes are \( \vec{e}_\parallel, \vec{e}_\perp \) and \( \vec{e}_z \) (see Fig. 2). The component of the electric vector \( \vec{E} \) parallel to \( \vec{e}_\parallel \) is \( E_\parallel \), and \( E_\perp \) is defined similarly. Circular polarization states are defined by the components

\[
E_\pm = \frac{1}{\sqrt{2}} (E_\parallel \mp iE_\perp) .
\]

When the complete orbit of the circulating electrons is a perfect circle and radiation emitted at every point along the orbit is permitted to reach a certain point A, then, due to the circular symmetry, the observed synchrotron radiation will depend only on the distance \( (A, C) \) and on the \( z \) coordinate of the position vector of A. It will not depend on the azimuthal angle of A, which determines the position of the projection of A onto the \((x, y)\) plane. When the orbit is not a perfect circle, or if radiation from some orbit points is not permitted to reach A, then the observed synchrotron radiation will depend on the azimuthal angle.

For cases of practical interest, the synchrotron radiation emitted at any point B is strongly peaked along \( y \), so that only a relatively small portion of the orbit contributes significantly to the radiation observed at any particular point A.

If we are interested in observing the synchrotron radiation in any finite region \( T \), and we limit the possible positions of A to \( T \), then within \( T \) the observed radiation will be independent of the azimuthal angle of A, provided only that radiation is permitted to reach \( T \) from a large enough circular section of the particle orbit.

For cases of practical interest, it is easy to fulfill the latter condition, and in the rest of this paper we assume that this is done. The \( \phi \) dependence of the radiation emitted at B can then be neglected.

We will use the following conventions and notation:

<table>
<thead>
<tr>
<th>Term</th>
<th>Meaning</th>
</tr>
</thead>
<tbody>
<tr>
<td>&quot;local&quot;</td>
<td>means value calculated at point B</td>
</tr>
<tr>
<td>&quot;electron&quot;</td>
<td>means either electron or positron, unless</td>
</tr>
<tr>
<td></td>
<td>explicitly stated otherwise</td>
</tr>
<tr>
<td>e</td>
<td>electron charge</td>
</tr>
<tr>
<td>c</td>
<td>velocity of light in vacuum</td>
</tr>
<tr>
<td>m</td>
<td>rest mass of the electron</td>
</tr>
</tbody>
</table>

\[
(2)
\]
\( \nu \) : velocity of circulating electrons
\( \mathcal{E} \) : local energy of the circulating electron
\( \gamma \) : \( \mathcal{E}/mc^2 \)
\( R \) : local radius of curvature of circulating electrons
\( \omega \) : \((2\pi/\text{period})\) of emitted synchrotron radiation
\( \mathbf{k} \) : photon wave vector (i.e. momentum/h)
\( \omega_c \) : \( \frac{3}{2} \frac{c}{R} \gamma^3 \)
\( \xi \) : \( \frac{1}{2} \frac{\omega}{\omega_c} (1 + \gamma^2 \psi)^{3/2} \)
\( P_\parallel, P_\perp \) : power radiated by one circulating electron in the form of synchrotron radiation polarized, respectively, along \( \hat{\mathbf{e}}_\parallel \) and \( \hat{\mathbf{e}}_\perp \)
\( \rho_{\nu} (\mathbf{r}_A, \omega), \rho_{\nu} (\mathbf{r}_A, \omega) \) : density at point \( A \) of photons of frequency \( \omega \), emitted by beam No. \( \nu \), polarized, respectively, along \( \hat{\mathbf{e}}_\parallel \) and \( \hat{\mathbf{e}}_\perp \); the position vector of point \( A \) is measured from point \( C \)
\( \rho_{\nu} (\mathbf{r}_A, \omega), \rho_{\nu} (\mathbf{r}_A, \omega) \) : density at point \( A \) of photons of frequency \( \omega \), emitted by beam No. \( \nu \), with polarization \( E_+ \) and \( E_- \), respectively
\( n \) : number of electrons in beam per unit time, passing through a surface perpendicular to the beam
subscript \( \nu \) : \( (\nu = 1, 2) \) means that the subscripted quantity refers to beam No. \( \nu \), or to photons emitted by that beam
\( ' \) : a prime indicates that the quantity is evaluated in the reaction centre of mass frame (see below).

