STUDIES OF THE ELECTRON-RING DYNAMICS IN A MODIFIED BETATRON FOR LARGE RING DISPLACEMENTS FROM THE MINOR AXIS

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A spatially two dimensional \((r, z)\) with three velocity components \((v_r, v_\theta, v_z)\) particle-in-cell (PIC) computer-simulation code is used to study the electron-ring dynamics in a magnetic-field configuration that is very similar to that of the Naval Research Laboratory (NRL) modified-betatron experiment. The electron-ring dynamics have been simulated over approximately 75 revolutions around the major axis, i.e., over several poloidal bounce periods. By comparing the electron-ring dynamics in an idealized magnetic field with that in the experiment, it has been determined that the field-index spatial fluctuations that occur in the experiment are harmless to high-current rings. In addition, the computer-simulation results have confirmed our theoretical predictions concerning the variation of the ring's equilibrium position with the vertical field as well as the existence, in the ring-centroid instability gap, of ring orbits having figure-eight shape.

1. INTRODUCTION

The modified betatron\(^1\) is a toroidal accelerator that has the potential to generate high-current electron beams. Its field configuration includes a strong toroidal magnetic field, in addition to the time-varying betatron field that is responsible for the acceleration. The toroidal magnetic field substantially improves the stability of the circulating electron ring, in particular during injection, i.e., when the ring energy is low.

The modified-betatron concept has been studied extensively,\(^1\)\textsuperscript{25} both analytically and numerically, during the last few years. These studies have addressed almost all the known critical issues of the concept, such as injection, equilibrium, resonances, and collective instabilities. The only major issue that has not been addressed yet in detail is the ring extraction from the modified-betatron field configuration. Recently, a program has been initiated at NRL to develop ring-extraction schemes, with subsequent implementation of these schemes to the experiment.

Up to now, most of the analytical studies have been carried out in the linear regime assuming idealized magnetic-field configurations. Similarly, most of the
numerical studies have assumed idealized magnetic-field configurations and have been carried out in the time scale of the ring-bounce (poloidal) period.

In this paper, we use a particle-in-cell (PIC) computer-simulation code named MOBE-PIC (MOdified BEtatron–Particle In Cell) to study the ring dynamics in a magnetic-field configuration that is very similar to that of the NRL modified-betatron experiment. The electron-ring dynamics have been simulated over approximately 75 revolutions around the major axis, i.e., over several bounce periods. Furthermore, the electron-ring parameters have been selected to be the same as those of the experiment.

By comparing the electron-ring dynamics in idealized magnetic fields with those in the experiment, it has been determined that the field-index spatial fluctuations that occur in the experiment are harmless to high-current rings. This conclusion is the same as that we reached previously for simulations that lasted only a fraction of the bounce period.

In addition, the computer-simulation results have confirmed our theoretical predictions concerning the variation of the ring’s equilibrium position with the vertical (betatron) field. The theoretical predictions are also in good agreement with recent results from the NRL modified-betatron experiment. Furthermore, the computer-simulation results have verified the predicted high sensitivity of the equilibrium position on the vertical field when the bounce frequency is near zero. This result has also been confirmed recently by the NRL experiment.

Finally, the simulations have confirmed the existence of ring orbits having figure-eight shape. Such orbits occur in the middle of the ring-centroid instability gap, i.e., when one of the betatron frequencies is real and the other is imaginary. In agreement with linear theory, these orbits are open near the minor axis, but they close before reaching the vacuum-chamber wall because of the nonlinearities of the image fields.

2. EQUILIBRIUM POSITION

The initialization of the electron ring in the simulations is facilitated by knowing the ring-equilibrium position as a function of the vertical field for the energies of interest. At its equilibrium position the electron ring is motionless in the transverse plane. Therefore, the equilibrium position can be determined from the equations of motion of the ring centroid under the conditions that $v_r = v_z = 0$, i.e.,

$$- \frac{v_0^2}{R_{eq}} = - \frac{|e|}{\gamma m} (E_r + \beta_0 B_z). \quad (1)$$

In Eq. (1) $R_{eq}$ is the equilibrium position, $v_0$ is the toroidal velocity, $\beta_0 = v_0/c$, $\gamma$ is the relativistic factor, $E_r$ is the radial electric field, and $B_z$ is the total vertical magnetic field, which includes both the external field and the self-field that acts on the ring centroid.

