Introduction

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expense of changing the meaning of gauge transformation for the non-relativistic system. In practice this option is never used.

Two different pion-nucleon Lagrangians with an approximate global symmetry and PCAC are examined. Unfortunately in neither case does the invariance restrict the Foldy-Wouthuysen transformation. In the pseudovector coupling model all unitarily equivalent FW transformations preserve the approximate symmetry of the Lagrangian. On the other hand for the sigma model\(^{17,15}\) it is impossible to find an FW transformation such that the invariance of the non-relativistic theory is the same as of the relativistic theory\(^{18}\). It is shown however, that PCAC is preserved in the Foldy-Wouthuysen reduction and the low-energy theorem which follows from it is used to restrict the form of the O(k) pieces of the effective non-relativistic pion-nucleon vertex.

The Foldy-Wouthuysen transformation technique is reviewed in sect. 2 and in sect. 3 it is applied to the pseudovector and sigma model Lagrangians. The approximate symmetry of the Lagrangian is discussed and it is shown that it does not restrict the form of the FW transformation. The form of terms generated by unitary transformation is also given. In sect. 4 the low-energy theorem following from PCAC is reviewed. It is shown that the reduced Lagrangians also have a partially conserved axial-vector current and the implications of this on the form of the effective pion-nucleon vertex are discussed. Conclusions are given in sect. 5.

2. The Foldy-Wouthuysen Transformation

In this section the reduction of a relativistic Lagrangian by the Foldy-Wouthuysen technique\(^{1,2}\) is reviewed. We prefer to work with the Lagrangian rather than with the Hamiltonian since it is the interaction part of the Lagrangian that enters the perturbation expansion of the S-matrix\(^{13}\). Consider a Lagrangian density

\[ \mathcal{L} = \bar{\psi} \left( i \frac{\partial}{\partial t} - m - \Gamma[\phi] \right) \psi + \mathcal{L}_B \]  

(2.1)

where \(\Gamma[\phi]\) is the coupling term and \(\mathcal{L}_B\) is the boson part of the Lagrangian. The terms in the fermion part of the Lagrangian can be separated into odd and even operators\(^2\)

\[ \mathcal{L}_F = \psi^\dagger \left( i \frac{\partial}{\partial t} - m - \gamma_0 m - \mathcal{E} - 0 \right) \psi, \]  

(2.2a)

\[ = \psi^\dagger \left( i \frac{\partial}{\partial t} - L \right) \psi. \]  

(2.2b)

Transforming the fermion field \(\psi' = \exp(\ii S) \psi\),

\[ \mathcal{L}_F = \psi'^\dagger \left\{ i \frac{\partial}{\partial t} - \mathcal{E} + \ii \left[ 1, \0 \right] e^{\text{i} S} \left[ 1, \0 \right] e^{-\text{i} S} \right\} \psi' \]  

(2.3a)

\[ = \psi'^\dagger \left( i \frac{\partial}{\partial t} - L' \right) \psi'. \]  

(2.3b)

Taking \(S = -\ii \gamma_0 / 2m\) reduces the odd operators in \(L'\) by order \(1/m\) compared to those in \(L\). Repeated application of this procedure eliminates the odd terms up to an arbitrary order in \(1/m\). The result to \(O(1/m^2)\) is

\[ \mathcal{L}_F = \psi'^\dagger \left\{ i \frac{\partial}{\partial t} - \gamma_0 m - \frac{\gamma_0}{2m} \left[ 0, \0 \right] + \frac{1}{8m^2} \left[ 0, \left[ 0, \mathcal{E} \right] \right] \right\} \psi', \]  

(2.4)

where \(\psi'\) is the appropriately transformed fermion field. If \(\Gamma[\phi]\) contains odd pieces linear in the boson field the term \([0, \0]\) will not appear in the Hamiltonian. This is in agreement with the results of
vector current.\( \mathbf{J} \). The Lagrangian is given by

\[
\mathcal{L} = \mathcal{L}_{\text{kin}} - \mathcal{L}_{\text{int}}
\]

where

\[
\mathcal{L}_{\text{kin}} = \frac{1}{2} m \left( \partial \phi \right)^2 - \mathcal{L}_{\text{int}}
\]

and

\[
\mathcal{L}_{\text{int}} = \left( \partial^2 \phi \right)^2 - \mathcal{L}_{\text{int}}
\]

where \( m \) is a constant and \( \mathcal{L}_{\text{int}} \) is the interaction term.