The intensity of the synchrotron radiation emitted at \( B \) has been calculated and evaluated for various values of the parameters\(^1 \)\(^{-12} \). The result is

\[
\frac{\partial}{\partial \psi} \frac{\partial}{\partial \omega} P_\parallel = \frac{3}{4\pi^2} \frac{e^2}{R} \left( \frac{\omega}{\omega_c} \right)^2 \gamma^2 (1 + \gamma^2 \psi)^2 K_i^2(\xi), \tag{3a}
\]

\[
\frac{\partial}{\partial \psi} \frac{\partial}{\partial \omega} P_\perp = \frac{3}{4\pi^2} \frac{e^2}{R} \left( \frac{\omega}{\omega_c} \right)^2 \gamma^2 (1 + \gamma^2 \psi)^2 \frac{\gamma^2 \psi^2}{1 + \gamma^2 \psi^2} K_i^2(\xi), \tag{3b}
\]

\[
\frac{E_\parallel}{E_\perp} = i \frac{(1 + \gamma^2 \psi)^{1/2}}{\gamma \psi} \frac{K_i[\xi]}{K_i[\xi]} \equiv i \bar{Q}. \tag{4}
\]
Both the $P_+$ and $P_-$ are found to go to zero rapidly for $\psi \geq 1/\gamma$.

The number of photons emitted per unit length of beam No. $\nu$, is inversely proportional to $\nu$. In all cases of interest $\nu_1 \approx \nu_2 \approx c$, and we will assume this. The emitted photons also travel with velocity $c$, so that

$$
\rho_{\nu}(\vec{P}_\nu, \omega_\nu) = \frac{1}{c^2} n_\nu \frac{1}{\hbar \omega_\nu} \int d\psi_\nu \frac{\partial}{\partial \psi_\nu} \frac{\partial}{\partial \omega_\nu} P_{\nu}(\omega_\nu, \psi_\nu) G_{\nu}(\psi_\nu, \vec{P}_\nu). 
$$

(5)

where $\int d\psi_\nu$ is an integration over all those values of $\psi_\nu$ which contribute (from various points in the beam) photons with frequency $\omega_\nu$ to point $A$, and $G_{\nu}(\psi_\nu, \vec{P}_A)$ is a geometrical factor.

For simplicity, let us make the assumption that the beam has zero cross-section. This is a good approximation, if the distance of point $A$ from all points with position vectors $\vec{R}_i$ of the beam is much larger than the largest diameter of the beam. Furthermore, let us also assume that $\gamma_\nu$ is high enough, so that only a very narrow $\psi_\nu$ range, $\Delta \psi_\nu$, will contribute to the $\int d\psi_\nu$. If the integrand in Eq. (4) varies only negligibly over the range $\Delta \psi_\nu$, then it can be taken outside the integral sign, and the integral performed immediately

$$
\rho_{\nu}(\vec{P}_\nu, \omega_\nu) = \frac{1}{c} n_\nu \frac{1}{\hbar \omega_\nu} \frac{\partial}{\partial \psi_\nu} \frac{\partial}{\partial \omega_\nu} P_{\nu}(\omega_\nu, \psi_\nu) \Delta \psi_\nu G_{\nu}(\psi_\nu, \vec{P}_A). 
$$

(6)

In this approximation, only a small interval $[R - \Delta R/2, R + \Delta R/2]$ along the beam contributes photons to point $A$, and all contributing points are approximately a distance of $|\vec{r}_A - \vec{R}|$ away from point $A$. The geometrical factor $G(\psi, \vec{r}_A)$ is easy to evaluate in this approximation. For example, if $A$ is located in the plane of the circle tangent to the beam at $B$, then

$$
G(\psi, \vec{r}_A) = \frac{u}{|\vec{r}_A - \vec{R}|} \frac{R}{\left[ R^2 + \left| \vec{r}_A - \vec{R} \right|^2 \right]^{3/2}}, 
$$

(7)

where $u$ is the unit length.

