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APPLICATION OF THREE-BODY TECHNIQUES TO PION-NUCLEUS SCATTERING*†

A.W. Thomas

TRIUMF at U.B.C., Vancouver, B.C., Canada V6T 1M5

LECTURE I

There is a considerable number of nuclear physicists who would argue that the title of these lectures is very misleading. In particular, the few-body problem has long been considered a small, very specialized, area of nuclear physics that could safely be ignored by the majority. I will have succeeded in my task if in the course of this presentation I can convince you that in pion physics, the pion few-nucleon system has provided several striking theoretical insights, and may continue to do so.

Our guiding theme taken from few-body physics, is an emphasis on exact unitarity—Amado's remark that unitarity is 'the hobgoblin of small minds' not withstanding! This concept is carefully introduced in the introductory section on the non-relativistic three-body problem. (Those readers strictly interested in pion-nucleus phenomena could skip §A completely on first reading.) The significance of the two-body t -matrix, which is the exact solution of the two-body scattering problem, is also stressed there. Later it will be seen that some of the sharpest divisions in the theory of π -nucleus scattering centre on the choice of off-shell behaviour for this object.

After the elimination of a two-body potential for the relativistically preferred scattering amplitude (or t -matrix), we show how (guided by unitarity) one can obtain practical relativistic three-body equations. Such equations have been applied to the πNN system, and particularly πD scattering with some success and we shall briefly review those results. In comparison with the π -many-nucleon problem, where one must make many approximations in order to carry

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out a calculation, the nD system allows essentially exact calculation once the nN t -matrices are given. Thus agreement with data in the nD system should be regarded as a pre-requisite for trust in any theory of π -nucleus scattering! Of course, agreement with experiment aside, from a purely theoretical point of view, the more rigorous treatment of relativistic effects in the three-body systems can tell one a good deal about directions for improving the usual, essentially non-relativistic, π -nucleus scattering theories.

We have already hinted at the controversial question of the nN range—that is, the most reliable off-shell behaviour of the nN t -matrix. This question has recently been asked much more clearly by Hybner and Thomas, who consider the nN interaction in a system with an extra nucleon—and particularly the implications of two and three-body unitarity in such a system for the off-shell behaviour of t_{nN} . We shall present these arguments rather briefly in §3, relying on Prof. Kisslinger for the nN field theory.

Having justified our off-shell model for the nN t -matrix, we turn to the formulation of the π -many-nucleon problem in terms of this object. The general theory outlined in §4 is that of Kerman *et al.*, and Watson. From the general theory it is relatively easy to write down all the conventional π -nucleus optical potentials (Kisslinger, Ericson-Ericson, Landau - Phatak - Tabakin, etc.). Although there are alternative ways to obtain the same result, we show how few-body ideas, and particularly the formulation of a three-body model for the first-order optical potential, have led to a deeper understanding of the π -nucleus system. In fact, many of the conventional theories drastically overestimate the cross-section for quasi-elastic scattering, and omit real pion absorption. While there are possibly important consequences for other reactions whose description needs distorted pion waves (as we mention), the most dramatic consequences may still await discovery in the (3,3) resonance region. Towards this end we mention the self-consistency question first raised by Ericson and Hulthén.

Finally, if time permits, we shall describe the efforts made in the πNN system to formally separate the 'effects of absorption' on elastic scattering, from the usual multiple scattering series. The implications of these formal results for the π -nucleus case will be discussed briefly.

A. Non-Relativistic Three-Body Scattering Theory

For background material we refer the interested reader to references (1-6), and particularly the classic paper of Lovelace and the most recent text (Ref. 4).