The image fields acting on the ring centroid have been derived previously, correct to order $(\Delta/a)^2$ and $a/r_0$. As shown in Fig. 1, $\Delta$ is the displacement of the
ring centroid from the minor axis of the torus, $a$ is the minor radius, and $r_0$ is the major radius of the torus. These fields are valid when $\Delta/a \leq \frac{1}{2}$. To compute the equilibrium position for larger ring displacements, we will make the \textit{ad hoc} assumption that the image fields at the ring centroid are given by the expressions

$$E_r = -2 |e| N_l \left[ \frac{(R - r_0)}{a^2 - \Delta^2} + \frac{1}{2R} \ln \frac{a}{r_b} + \frac{r_b^2}{8Ra^2} \right],$$  

$$B_z = 2 |e| N_l \left[ \frac{(R - r_0)}{a^2 - \Delta^2} - \frac{1}{2R} \left( \ln \frac{a}{r_b} + 1 \right) + \frac{r_b^2}{8Ra^2} \right] \beta_\theta,$$

where $N_l$ is the linear electron-ring density. These fields are similar to those derived previously, except that $a^2$ in the denominator of the first term inside the brackets has been replaced with $a^2 - \Delta^2$. Such an assumption appears to be reasonable, provided that the ratio $a/r_0$ is considerably smaller than unity.

Furthermore, it has been assumed that the toroidal magnetic field that acts on the ring centroid is given by

$$B_\theta = B_{\theta 0} r_0 / R,$$  

and the betatron field components are described by the equations

$$B_z = \frac{1}{r} \frac{\partial}{\partial r} (rA^b_0) \bigg|_{r=R}$$  

and

$$B_r = -\frac{1}{r} \frac{\partial}{\partial z} (rA^b_0) \bigg|_{r=R},$$
where the magnetic-vector potential $A^b_0$ is given by

$$A^b_0(r, z) = B_z o \left[ \left( \frac{r_0}{r} \right)^n \left( \frac{r}{2-n} \right) + \frac{r_0^2 (1-n)}{r (2-n)} + \frac{n z^2}{2r} \right]$$  \hspace{1cm} (5)

In Eq. (5) $B_z o$ is the magnetic field at $r = r_0$, $z = 0$, and $n$ is the external-field index, i.e.,

$$n = \frac{r_0}{B_z o} \left( \frac{\partial B_z}{\partial r} \right)_{r_0 o}.$$

The $\gamma$ in Eq. (1) is the normalized kinetic energy of the reference electron that is located on the centroid of the ring. In the computer simulations this $\gamma$ is taken as equal to the average gamma $\langle \gamma \rangle$. Therefore, to make a meaningful comparison between the theory and the simulation it is necessary to replace $\gamma$ in Eq. (1) with $\langle \gamma \rangle$. As a result, an expression is needed that relates the average gamma of the ring with the gamma of electrons at the diode of the injector.

Consider an electron beam emitted from a diode. It is assumed that at the anode all the electrons have the same energy $\gamma_d$. During the formation of the electron ring inside the torus the kinetic energy of the electrons is reduced in order to provide the necessary energy to build up the electromagnetic fields inside the torus. The reduction of the beam’s kinetic energy may be computed from the conservation of energy, i.e.,

$$N(\gamma_d - 1)mc^2 = N \langle \gamma - 1 \rangle mc^2 + \frac{1}{2c} \int \mathbf{j} \cdot \mathbf{A} dV - \frac{|e|}{2} \int n_0 \Phi dV,$$  \hspace{1cm} (6)

where $N$ is the total number of electrons in the beam, $(\gamma_d - 1)mc^2$ is the kinetic energy of electrons at the anode, and $(\gamma - 1)mc^2$ is the average kinetic energy of the electrons after equilibrium has been established. The last two terms in Eq. (6) represent the magnetic and electric field energies, respectively.

For uniform particle density $n_0$ and current density $j_b$, Eq. (6) becomes

$$\gamma_d = \langle \gamma \rangle - \frac{|e|}{2mc^2(2\pi R)(\pi r_b^2)} \left[ \int \beta_\theta A_\theta dV + \int \Phi dV \right].$$  \hspace{1cm} (7)

The potential inside the beam, i.e., for $\rho / r_b \leq 1$, is given by

$$\Phi(\rho, \phi) = \Phi_0 - \frac{\alpha r_b^2}{8} (1 - \rho^2 / r_b^2) + \sum_{n=1}^{\infty} A_n \left( \frac{\rho}{r_b} \right)^{n+1} e^{i(n-1)\phi} + \text{c.c.},$$  \hspace{1cm} (8a)

where

$$\Phi_0 + \Phi_0^* = - \frac{\alpha r_b^2}{2} \left[ \ln \frac{a}{r_b} - \frac{\Delta^2}{a^2} - r_b^2 \frac{\Delta}{8Ra} \frac{(\Delta)}{a} \cos \delta \right],$$  \hspace{1cm} (8b)

$$A_1 = \left( \frac{r_b}{a} \right) \left( \frac{\alpha r_b^2}{4} \right) \left( \frac{\Delta}{a} \right) e^{-i\delta} + \frac{a}{2R} \ln \frac{a}{r_b} + \frac{r_b^2}{8Ra},$$  \hspace{1cm} (8c)

$$A_2 = \left( \frac{r_b}{a} \right) \left( \frac{\Delta}{2a} \right) \left( \frac{\alpha r_b^2}{4} \right) \left( \frac{\Delta}{a} \right) e^{-i\delta} - \frac{a}{2R} \frac{r_b^2}{4aR} e^{-i\delta}.$$  \hspace{1cm} (8d)
and
\[ \alpha = 4\pi |e| n_0. \] (8e)