The Lagrangian space consists of the effective vertex functions \( \Gamma \) and \( \Phi \), which are

\[
\Gamma = \frac{1}{2} \phi \left( \partial^2 \phi \right) - \mathcal{L}_{\text{int}}
\]

and

\[
\Phi = \left( \partial^2 \phi \right)^2 - \mathcal{L}_{\text{int}}
\]

so that

\[
\phi \cdot \phi = \frac{1}{2} \phi \left( \partial^2 \phi \right) - \mathcal{L}_{\text{int}} + \left( \partial^2 \phi \right)^2 - \mathcal{L}_{\text{int}}
\]

for constant \( \mathcal{L}_{\text{int}} \). Separating the odd and even terms, we can write

\[
\phi = \phi_1 - \phi_2
\]

where \( \phi_1 \) and \( \phi_2 \) are odd and even functions, respectively.

The difference between the two Lagrangians can be written as

\[
\mathcal{L}_{\text{int}} = \mathcal{L}_{\text{int}} + \mathcal{L}_{\text{int}}
\]

where \( \phi_1 \) and \( \phi_2 \) come from the Feinberg-Nambu theorem and \( \phi \) is the interaction term.

The operator \( \mathcal{O} = \phi_1 + \phi_2 \) can be written as

\[
\mathcal{O} = \phi_1 + \phi_2
\]

except for the last term which is invariant under the transformation.
where $\omega_0$ is the pion energy and $p_i (p_f)$ is the initial (final) nucleon momentum. This coupling is $O(k^3)$ in pion four-momentum. Within the pseudovector coupling model the $O(k)$ terms of the effective vertex, which includes the so-called Galilean correction term, are unambiguous and not subject to phenomenological adjustment.

We next consider the sigma model given by the Lagrangian

$$\mathcal{L} = \overline{\psi} \left( \gamma \gamma_0 - m - g(s + i \gamma \cdot \phi \gamma_5) \right) \psi$$

$$+ \frac{1}{2} \left( \partial_\mu \phi \cdot \partial_\mu \phi - \mu_0^2 \phi \cdot \phi \right) + \frac{1}{2} \left( \partial_\mu s \cdot \partial_\mu s - \mu_0^2 s \cdot s \right)$$

$$- \lambda^2 \gamma_5 \cdot (s \cdot \phi^2) - \frac{1}{4} \lambda^2 (s \cdot \phi^2)^2 - v \mu_0^2 s + c \cdot c \cdot s$$

(3.7)

where $\gamma$ is the expectation of the sigma field and $s = \sigma \cdot v$. Except for the last term $\mathcal{L}$ is invariant under the transformation

$$\begin{align*}
T_0: & \quad \psi \rightarrow \psi \left( 1 - i \gamma(x) \cdot \gamma_5 \right) \psi, \\
\psi \rightarrow & \quad \psi + 2 \gamma(x) (s + v), \\
s + v & \rightarrow s + 2 \gamma(x) \psi,
\end{align*}

(3.8)

for $\gamma(x)$ = constant. The even and odd parts of the fermion Lagrangian are

$$\mathcal{E} = g \gamma_0 s,$$

(3.9)

$$\gamma_0 = -i \gamma_0 \gamma \cdot \gamma_5,$$

(3.9)

and

$$\gamma_0 = i \gamma_0 \gamma_5 \gamma \cdot \phi.$$

(3.9)

For $T_0$ involves $\gamma_5$ and mixes pion and sigma fields it is clear that the reduced theory can not be invariant under the same transformation as the relativistic theory. Unlike the pseudovector model there is no invariant combination of $O_1$ and $O_2$.

To $O(1/m)$ the reduced Lagrangian is

$$\mathcal{L} = \overline{\psi} \left( \frac{\gamma_0}{\gamma_0^2 - m^2} - \gamma_0 n - g \gamma_0 s - \frac{\gamma_0}{2m} \left(-1 \gamma_0 \gamma \cdot \gamma_5 + i \gamma_0 \gamma \gamma_5 \gamma \cdot \phi \right)^2 \right) \psi$$

$$+ \mathcal{L}_B + c \cdot c \cdot s - 2 \gamma(x) \cdot \phi)$$

$$+ \mathcal{L}_B + c \cdot c \cdot s - 2 \gamma(x) \cdot \phi.)$$

(3.10)

The lowest order $\lambda$-dependent pion-nucleon coupling is $O(1/m^2)$ and is given by

$$\int \frac{\lambda}{8m^2} \overline{\psi} \gamma \gamma_0 \gamma_5 \gamma \cdot \phi \gamma \cdot \phi \gamma_5 \gamma \cdot \phi \psi$$

(3.11)

which in momentum space contributes

$$-i \frac{\lambda}{8m^2} \gamma_0 \gamma_5 \gamma \cdot (p_i + p_f)$$

(3.12)

to the non-relativistic vertex. This has the form of the Galilean correction term. As we have seen there is nothing that chiral invariance of the sigma model can tell us about its presence or absence.