We will describe elastic photon-photon scattering in the reaction centre-of-mass frame. In this frame, we denote the wave vectors of the two incoming photons by $\vec{k}_1'$ and $\vec{k}_2'$, those of the outgoing photons by $\vec{k}_1''$ and $\vec{k}_2''$. We define the unit vector $\vec{e}_\mu'' = \vec{k}_\mu' \times \vec{k}_\mu''$, and the unit vectors $\vec{e}_\mu' = \vec{k}_\mu' \times \vec{e}_\mu''$. The scattering angle $\theta'$
is defined as the angle between $\vec{k}_1'$ and $\vec{k}_4'$, and takes $\vec{k}_1'$ into $\vec{k}_4'$, through a right-handed rotation around $\vec{e}_{1}'$.

We describe the polarization state of the photon, whose wave vector is $\vec{k}_\mu'$, in the right-handed Cartesian frame whose three axes are defined by $\vec{e}_{1}'$, $\vec{e}_{\mu}'$, and $\vec{e}_{\mu}'$. For example, the electric field $\vec{E}_1$ emitted by the first beam has two components, $E_{1\mu}'$ parallel to $\vec{e}_{1}'$, and $E_{\mu}'$ parallel to $\vec{e}_{\mu}'$. These describe photons polarized linearly along $\vec{e}_{1}'$ and $\vec{e}_{\mu}'$, respectively. Circular polarization states are defined by the components

$$E_{2\mu}' = \frac{1}{\sqrt{2}} \left( E_{1\mu}' + iE_{\mu}' \right).$$  

(8)

In the following, we will represent all photon polarization states as a linear superposition of circular polarization states. We denote by $\lambda_{\mu}'$ the circular polarization state of the photon whose wave vector is $\vec{k}_\mu'$, where $\lambda_{\mu}' = +$ or $-$ for $\mu = 1, 2, 3, 4$.

The differential (in $\theta'$) elastic cross-section of photons with frequency $\omega'$ in the reaction centre-of-mass frame, is

$$\frac{d}{d\theta'} \sigma_{\lambda_{1}'\lambda_{2}'\lambda_{3}'\lambda_{4}'}(\theta', \omega') = \frac{\alpha^2 \hbar^2}{4\pi^2 m^2 c^4} \left\lvert \frac{1}{\omega'} M_{\lambda_{1}'\lambda_{2}'\lambda_{3}'\lambda_{4}'}(\theta', \omega') \right\rvert^2. $$

(9)

The functions $M$ have been calculated by Karplus and Neuman in the special cases $\theta' = 0$, and $\theta' = \pi/2$, in terms of three transcendental functions, $B$, $T$ and $I$, defined by them. De Tollis succeeded in expressing $M$ as a function of $B$, $T$ and $I$ for all values of $\theta'$. The definitions of $B$, $T$ and $I$ in terms of the kinematical variables of the scattering, as well as the expression of $M$ in terms of $B$, $T$ and $I$, are lengthy, and will not be reproduced here. They can be found in Refs. 5 and 8 (see Appendix for certain misprints).

Suppose that two beams of synchrotron radiation collide and scatter in an interaction region $T$. The number of events per unit time, in which two photons with polarizations $\lambda_3$ and $\lambda_4$ are produced, is

$$N_{\lambda_3\lambda_4} = \frac{n_1 n_2 \alpha^4}{4\pi^2 c^2 m^2} \int_{-\pi/2}^{+\pi/2} d\theta' \int_0^{\pi/2} d\omega_1 \int_0^{\pi/2} d\omega_2 \int_0^{\pi/2} d\psi_1 \int_0^{\pi/2} d\psi_2 \left\lvert \frac{1}{\omega'} M_{\lambda_1\lambda_2\lambda_3\lambda_4}(\theta', \omega') \right\rvert^2 \times$$

$$\times \left[ \left\lvert \frac{\partial}{\partial \omega_1} \frac{\partial}{\partial \psi_1} P_{\lambda_1}(\omega_1, \psi_1) \right\rvert^2 + \left\lvert \frac{\partial}{\partial \omega_2} \frac{\partial}{\partial \psi_2} P_{\lambda_2}(\omega_2, \psi_2) \right\rvert^2 \right] \times \int d\overline{\Phi}_B G_1(\psi_1, \overline{\Phi}_A) G_2(\psi_2, \overline{\Phi}_A),$$