Substituting Eq. (8) into the last term of Eq. (7), we obtain
\[ \int \Phi \, dV = -(2\pi)^2 R b^2 |e| N_b \left[ \frac{1}{4} + \ln \frac{a}{r_b} - \left( \frac{\Delta}{a} \right)^2 \right]. \] (9)

Since \( r = R + \rho \cos \phi \), the integral \( \int A_\theta \, dV \) can be written as
\[ \int A_\theta \, dV = \frac{1}{R} \int \psi \left( 1 - \frac{\rho}{R} \cos \phi \right) \, dV, \] (10)
where \( \psi (\rho, \phi) = r A_\theta \) is the magnetic stream function.

Inside the ring, i.e., for \( \rho/r_b \ll 1 \) and \( J_e = \text{constant} \), the stream function is given by
\[
\psi (\rho, \phi) = \psi_0 - \frac{\alpha' r_b^2}{8} \left( 1 - \frac{\rho^2}{r_b^2} \right) + \sum_{n=1}^{\infty} A'_n \left( \frac{\rho}{r_b} \right)^n e^{i n \phi} \\
+ \frac{1}{R} \left[ \frac{3 \alpha' \rho^3 \cos \phi}{32} + \sum_{n=1}^{\infty} A''_n \left( \frac{\rho}{r_b} \right)^{n+1} e^{i(n-1)\phi} \right] + \text{c.c.,} \] (11a)
where, to the order \((\Delta/a)^2\) and \(a/r_0\), the coefficients \(A'_n\) and \(A''_n\) are given by
\[
A'_1 = \left( \frac{r_b}{a} \right) \left( \frac{\alpha' r_b^2}{4} \right) \left[ \frac{\Delta}{a} e^{-i \delta} - \frac{a}{2R} \left( \ln \frac{a}{r_b} + 1 \right) + \frac{r_b^2}{8Ra} \right], \] (11b)
\[
A''_2 = \left( \frac{r_b}{a} \right)^2 \left( \frac{\Delta}{2a} \right) \left( \frac{\alpha' r_b^2}{4} \right) \left[ \frac{\Delta}{a} e^{-i \delta} + \frac{a}{2R} + \frac{r_b^2}{4Ra} \right] e^{-i \delta}. \] (11c)

The sum of constant \(\psi_0\) and its complex conjugate is given by
\[
\psi_0 + \psi_0^* \equiv - \frac{\alpha' r_b^2}{2} \left[ \ln \frac{a}{r_b} - \frac{\Delta^2}{a^2} - \frac{r_b^2}{8Ra} \left( \frac{\Delta}{a} \right) \cos \delta \right], \] (11d)
and
\[
\alpha' = 4\pi |e| n_0 R \beta_0. \] (11e)

Substituting Eqs. (11) into Eq. (10) and carrying out the integral, then substituting the resulting expression, as well as Eq. (9), into Eq. (7), we obtain the desired result:
\[
\gamma_d = \langle \gamma \rangle + 2\nu \left[ \frac{1}{4} + \ln \frac{a}{r_b} - \left( \frac{\Delta}{a} \right)^2 \right] - \left( \nu + \langle \gamma \rangle \right) \left[ \frac{1}{4} + \ln \frac{a}{r_b} - \left( \frac{\Delta}{a} \right)^2 \right], \] (12)
where \(\nu\) is the Budker parameter.

Table I shows values of \(\Delta \gamma = \gamma_d - \langle \gamma \rangle\) obtained from the PIC code and also from Eq. (12). The agreement is very good. Since the total number of particles in the ring remains fixed as its major radius changes, the electron density in the code varies as \(r_0/R\). This effect is taken into account by replacing \(\nu\) with \(\nu_0 r_0/R\), where \(\nu_0\) is the Budker parameter on the minor axis. In addition, the beam radius in Eq. (7) has been replaced by \(r_b + r_c/4\), where \(r_b\) is the initialization radius of the ring.
TABLE I
Reduction in the Beam Kinetic Energy ($\Delta \gamma$) During Injection from the Simulation Code (PIC) and from Eq. (12). The Energy Change Has Been Computed for Several Radial Injection Positions (R) and Beam Radii ($r_b$).