There is however a way of determining whether it is allowable as a phenomenological effective operator.

4. Constraints from the low-energy theorem

No use has yet been made of the fact that the pion-nucleon Lagrangians lead to PCAC. Using PCAC the leading term (in pion four-momentum) of the pion emission amplitude and the axial current of the nucleon can be related. In this section we derive the axial current for the non-relativistic pseudovector and sigma models and discuss the implications on the form of the effective pion-nucleon vertex.

A detailed derivation of the low-energy theorem using PCAC may be found in ref.21). Let $\mathcal{W}_j = <b|J_j^a|a>$ be the matrix element of the axial current (isospin index $j$) between states $a$ and $b$. Now $\mathcal{W}_j$ can be written as $\mathcal{W}_j^{\text{ext}} + \Delta \mathcal{W}_j$ where the matrix element for the coupling of the axial-vector current (carrying momentum $k$) to an external line is given by
(46.4)
\[
\left( \phi \cdot (x) \right) \frac{2z - s}{c} \eta \xi + 2 \xi \eta + \left( \xi \cdot (x) \right) \frac{2z - s}{c} \xi + \left( \phi \cdot (x) \right) \frac{2z - s}{c} \eta
\]

(46.5)
\[
(\xi \cdot (x) \eta) + \left( \phi \cdot (x) \eta \right) = \left( \phi \cdot (x) \eta \right) = \left( \phi \cdot (x) \eta \right) \cdot \left( \phi \cdot (x) \eta \right)
\]

(46.6)
\[
\left( \phi \cdot (x) \eta \right) = 0
\]

(46.7)
\[
\left( \phi \cdot (x) \eta \right) = 0
\]

(46.8)
\[
\left( \phi \cdot (x) \eta \right) = 0
\]

(46.9)
\[
\left( \phi \cdot (x) \eta \right) = 0
\]

But of the Lagrangian which has been transformed by $\phi$, that is, of

(47.1)
\[
\left( \phi \cdot (x) \eta \right) = 0
\]

obtained by making my reduction of the Lagrangian of the equation of\n
(48.1)
\[
\left( \phi \cdot (x) \eta \right) = 0
\]

the properties of the non-relativistic Lagrangian. Equivalent results are

(49.1)
\[
\left( \phi \cdot (x) \eta \right) = 0
\]

formally, it is not necessary to know explicitly the transformation

(50.1)
\[
\left( \phi \cdot (x) \eta \right) = 0
\]

for is obtained by applying the reduction to the equation in\n
(51.1)
\[
\left( \phi \cdot (x) \eta \right) = 0
\]

and that the appropriate non-relativistic version transformation operator

(52.1)
\[
\left( \phi \cdot (x) \eta \right) = 0
\]

show that there is still no invariance of the non-relativistic theory

(53.1)
\[
\left( \phi \cdot (x) \eta \right) = 0
\]

formally one can

(54.1)
\[
\left( \phi \cdot (x) \eta \right) = 0
\]

The sign model is less straightforward since the chiral trans-

(55.1)
\[
\left( \phi \cdot (x) \eta \right) = 0
\]

forms.

(56.1)
\[
\left( \phi \cdot (x) \eta \right) = 0
\]

As is expected from the form of (3.1) and supports independen-

(57.1)
\[
\left( \phi \cdot (x) \eta \right) = 0
\]

of $\phi$ (3.1) does not contribute to $\eta$ (1.1). Hence the forms

(58.1)
\[
\left( \phi \cdot (x) \eta \right) = 0
\]

current to the problem is

(59.1)
\[
\left( \phi \cdot (x) \eta \right) = 0
\]

current in momentum space is the lowest order $\phi$-dependent current of the axial

(60.1)
\[
\left( \phi \cdot (x) \eta \right) = 0
\]

from which one obtains

(61.1)
\[
\left( \phi \cdot (x) \eta \right) = 0
\]

or (1.1) which is is

(62.1)
\[
\left( \phi \cdot (x) \eta \right) = 0
\]

The non-relativistic axial-vector current is then generated as a par-

(63.1)
\[
\left( \phi \cdot (x) \eta \right) = 0
\]

To use the low-energy theorem it is necessary to generate another

(64.1)
\[
\left( \phi \cdot (x) \eta \right) = 0
\]

The (0) contribution to $\eta$ is thus determined by the (0) terms of

(65.1)
\[
\left( \phi \cdot (x) \eta \right) = 0
\]

or (0) $\eta$.

(66.1)
\[
\left( \phi \cdot (x) \eta \right) = 0
\]

the non-relativistic axial-vector current.