(10)
where \( \int d\vec{r}_A \) is taken over the whole interaction volume, \( \omega' = \sqrt{\omega_1 \omega_2} \), and using Eqs. (4) and (8) we define (ignoring an uninteresting over-all phase) \( P_{\lambda} \) to be proportional to \( \xi^2 \):

\[
\frac{\partial}{\partial \psi_u} \frac{\partial}{\partial \omega_u} P_{\lambda_u} = \frac{\partial}{\partial \psi_v} \frac{\partial}{\partial \omega_v} P_{\lambda_v} = \frac{P_\lambda + P_\lambda}{2(1 + Q^2_\lambda)} \left( 1 + \lambda_\lambda Q_{\lambda\lambda} \right)^2, \quad (\lambda_v = \pm). \tag{11}
\]

The scattering physics is contained in the first two rows of Eq. (10). The quantity in the third row in Eq. (10) is a purely geometrical factor and can be evaluated for any experimental arrangement. In this paper we will do so only for some of the most likely arrangements.

If the conditions stated before Eq. (7) are satisfied, then

\[
G \equiv \int d\vec{r}_A \left[ G_1(\psi_1, \vec{r}_A) G_2(\psi_2, \vec{r}_A) \right] = \frac{u^2}{|\vec{r}_A - \vec{R}_1||\vec{r}_A - \vec{R}_2|} \frac{R_1 R_2 V}{\left[ R_1^2 + |\vec{r}_A - \vec{R}_1|^2 \right]^{3/2} \left[ R_2^2 + |\vec{r}_A - \vec{R}_2|^2 \right]^{3/2}}, \tag{12}
\]

where \( V \) is the volume of the interaction region. If, in addition,

\[
R_1 = R_2 \equiv R, \quad |\vec{r}_A - \vec{R}_1| = |\vec{r}_A - \vec{R}_2| \equiv |\vec{r}_A - \vec{R}|, \tag{13}
\]

and if both beams are electron beams or both beams are positron beams, then \( \Delta \psi_1 = -\Delta \psi_2 \), but if one of the beams contains electrons and the other positrons, then \( \Delta \psi_1 = +\Delta \psi_2 \). Equation (12) now reduces to

\[
G = g_0 \delta[\psi_1 - (\pm 1)\psi_2] \tag{14}
\]

where \( \delta(x) \) is the usual delta function of \( x \), and

\[
G = \frac{u^2 R^2 V}{|\vec{r}_A - \vec{R}|^2 \left[ R^2 + |\vec{r}_A - \vec{R}|^2 \right]} \tag{15}
\]

Here the + sign holds if one beam is an electron beam and the other is a positron beam, otherwise the - sign holds. For example, consider the case when both beams have a vertical emittance of \( 5 \times 10^{-4} \) cm rad. At points \( B_1 \) and \( B_2 \) focus the beam to a vertical height of \( 10^{-2} \) cm. Then \( 3 \times 10^{-1} \) away from these points, the beam
height will be $\approx 2.5 \times 10^{-2}$ cm. Let $\gamma = 1.2 \times 10^4$, so that $\Delta \psi \approx 1/\gamma \approx 8 \times 10^{-5}$.

Choose $|\vec{r}_A - \vec{R}| = 10^3$ cm, so that at the interaction region the synchrotron radiation will be appreciable within a distance $(2.5 \times 10^{-2} + 10^3 \times 8 \times 10^{-5}) \approx 10^{-1}$ cm from the common plane of the two beams. Let the interaction region extend $10^{-1}$ cm above and below this plane, and let its projection onto this plane be a circle with radius 3 cm. Choose $R = 10^2$ cm. Then $g \approx 2 \times 10^{-8}$.