<table>
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<tr>
<th>$R$ (cm)</th>
<th>$I$ (kA)</th>
<th>$\gamma_d$ (cm)</th>
<th>$r_b$ (cm)</th>
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in the code and $r_c$ is the width of the cell. This correction takes into account the fact that when the electrons are placed with their centers in the boundary, they "stick out" 1/2 cell, and thus the radius of the ring, on the average, will be 1/4 cell larger. At the smallest beam radius of 1 cm, the 1/4-cell correction (0.125 cm) is quite substantial, but at the larger radii the effect is minimal.

Recently,\textsuperscript{25} by integrating the energy-rate equation, we have obtained an equation that describes the variation of $\gamma_{rc}$ of the reference electron at the ring centroid as the ring moves along its orbit. Assuming that $\gamma_0 = \gamma$, where $\gamma_0 = (1 - \beta^2_\theta)^{-1/2}$, omitting a small term that is proportional to $v/\gamma^2$ and the subscript in the gamma, we obtain

$$\gamma + 2 \nu \left[ \frac{1}{2} \ln \frac{a}{r_b} + \ln \left(1 - \frac{\Delta^2}{a^2}\right) - \frac{r_b^2 (R - r_0)}{8a^2 R} \right]$$

$$- \frac{\nu}{\gamma^2} \left[ \frac{1}{2} \ln \frac{a}{r_b} + \ln \left(1 - \frac{\Delta^2}{a^2}\right) + \ln \frac{a}{r_b} \ln \frac{R}{r_0} \right] = \gamma_c, \quad (13)$$

where $\gamma_c$ is a constant determined from the injection conditions.

Substituting Eqs. (2), (3), (4), and (13) into Eq. (1), and since $\beta_\theta = \beta$ at the equilibrium position, the resulting expression is solved numerically. Results are shown in Fig. 2 for several beam currents. The values of the remaining parameters are listed in Table II. As expected when the ring current is zero, the magnetic field required to keep the ring at its equilibrium position decreases with increasing $R_{eq}$. However, as the ring current increases the single-particle picture is modified dramatically. We observe that the curve rotates counterclockwise while it deforms near the wall. This deformation is due to the nonlinear effects.

![FIGURE 2](image.png)

**FIGURE 2** Electron–ring equilibrium position $R_{eq}$ at a function of the vertical magnetic field $B_{zo}$ (at the minor axis), for five values of the ring current. For all curves in this figure, the beam was injected at 9 cm from the minor axis with $\gamma_d = 2.96$. The gamma of the reference electron at the ring centroid $\gamma_{rc}$ is also shown on each curve. The values of the various parameters are listed in Table II.
Table II

Values of the Various Parameters Used in the Runs Shown in Fig. 2

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
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<tr>
<td>Total beam energy (diode) ( \gamma_d )</td>
<td>2.96 ( (E = 1.0 \text{ MeV}) )</td>
</tr>
<tr>
<td>Torus major radius ( r_0 )</td>
<td>100 cm</td>
</tr>
<tr>
<td>Beam radius ( r_b )</td>
<td>1.0 cm</td>
</tr>
<tr>
<td>Torus minor radius ( a )</td>
<td>16 cm</td>
</tr>
<tr>
<td>Centroid position at injection</td>
<td>109 cm</td>
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<td>Toroidal magnetic field at ( r_0, z = 0 ), ( B_{z0} )</td>
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<td>External field index ( n )</td>
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Figure 3 shows a comparison between the predictions of the theory and the results of the simulation code. The values of the various parameters are listed in Table III. In Fig. 3a we used Eq. (13) to compute \( \gamma \), the Budker parameter was assumed constant, i.e., \( \nu = \nu_0 \), and the image fields were computed from Eq. (2). These fields are based on the assumption that both \( n_0 \) and \( J \) are uniform across the beam. However, in the code both \( n_0 \) and \( J \) vary, at least initially, as \( 1/R \). To take this effect, at least partially, into account, we have replaced the \( \ln (a/r) + 1 \) term in Eq. (2b) with \( \ln (a/r_b) + \frac{1}{2} \). The numerical factor that follows the \( \ln \) in Eq. (2b) depends on the density distribution. It is unity for uniform density, two when the density is proportional to the major radius, and zero when the density is inversely proportional to the radius. In addition, in Eq. (1), we replaced \( \gamma \) with \( \langle \gamma \rangle \) and \( \nu \) with \( \nu_0 r_0/R \). The results are shown in Fig. 3b. It is apparent that the agreement has been improved. However, the agreement appears to be satisfactory in both cases.

3. LONG-TIME SIMULATION RESULTS

The main purpose of our work was to study the long-time dynamics of the electron ring in a magnetic-field configuration that is very similar to that of the NRL modified-betatron experiment. In these simulations the electron-ring dynamics has been studied over approximately 75 revolutions around the major axis, i.e., over several bounce periods.

The methodology that we followed is very simple. For a selected set of parameters, we studied the electron-ring dynamics using an idealized magnetic field described by a magnetic vector potential. In such a configuration the field index was uniform. Subsequently, we repeated the run with the same parameters but using fields calculated from filaments placed at the same positions as the coils in the experiment.