(67.1)
\[
\left( \phi \cdot (x) \eta \right) = 0
\]

containing an external non-zero. The PCF requires

(68.1)
\[
\left( \phi \cdot (x) \eta \right) = 0
\]

where $\phi$ is the matrix element for non-perturbative axial-vector current

(69.1)
\[
\left( \phi \cdot (x) \eta \right) = 0
\]

$\left( \phi \cdot (x) \eta \right) = 0$.
where
\[ \mathcal{L}'_B = \mathcal{L}_B - 2 \frac{\partial}{\partial x} \partial^\mu q(x) + 2(s^u \partial^\mu \partial_\mu q(x). \quad (4.10) \]

Carrying out the FW transformation with
\[ \mathcal{E} = g \gamma_0 \gamma^5 - \gamma_0 \gamma^\alpha \gamma_5 \gamma^\alpha \gamma_5 \quad (4.11a) \]
and
\[ 0 = -i \gamma_0 \gamma^\alpha \gamma^\beta + i g \gamma_0 \gamma^\beta \gamma_5 \gamma^\alpha \gamma_5 \quad (4.11b) \]
gives the reduced Lagrangian
\[ \mathcal{L}'_\alpha = \psi' \left\{ \frac{\partial}{\partial t} - \gamma_0 \gamma^\alpha \gamma^\beta + \gamma_0 \gamma^\alpha \gamma_5 \gamma^\beta \right\} \psi' + \text{terms higher order in } L/m \]
\[ + \mathcal{L}'_B + \frac{c}{c} (s^u \partial^\mu \partial_\mu. \quad (4.12) \]

The axial-vector current is then
\[ J^\mu_j = \psi' \left\{ \frac{1}{2} \frac{1}{2m} \left[ i \gamma^\mu \gamma_5 \gamma^\nu + i \gamma^\nu \gamma_5 \gamma^\mu \right] \right\} \psi' + \text{terms higher order in } L/m. \quad (4.13) \]

The divergence of \( J^\mu_j \) is easily obtained from \[ \partial_\mu J^\mu_j = \delta \mathcal{L}'_B / \delta a_j(x). \]
This is zero except for the contribution of the last term of (4.12) which is proportional to \( \phi_j \) so that PCAC is satisfied by all unitarily equivalent reduced Lagrangians.

The fermion part of the axial current (4.13) involves only even operators. The terms describing the coupling of the axial current to the nucleon (i.e., terms without a boson field) are identical to those in the pseudovector model. Using the low-energy theorem this means that the leading term in the pion emission amplitude \( \mathcal{M}_a(0) \) is identical in non-relativistic pseudovector and sigma models just as it is for the relativistic theories.

In higher orders of \( 1/m \) the FW transformed Lagrangian and hence the axial current will depend on the order in which the odd operators are removed from (4.9c). The form of the lowest order term in which an ambiguity arises is
\[ i \frac{1}{2m^2} \psi' \left\{ - i \gamma_0 \gamma^\alpha \gamma^\beta + \gamma^\alpha \gamma_5 \gamma^\beta \right\} \psi', \quad (4.14) \]
This makes a contribution to \( J^\mu_j \) which is identical to the \( \lambda \)-dependent term (4.7) found for the pseudovector model. Since (4.14) is \( O(k) \) in momentum space it does not contribute to \( \mathcal{M}_a(0) \). The same argument applies to \( \lambda \)-dependent terms of higher order in \( 1/m \).

Taking PCAC into account we see that \( \mathcal{M}_a(0) \) must be independent of \( \lambda \). This means that the contribution of the \( \lambda \)-dependent pion-nucleon vertex eq. (3.12) to processes in which a pion is emitted or absorbed on an external nucleon line must be cancelled by other \( \lambda \)-dependent terms in the amplitude. In such processes the contribution of \( O(k) \) terms in the non-relativistic pion-nucleon vertex (this includes the Galilean correction term) is fixed by the soft pion theorem.

5. Conclusion

For theories with local gauge invariance (e.g., electrodynamics) the requirement that the non-relativistic and relativistic Lagrangians have the same transformation properties uniquely fixes the Foldy-Wouthuysen transformation. This is not the case for two models of the pion-nucleon interaction which have an approximate global symmetry and satisfy PCAC. With exception of the pion mass term the pseudovector coupling pion-nucleon Lagrangian is invariant under addition of a constant to the pion field. This invariance is preserved by all unitarily
It is a pleasure to thank K.W. Hepp, and J.M. Gubser, for helpful discussions.

(1) References

(2) J.H. Bethe, and W. Drell, "Relativistic Quantum Mechanics,