In practice, one will want to increase the counting rate by increasing the interaction region. It may then happen that $\psi_1$ and $\psi_2$ can no longer be considered constant over the whole interaction region. In this case, if the interaction is such that all points $A$ in this region satisfy Eq. (13) to a good approximation (this happens if the region is half-way between $B_1$ and $B_2$, and has sufficiently small dimensions), and furthermore if the dimensions of the interaction region are small enough, so that $|\vec{r}_A - \vec{R}| \approx$ constant over the whole region, then from Eq. (14) we find

$$N_{\lambda_3 \lambda_4} = \frac{n_3 n_4 a^4}{4 \pi m c^2} \frac{\pi}{2 \alpha} \int_{-\pi/2}^{\pi/2} d\theta' \int_{-\pi/2}^{\pi/2} d\omega_1 \int_{-\pi/2}^{\pi/2} d\omega_2 \int_{-\pi/2}^{\pi/2} d\psi_1 \left| \sum_{\lambda_1 \lambda_2} \frac{1}{\omega_1 \omega_2} M_{\lambda_1 \lambda_2 \lambda_3 \lambda_4} (\theta'_1, \omega'_1) \times \right| \frac{\partial}{\partial \omega_1} P_{\lambda_1} (\omega_1, \psi_1) \left| \frac{\partial}{\partial \psi_1} P_{\lambda_2} (\omega_2, \psi_2 = \pm \psi_1) \right|^{1/2} g$$

(16)

Since the polarization of the emitted synchrotron photons depends on $\psi$, the $\int d\psi$ was not neglected in Eq. (16). However, if the dimensions of the interaction region are sufficiently small compared to $|\vec{r}_A - \vec{R}|$, then all photons which are emitted at one common point and reach the interaction region can be considered as having essentially parallel momenta (narrow beam approximation). If the two photon beams collide head on in the laboratory, then the number of photon scattering events\textsuperscript{11} per unit time for a certain set of final polarizations is obtained from $N_{\lambda_3 \lambda_4}$ by dividing by $2\pi$, omitting $\int d\theta'$, and evaluating $M$ at $\theta' = 0$:

$$\frac{\partial}{\partial \omega'} N_{\lambda_3 \lambda_4} (\omega' = 0) = \frac{\partial}{\partial \omega'} \mathcal{N}_{\lambda_3 \lambda_4} (\theta' = 0) g,$$

(17)

where
\[
\frac{3}{2} \frac{\partial}{\partial \omega} \sigma_{\lambda \lambda' q}(\theta' = 0) \equiv \frac{n_1 n_2 \alpha^4}{4 \pi^2 m^2 c^5} \int_0^\infty \frac{d \omega_1}{\omega_1} \int_{-\pi/2}^{\pi/2} d \psi_1 \int_{-\pi/2}^{\pi/2} d \psi_2 \left| \sum_{\lambda_1 \lambda_2} \frac{1}{\omega_1} M_{\lambda_1^* \lambda_2^* \lambda' q}(\theta' = 0, \omega') \right|^2 \\
\times \left[ \frac{\partial}{\partial \omega_1} \frac{\partial}{\partial \psi_1} P_{\lambda_1}(\omega_1, \psi_1) \right] \left[ \frac{\partial}{\partial \omega_2} \frac{\partial}{\partial \psi_2} P_{\lambda_2}(\omega_2, \psi_2 = (\pm 1) \psi_2) \right] \right|^{1/2} \quad (17a)
\]

The number of photons with a certain frequency and given polarization state scattered in a direction parallel to the momenta in one of the incoming photon beams, say the first one\(^{1a}\), is

\[
\frac{3}{2} \frac{\partial}{\partial \omega} \frac{\partial}{\partial \theta} N_{\lambda q}(\theta' = 0, \omega) = \sum_{\lambda_\alpha} \frac{\partial}{\partial \omega_1} \frac{\partial}{\partial \theta} \sigma_{\lambda_\alpha q}(\theta' = 0, \omega_1) g , \quad (18)
\]