Figure 4a shows the orbit of the electron-ring centroid in an idealized magnetic field. The values of the various parameters for this run are listed in Table IV. This simulation lasted 1.5 \( \mu \text{sec} \). During this time the ring made about 2.5 bounce oscillations. The "+" sign on the figure is plotted each time the beam completes an orbit around the major axis (\( \sim 24 \text{nsec} \)). The configuration and
FIGURE 3 Theoretical and computational predictions of the electron-ring equilibrium position $R_{eq}$ as a function of the vertical magnetic field $B_{z0}$ (at the minor axis). In (a), the theoretical results were obtained from Eq. (1) with $\gamma = \gamma_c$ and the fields from Eq. (2). In (b), $\gamma = \gamma$ and the 1 in Eq. (2b) was replaced with 0.5.
phase spaces at the end of the orbit are shown in Fig. 4b. In this figure \( x = r - R,\)
\( y = z - Z,\) \( x' = (v_r - \Omega_L y)/v_\theta,\) \( y' = (v_z + \Omega_L x)/v_\theta,\) and \( \Omega_L = (\Omega_\theta/2\gamma)(r_0/R).\) The two phase-space plots \( x, x'\) and \( y, y'\) start symmetric and remain symmetric over the entire duration of the run. In contrast, the \( \gamma v_z \) vs \( x\) plot is initially symmetric with respect to the vertical \((\gamma v_z)\) axis, but rapidly after the initialization of the run the plot rotates \(-30^\circ\) around the center. This rotation is a manifestation of electron rotation around the ring center.

The corresponding ring-centroid orbit in the coil-generated field is shown in Fig. 5a, and the configuration and phase spaces at the end of the orbit are shown in Fig. 5b. By comparing Figs. 4 and 5, it becomes apparent that the macroscopic orbits as well as the configuration and phase spaces are almost identical in the two cases. Therefore, we may conclude that the field-index spatial fluctuations that occur in the experiment are harmless to high-current rings, at least over the first several bounce periods. As shown in Fig. 6, the external field-index fluctuations are substantial in the coil-generated fields.

It is apparent from Fig. 2 that when the ring current is approximately 1 kA the equilibrium position is very sensitive to the vertical magnetic field. This sensitivity results from the balancing of the external forces by the image forces of the wall. As a consequence, the bounce frequency \( \omega_B \) becomes very small.

To check the predicted sensitivity of the equilibrium position on the vertical magnetic field \( B_{z0},\) we made four runs with slightly different values of the magnetic field. The results are shown in Fig. 7. The values of the various parameters are listed in Table V. By increasing the values of \( B_{z0}\) from 43 G to 45 G, the equilibrium position increased from approximately 98 cm to 111 cm. Even more dramatic is the change in \( R_{eq}\) when \( B_{z0}\) increased from 43.9 G to 44.0 G. In this case the equilibrium position increased by more than 6 cm and the direction of the poloidal motion was reversed.
(a) Orbit of electron ring centroid (PIC)

(b) FIGURE 4(a) Orbits of the electron-ring centroid in the transverse plane from the MOBE codes, when the field index is uniform. (b) Configuration space $r$, $z$, phase spaces $x, x'$ and $y, y'$, and plot of $\gamma u_z$ vs $x$ at $t = 1.5 \mu$sec (end of the orbit). The values of the various parameters for this run are listed in Table IV.
TABLE IV
Values of the Various Parameters Used in Figs. 4 and 5

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Total beam energy (diode) ( \gamma_d )</td>
<td>2.76 MeV</td>
</tr>
<tr>
<td>Average kinetic energy ( \langle \gamma \rangle )</td>
<td>2.14</td>
</tr>
<tr>
<td>Ring centroid energy ( \gamma_{rc} )</td>
<td>2.05</td>
</tr>
<tr>
<td>Beam current ( I_0(\beta = 1) )</td>
<td>3 kA</td>
</tr>
<tr>
<td>Actual current ( I )</td>
<td>2.65 kA</td>
</tr>
<tr>
<td>Torus major radius ( r_0 )</td>
<td>100 cm</td>
</tr>
<tr>
<td>Beam radius ( r_b )</td>
<td>1.5 cm</td>
</tr>
<tr>
<td>Corrected beam minor radius ( r^* )</td>
<td>1.625</td>
</tr>
<tr>
<td>Torus minor radius ( a )</td>
<td>16 cm</td>
</tr>
<tr>
<td>Centroid position at injection ( \gamma )</td>
<td>109 cm</td>
</tr>
<tr>
<td>Betatron magnetic field at ( r_0, z = 0 ), ( B_{zo} )</td>
<td>44.3 G</td>
</tr>
<tr>
<td>Toroidal magnetic field at ( r_0, z = 0 ), ( B_{eo} )</td>
<td>2000 G</td>
</tr>
<tr>
<td>External field index ( n )</td>
<td>0.5 (uniform in Fig. 4)</td>
</tr>
<tr>
<td>External field index ( n )</td>
<td>0.41 (coils in Fig. 5)</td>
</tr>
<tr>
<td>Timestep ( \Delta t )</td>
<td>5.0 psec</td>
</tr>
<tr>
<td>Number of particles</td>
<td>1024</td>
</tr>
</tbody>
</table>

Since the bounce frequency is very small, the computer simulation of these orbits is very expensive. For example, when \( B_{zo} = 44 \) G, it requires about 6 \( \mu \)sec to complete slightly more than half a bounce orbit, which corresponds to three hours on a Cray XMP-12 computer.

Since the fields of the modified-betatron configuration are independent of the toroidal angle \( \theta \), the canonical angular momentum \( P_\theta \) for the reference electron at the ring centroid is a constant of the motion, i.e.,

\[
\frac{P_\theta}{mc} = \gamma R \beta_\theta - \frac{|e|}{mc^2} R (A^b_\theta + A^s_\theta) = \text{constant},
\]

(14)

where \( A^b_\theta \) is the betatron-field magnetic vector potential and is given in Eq. (5), and the self-magnetic potential is

\[
A^s_\theta = -2N_e |e| \beta_\theta \left[ \frac{1}{2} + \ln \frac{a}{r_b} + \ln \left( 1 - \frac{\Delta^2}{a^2} \right) - \frac{r_b^2}{8a^2} \frac{(R - r_0)}{R} \right].
\]

(15)

Equation (14), with \( \gamma \) given by Eq. (13) and \( A^s_\theta \) and \( A^s_\theta \) given by Eqs. (5) and (15), respectively, describes the nonlinear orbits of the ring centroid in the plane transverse to the minor axis. Results are shown in Fig. 8. The values of the various parameters in this run are the same as those of Fig. 7, except for the value of \( B_{zo} \), which is a fraction of a gauss higher. The results of Fig. 8 have been obtained by replacing the numerical term \( \frac{1}{2} \) appearing inside the brackets in Eqs. (13) and (15) by \( \frac{1}{4} \), in order to take into account the average \( \gamma \), and the fact that \( J_\theta \) is not uniform. By comparing Figs. 7 and 8, it is clear that Eq. (14) predicts rather accurately the orbits of the ring centroid even in this very complex region, i.e., when \( \omega_\theta \) is very small.

Figure 9 shows the vertical magnetic field required to confine the ring at its equilibrium position. The values of the various parameters are the same as those
FIGURE 5(a) Orbit of the electron-ring centroid in the transverse plane from the MOBE code, when the betatron field is generated by coils that are located in the same position as the coils in the NRL modified-betatron experiment. (b) Configuration space \( r, z \), phase spaces \( x, x' \) and \( y, y' \), and plot of \( \gamma v_z \) vs \( x \) at \( t = 1.5 \mu \text{sec} \) (end of the orbit). The values of the various parameters for this run are listed in Table IV.
FIGURE 6 Field index used in the run shown in Fig. 5. As the electron ring drifts on its poloidal orbit of 9 cm radius, the field index varies approximately by a factor of two.

FIGURE 7 Orbits of the electron-ring centroid in the transverse plane from the MOBE code for small changes in the vertical magnetic field. The values of the various parameters for this run are listed in Table V. Substantial particle losses are observed when the ring moves along the 43.0- and 43.9-G orbits.
TABLE V
Values of the Various Parameters Used in Fig. 7

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Total beam energy (diode) $\gamma_d$</td>
<td>2.76 ($E = 0.9$ MeV)</td>
</tr>
<tr>
<td>Average kinetic energy $\langle \gamma \rangle$</td>
<td>2.55</td>
</tr>
<tr>
<td>Ring centroid energy $\gamma_{rc}$</td>
<td>2.52</td>
</tr>
<tr>
<td>Beam current $I_v (\beta = 1)$ = 1 kA</td>
<td>1 kA</td>
</tr>
<tr>
<td>Actual current $I = 0.92$ kA</td>
<td></td>
</tr>
<tr>
<td>Torus major radius $r_0 = 100$ cm</td>
<td></td>
</tr>
<tr>
<td>Beam radius $r_b = 1.5$ cm</td>
<td></td>
</tr>
<tr>
<td>Corrected beam minor radius = 1.625</td>
<td></td>
</tr>
<tr>
<td>Torus minor radius $a = 16$ cm</td>
<td></td>
</tr>
<tr>
<td>Centroid position at injection = 109 cm</td>
<td></td>
</tr>
<tr>
<td>Betatron magnetic field at $r_0$, $z = 0$, $B_{0z} = 43-45$ G</td>
<td>43-45 G</td>
</tr>
<tr>
<td>Toroidal magnetic field at $r_0$, $z = 0$, $B_{00} = 1000$ G</td>
<td>1000 G</td>
</tr>
<tr>
<td>External field index $n = 0.5$ (uniform)</td>
<td></td>
</tr>
<tr>
<td>Timestep $\Delta t = 10.0$ psec</td>
<td></td>
</tr>
<tr>
<td>Number of particles = 1024</td>
<td></td>
</tr>
</tbody>
</table>