where

\[
\frac{3}{2} \frac{\partial}{\partial \omega_1} \frac{\partial}{\partial \theta} \sigma_{\lambda_\alpha q}(\theta' = 0, \omega_1) = \frac{n_1 n_2 \alpha^4}{4 \pi^2 m^2 c^5} \int_0^\infty \frac{d \omega_2}{\omega_2} \int_{-\pi/2}^{\pi/2} d \psi_1 \int_{-\pi/2}^{\pi/2} d \psi_2 \left| \sum_{\lambda_1 \lambda_2} \frac{1}{\omega_1} M_{\lambda_1^* \lambda_2^* \lambda' q}(\theta' = 0, \omega') \right|^2 \\
\times \left[ \frac{\partial}{\partial \omega_1} \frac{\partial}{\partial \psi_1} P_{\lambda_1}(\omega_1, \psi_1) \right] \left[ \frac{\partial}{\partial \omega_2} \frac{\partial}{\partial \psi_2} P_{\lambda_2}(\omega_2, \psi_2 = (\pm 1) \psi_2) \right] \right|^{1/2} \quad (18a)
\]

The scattering physics is contained in

\[
\frac{\partial}{\partial \theta} \sigma_{\lambda_\alpha q}(\theta' = 0) ,
\]

and in

\[
\frac{\partial}{\partial \omega_1} \frac{\partial}{\partial \theta} \sigma_{\lambda_\alpha q}(\theta' = 0, \omega_1) .
\]

These quantities have been computed for certain values of the parameters, and are shown in Figs. 4, 5 and 6. They are normalized so that \(n_1 n_2 = (6.24)^2 \times 10^{36}\), corresponding to two beams, each with an intensity of 1 A.

4. COMPUTATION

The integrals in Eqs. (17a) and (18a) have been evaluated by numerical integration on a CDC 7600 computer at CERN. Because of the fact that for \(\theta' = 0\)

\[
M_{++} = M_{--} , \quad M_{++} = M_{--} , \quad M_{+-} = M_{-+} ,
\]

- **The number of photons with a certain frequency and given polarization state scattered in a direction parallel to the momenta in one of the incoming photon beams, say the first one, is...**

- **The scattering physics is contained in...**

- **These quantities have been computed for certain values of the parameters, and are shown in Figs. 4, 5 and 6. They are normalized so that...**

- **The integrals in Eqs. (17a) and (18a) have been evaluated by numerical integration on a CDC 7600 computer at CERN. Because of the fact that for \(\theta' = 0\)...**
and that all other $M$ are zero, the sum over $\lambda_1$ and $\lambda_2$ simplifies considerably. In fact, it reduces to two terms in the cases $\lambda_3 = +, \lambda_4 = +; \lambda_3 = -, \lambda_4 = -,$ and to one term for $\lambda_3 = +, \lambda_4 = -; \lambda_3 = -, \lambda_4 = +.$

Further, it is found that for both the electron-electron and electron-positron case

$$\frac{3}{\Omega} \sigma_{++}(\theta' = 0) = \frac{3}{\Omega} \sigma_{--}(\theta' = 0)$$

and

$$\frac{3}{\Omega} \sigma_{+-}(\theta' = 0) = \frac{3}{\Omega} \sigma_{-+}(\theta' = 0).$$

These equalities are also true for the quantity defined in Eq. (18a). After some experimentation in order to determine the behaviour of the integrand, the following computing procedure was found adequate. First, the integration was performed over $\omega_2$. Instead of the upper limit $\omega$, the straight line $\omega' = \omega^* - \omega_1$ was taken, where

$$\omega^* = 2\omega_c \xi(1 + \gamma^2\psi^2)^{1/2}$$

is defined in such a way that the argument $\xi = \xi^*$ in the modified Bessel function is large enough to make the contribution of the remaining integrand to the integral negligible. It was found that $\xi^* = 15$ is a safe value, which could be made smaller in most cases. Since $|M_{+++}|^2$ and $|M_{++-}|^2$ are not differentiable at $\omega' = (\omega_1 \omega_2)^{1/2} = 1,$ care was taken to integrate separately on both sides of the hyperbola $\omega_2 = 1/\omega_1.$ For $\lambda_3 = +, \lambda_4 = -, this separation is not necessary.