FIGURE 8 Orbits of the electron-ring centroid in the transverse plane from the constant of the motion for four different values of the vertical magnetic field. As in Fig. 7 the ring is injected at $z = 0$, $R = 109$ cm. These orbits are very similar to those shown in Fig. 7 but at slightly different fields. The values of the various parameters are shown in Table V. The number in the orbits is the value of $P_o/\mu c$. 
FIGURE 9 Electron-ring equilibrium position \( R_{eq} \) as a function of the vertical magnetic field \( B_{z0} \) (inferred to the minor axis) for the values of the parameters listed in Table V. This figure was obtained from Eq. (1) with \( \gamma = \langle \gamma \rangle \) and by replacing the 1 in Eq. (2b) with 0.5.

listed in Table V. The results of this figure are in good agreement with those of Fig. 7. For \( B_{z0} = 45 \text{ G} \), Fig. 9 predicts a single equilibrium position at about 112 cm, which is in agreement with the computer results of Fig. 7. For \( B_{z0} = 43 \text{ G} \), Fig. 9 predicts that there is not any equilibrium position. Figure 7 shows that at this value of the magnetic field the ring centroid initially moves vertically, which is a manifestation of the absence of an equilibrium position. The centroid trajectory curves after the ring loses particles, and shortly thereafter the entire ring strikes the wall. At \( B_{z0} = 44 \text{ G} \), Fig. 9 predicts three equilibrium positions, located at 103, 109, and 89.5 cm. The equilibrium position of the 44-G orbit in Fig. 7 is located at \( \sim 103 \text{ cm} \). Finally, for \( B_{z0} = 43.9 \text{ G} \) Fig. 9 predicts a single equilibrium position at 89.5 cm. In the simulation results of Fig. 7, the equilibrium position initially appears to be well to the left of the minor axis. However, as the ring drifts on this highly elliptical orbit, it starts losing particles at \( R = 97 \text{ cm}, Z = -14 \text{ cm} \), and the equilibrium position shifts to near the minor axis.

It may be seen from Fig. 9 that there is a range of magnetic fields for which three equilibrium positions occur simultaneously. The two equilibria near the wall exhibit space-charge-dominated behavior, while the third, near the minor axis, exhibits single-particle behavior. This suggests the existence of orbits that are shaped like the infinity symbol (\( \infty \)), or figure eight on its side. Results obtained
FIGURE 10  Electron-ring centroid orbit in the transverse plane from the constant of the motion. These orbits have been obtained by setting $2vr_0^2/\gamma a^2 = \frac{1}{2}$. This condition is similar to that of Eq. (17). The number on the orbit is the value of $p_0/mc$.

from Eq. (14) are shown in Fig. 10. The values of the various parameters for this run are listed in Table VI.

Figure-eight orbits exist because $\omega_B^2 < 0$ near the minor axis, but as a result of the nonlinear fields $\omega_B^2 > 0$ when the ring moves away from the minor axis. Thus, the ring changes rotation direction when it is located away from the minor axis, and a figure eight (on its side) is formed.

The existence of figure-eight orbits has been confirmed by the simulations, as shown in Fig. 11. Using the values of the parameters listed in Table VI, we made a series of runs with the PIC code, keeping the total energy ($\gamma_d$) the same but starting the ring at various positions near the minor axis. The superposition of six such runs is shown in Fig. 11. Although the size of the orbit is different in the two figures, their shapes are very similar.

The specific conditions under which the figure-eight trajectories appear may be determined as follows. From Eq. (14), with $P_\theta = 0$ and using the cylindrical approximation for $\gamma$ and $A^\theta_0$, i.e., omitting the terms that vary as $1/r_0$ in Eqs. (13) and (15), and also using the linear expression for the betatron magnetic vector potential

$$A^\theta_0 = B_{zo} r_0 \left[ 1 + \frac{(R - r_0)^2(1 - n)}{2r_0^2} + \frac{Z^2n}{2r_0^2} \right].$$
TABLE VI
Values of the Various Parameters Used in Fig. 10

- Total beam energy $\gamma_d = 2.76$ ($E = 0.9$ MeV)
- Average kinetic energy $\langle \gamma \rangle = 2.36$
- Ring centroid energy $\gamma_{rc} = 2.32$
- Beam current $L_v(\beta = 1) = 1.452$ kA
- Actual current $I = 1.316$ kA
- Torus major radius $r_0 = 100$ cm
- Beam radius $r_b = 1.5$ cm
- Corrected beam minor radius = 1.625
- Torus minor radius $a = 16$ cm
- Centroid position at injection = 100 cm
- Betatron magnetic field at $r_0$, $z = 0$, $B_{0z} = 44.3$ G
- Toroidal magnetic field at $r_0$, $z = 0$, $B_{00} = 1000$ G
- External field index $n = 0.41$ (uniform)
- Timestep $\Delta t = 10.0$ psec
- Number of particles = 1024

FIGURE 11 Electron-ring centroid orbit in the transverse plane from the MOBE code. These orbits have been obtained by setting $2v_0^2/\gamma_d^2 = \frac{1}{2}$. The values of the various parameters for the six runs shown in this figure are listed in Table VI.
we obtain
\[ \gamma_c + \frac{\nu}{\gamma^2} \left[ \frac{1}{2} + \ln \frac{a}{r_b} + \ln \left( 1 - \frac{\Delta^2}{a^2} \right) \right] = \frac{\Omega_{20} r_0}{c \beta \theta} \left[ 1 + \left( \frac{R - r_0}{2r_0^2} \right) \left( 1 - n \right) + \frac{Z^2 n}{2r_0^2} \right]. \] (16)

Expanding $1/\beta_0$ and $1/\gamma^2$ near $r_0$ using Eq. (13),
\[ \frac{1}{\beta_0} \equiv \frac{1}{\beta_{00}} + \left( 2\nu / \gamma_0^3 \beta_{00}^3 \right) \ln \left( 1 - \frac{\Delta^2}{a^2} \right) \]
and
\[ \frac{1}{\gamma^2} \equiv \frac{1}{\gamma_0^2} + \left( 2\nu / \gamma_0^3 \right) \ln \left( 1 - \frac{\Delta^2}{a^2} \right) \approx \frac{1}{\gamma_0^2}, \]
and setting
\[ - \frac{2\nu r_0^2}{\gamma_0^3 a^2} \left( \frac{1}{\lambda} - \frac{2}{\beta_{00}^2} \right) = \frac{1}{2}, \] (17)

Equation (16) becomes
\[ \left[ \left( \frac{R - r_0}{a} \right)^2 + \left( \frac{Z}{a} \right)^2 \right] + 4(\frac{1}{2} - n) \left[ \left( \frac{Z}{a} \right)^2 - \left( \frac{R - r_0}{a} \right)^2 \right] = 0 \]
\[ = - \frac{16\nu}{\gamma_0} \left( \frac{r_0}{a} \right)^2 \left( \frac{1}{2} + \ln \frac{a}{r_b} \right), \] (18)
where $\lambda = 1 + \left( 2\nu / \gamma_0 \right) \left( \frac{1}{2} + \ln \frac{a}{r_b} \right)$.

Equation (18) is the Lemniscate of Bernoulli, i.e., a figure eight on its side when $n < \frac{1}{2}$ or a figure eight when $n > \frac{1}{2}$. Therefore, such trajectories will appear when the relation of Eq. (17) is satisfied. It can be shown that this occurs in the middle of the ring-centroid instability gap.

When the beam is injected on the minor axis, $\gamma_c = \gamma_d$, and Eqs. (12) and (13) give
\[ \langle \gamma \rangle - \gamma = \frac{\nu}{2} (1 - \frac{1}{2} \gamma^2). \] (19)
For the values of the parameters listed in Table VI, Eq. (19) gives $\langle \gamma \rangle - \gamma = 0.04$, which is exactly the difference between $\langle \gamma \rangle = 2.36$ and $\gamma_{rc} = 2.32$, listed in Table VI.

4. CONCLUSIONS

We have carried out computer simulations of a multi-kiloampere electron ring confined in a realistic modified-betatron magnetic-field configuration. In these simulations the vertical magnetic field is generated by filaments that are located in the same positions as the coils in the NRL modified-betatron experiment. The electron-ring dynamics has been simulated over several $\mu$sec, i.e., over several
bounce periods. It has been determined that the field-index spatial fluctuations that inevitably occur under such conditions are harmless to the high-current electron ring.

In addition, the computer-simulation results have confirmed our theoretical predictions concerning the variation of the ring's equilibrium position with the vertical (betatron) field. The theoretical predictions are also in good agreement with recent results from the NRL modified-betatron experiment. Furthermore, the computer-simulation results have verified the predicted high sensitivity of the equilibrium position on the vertical field when the bounce frequency is near zero. This result has also been confirmed recently by the NRL experiment.

Finally, the simulations have confirmed the existence of ring orbits having figure-eight (8) shape. Such orbits occur in the halfway point of the ring-centroid instability gap. In agreement with linear theory, these orbits are open near the minor axis, but they close before reaching the vacuum-chamber wall because of the nonlinearities of the image fields.

REFERENCES