The next step, the integration over $\psi,$ was performed with a trapezoidal rule, using unequally spaced points in the range $|\psi| \leq 5 \times 10^{-4},$ instead of $|\psi| < \pi/2.$ This reduced range was found to be sufficient. The results obtained correspond to Eq. (18a).

In order to obtain the values of Eq. (17a), another trapezoidal rule integration was applied, taking in most cases $0.01 \leq \omega_1 \leq 30,$ except for $R \geq 100$ and $\gamma = 5870.5,$ where a range $0.001 \leq \omega_1 \leq 3$ is adequate. These results have been tested in some cases by integrating over both $\omega_1$ and $\omega_2$ with an adaptive Gaussian quadrature rule.

For the computation of $K_{\lambda_3}(x),$ $K_{\lambda_3}(x),$ the Chebyshev approximations given by Luke$^{16}$ were used.
ACKNOWLEDGEMENTS

We wish to thank R. Haensel, R. Karplus and F. Rohrlich for helpful correspondence concerning this problem. A detailed correspondence with B. De Tollis was particularly valuable in aiding us to locate and confirm misprints in the published literature.

One of us (P.L.C.) wishes to acknowledge the hospitality of the Theory Division at CERN, where part of this work was done.
Table II: 2nd row, last column: for $-1 + 2/\omega^2$ read $-1 + 2/\omega^3$; 
for $M_{+++}$ read $M_{++--}$
for $M_{++-}$ read $M_{++++}$.

Table III: 2nd row, 2nd column: for $-2/\omega^2$ read $-1/\omega^2$.

Fig. 2: this figure should be replaced by Fig. 7 of the present paper.

Table IV: there are several inaccurate numerical values in this table.

ii) In: B. De Tollis, Nuovo Cimento, 32, 757 (1964) (the following misprints are pointed out by B. De Tollis, in the second of Refs. 8, footnote 1)

p. 759: line 10, for "...interchanges 2 ↔ 3 and 3 ↔ 4..."
read "...interchanges 2 ↔ 4 and 3 ↔ 4..."

Eq. (II.4): lower limit of integration is 1 instead of 0.

Further, Eq. (II.3): for \{|s(a + b(r))^2\} read |s(a + b(r))^2|
for \{|r(a + b(s))^2\} read |r(a + b(s))^2|.

iii) In: B. De Tollis, Nuovo Cimento, 35, 1182 (1965) (pointed out by B. De Tollis and G. Violini in the third of Refs. 8, footnote 1)

Eq. (10): in the last term of the fourth question:
for \(M^{(1)}_{1122}(s, t)\) read \(M^{(1)}_{1122}(t, s)\).

p. 294: the footnote is in error, and the weight factors in Ref. 5 are correct.

v) In: J. Schwinger, Phys. Rev. 75, 1912 (1949),

Eq. (II.37): for \(\pi^2/4\) read \(\pi^2/8\).
Figure captions

Fig. 1: Two circulating (electron or positron) beams produce intense synchrotron radiation at the points where the local radius of curvature of the beam is small. The two beams may circulate in one ring, or (as drawn in the figure) in two rings. Two beams of intense synchrotron radiation are directed towards the interaction region, where photon-photon scattering takes place.

Fig. 2: The figure shows the axes \( \mathbf{x}, \mathbf{y} \) and \( \mathbf{z} \) of a Cartesian coordinate frame located at the point \( B \) of emission of the synchrotron radiation. The \( x \) axis is parallel to the position vector of \( B \) measured from the point \( C \) which is the centre of the circle tangent to the particle orbit at \( B \). When the circulating particle is an electron, then \( \mathbf{y} \) is parallel to the velocity of the circulating particle at \( B \). If the particle is a positron, \( \mathbf{y} \) is antiparallel to its velocity at \( B \). The \( \mathbf{z} \) is parallel to \( \mathbf{z} \). The unit vector \( \mathbf{e}_x \) is parallel to \( \mathbf{z} \), and \( \mathbf{e}_x = \hat{\mathbf{r}} \times \mathbf{e}_y \).

Fig. 3: Synchrotron radiation emitted at any point is concentrated within the angular interval \( \Delta \psi \). Only points between \( \left[ R - \Delta R / 2 \right] \) and \( \left[ R + \Delta R / 2 \right] \) contribute significantly to the radiation at \( A \). If \( R \gg \Delta R_+ \) \( \Delta R_- \), then \( \Delta R_+ \approx \Delta R_- \).

Fig. 4: \( \left( \partial / \partial \Omega' \right) \overline{\sigma}_{\lambda_3 \lambda_4} (\theta' = 0) \) as a function of \( R \), for \( \gamma = 5.8715 \times 10^3 \) and \( 1.17421 \times 10^9 \), when both circulating beams are 1 A electron beams \( \left[ \text{i.e. } \psi_2 = -\psi_1 \text{ in the argument of } P_{\lambda_2} \text{ in Eq. (16)} \right] \).

a) \( \lambda_3 = +, \lambda_4 = +, \) or \( \lambda_3 = -, \lambda_4 = - \),

b) \( \lambda_3 = +, \lambda_4 = -, \) or \( \lambda_3 = -, \lambda_4 = + \).

Fig. 5: \( \left( \partial / \partial \Omega' \right) \overline{\sigma}_{\lambda_3 \lambda_4} (\theta' = 0) \) as a function of \( R \), for \( \gamma = 5.8715 \times 10^3 \) and \( 1.17421 \times 10^9 \), when one of the two circulating beams is a 1 A electron beam, the other is a 1 A positron beam \( \left[ \text{i.e. } \psi_2 = +\psi_1 \text{ in the argument of } P_{\lambda_2} \text{ in Eq. (16)} \right] \).

a) \( \lambda_3 = +, \lambda_4 = +, \) or \( \lambda_3 = -, \lambda_4 = - \),

b) \( \lambda_3 = +, \lambda_4 = -, \) or \( \lambda_3 = -, \lambda_4 = + \).

Fig. 6: \( \left( \partial / \partial \Omega' \right) \left( \partial / \partial \omega_1 \right) \overline{\sigma}_{\lambda_3 \lambda_4} (\theta' = 0, \omega_1) \) as a function of \( \omega_1 \), for \( R = 10 \text{ cm and } 10^2 \text{ cm, and for } \gamma = 5.8715 \times 10^3 \) and \( 1.17421 \times 10^6 \), when one of the two circulating beams is a 1 A electron beam, the other is a 1 A positron beam \( \left[ \text{i.e. } \psi_2 = +\psi_1 \text{ in the argument of } P_{\lambda_2} \text{ in Eq. (16)} \right] \).

(Note the different scales.)

a) \( \lambda_3 = +, \lambda_4 = +, \) or \( \lambda_3 = -, \lambda_4 = - \),

b) \( \lambda_3 = +, \lambda_4 = -, \) or \( \lambda_3 = -, \lambda_4 = + \).
Fig. 7: The figure shows \( V = \left[ \text{Re} \left( \frac{M}{\omega} \right) \right]^2, \) \( R = \left[ \text{Im} \left( \frac{M}{\omega} \right) \right]^2 \) and \( T = \frac{|M|^2}{\omega^2} \) evaluated at \( \theta' = \pi/2. \) (Note the different scales.)

a) For \( \lambda_1 = +, \lambda_2 = +, \lambda_3 = +, \lambda_4 = + \),
b) for \( \lambda_1 = +, \lambda_2 = +, \lambda_3 = -, \lambda_4 = - \),
c) for \( \lambda_1 = +, \lambda_2 = -, \lambda_3 = +, \lambda_4 = - \),
d) for \( \lambda_1 = +, \lambda_2 = +, \lambda_3 = +, \lambda_4 = - \).

For the other \( \lambda \) values, the \( V, R \) and \( T \) can be obtained from the above ones with the help of the symmetry relations:

\[ M_{+++-} = M_{----}, \quad M_{++-+} = M_{----}, \quad M_{+-++} = M_{----} = M_{+++-} = M_{++--}, \]

and all the remaining \( M \)'s are equal to each other and to \( M_{+++-} \).

The \( \omega \) is measured in units of \( mc^2 \).
Fig. 1
Fig. 5
Fig. 7