(1) Integral equation for scattering—the two-body t -matrix. We assume that the reader is familiar with the Lippmann-Schwinger

formulation of the two-body scattering problem as a single integral equation equivalent to both the Schrödinger equation and appropriate boundary conditions. For a local potential $V(r)$, this integral equation is

$$\psi_k^{(+)}(\vec{r}) = (2\pi)^{-3/2} e^{i\vec{k}\cdot\vec{r}} - \frac{\mu_{12}}{2\pi} \int d\vec{r}' \frac{e^{i\vec{k}'\cdot\vec{r}'}}{|\vec{r} - \vec{r}'|} V(r') \psi_k^{(+)}(\vec{r}'), \quad (A.1)$$

with $\mu_{12} [= m_1 m_2 / (m_1 + m_2)]$ the reduced mass and $\psi_k^{(+)}$ the full solution of the two-body scattering problem for an incident projectile of momentum $\vec{k}$. Equation (A.1) may be shown to have the following asymptotic behaviour

$$\psi_k^{(+)}(\vec{r}) \xrightarrow{r \rightarrow \infty} (2\pi)^{-3/2} e^{i\vec{k}\cdot\vec{r}} + f(k, \theta) \frac{e^{ikr}}{r}, \quad (A.2)$$

with

$$f(k, \theta) = -4\pi^2 \mu_{12} \int d\vec{r}' \left\{ \frac{e^{-i\vec{k}'\cdot\vec{r}'}}{(2\pi)^{3/2}} \right\} V(r') \psi_k^{(+)}(\vec{r}'), \quad (A.3)$$

the scattering amplitude ($|\vec{k}'| = |\vec{k}|$, $\cos \theta = \hat{k} \cdot \hat{k}'$). That is, the differential cross-section for two-body scattering is

$$\frac{d\sigma(\theta)}{d\Omega} = |f(k, \theta)|^2. \quad (A.4)$$

If we write Eq. (A.1) in operator form as

$$|\psi_k^{(+)}\rangle = |\vec{k}\rangle + G_0(k^2/2\mu_{12} + i\epsilon) V |\psi_k^{(+)}\rangle, \quad (A.5)$$

it is in fact valid for any short-range potential (local or non-local). The free Green's function $G_0(E)$ is defined as usual

$$G_0(E) = (E - H_0)^{-1}, \quad (A.6)$$

with H_0 the kinetic energy operator, and the fully interacting Green's function $G(E)$ is

$$G(E) = (E - H_0 - V)^{-1}. \quad (A.7)$$

The so-called resolvent equation for $G(E)$ is readily derived using the operator identity

$$B - A = A(A^{-1} - B^{-1})B, \quad (A.8)$$

and hence

$$G(E^+) = G_0(E^+) + G_0(E^+) V G(E^+). \quad (A.9)$$

The two-body t -matrix, $t(E)$, corresponding to the potential V is defined in terms of this Green's function as

$$G(E) = G_0(E) + G_0(E) t(E) G_0(E). \quad (A.10)$$

At this stage it is worth pointing out that the easiest derivation of Eq. (A.5) follows from (A.9). Since one can define the scattering wave function as

$$|\psi_k^{(+)}\rangle = \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} G(E + i\epsilon) |\vec{k}\rangle, \quad (A.11)$$

(with $E \equiv k^2/2\mu_{12}$), and

$$\lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} G_0(E + i\epsilon) |\vec{k}\rangle = |\vec{k}\rangle, \quad (A.12)$$

multiplying (A.9) to the right by $|\vec{k}\rangle$, and throughout by $i\epsilon$, leads to Eq. (A.5). Using Eq. (A.10), however, leads to the alternative expression

$$|\psi_k^{(+)}\rangle = |\vec{k}\rangle + G_0(E^+) t(E^+) |\vec{k}\rangle. \quad (A.13)$$

In the momentum representation, using momentum states normalised so that

$$\langle \vec{p} | \vec{p}' \rangle = \delta(\vec{p} - \vec{p}'), \quad (A.14)$$

Eq. (A.13) becomes

$$|\psi_k^{(+)}(\vec{p})\rangle = \delta(\vec{p} - \vec{k}) + (k^2 + i\epsilon - p^2)^{-1} \langle \vec{p} | t(E) | \vec{k} \rangle. \quad (A.15)$$

(Alternatively, one may replace

$$(k^2 + i\epsilon - p^2)^{-1} = p / (k^2 - p^2) - i\pi \delta(k^2 - p^2), \quad (A.16)$$

where ' p ' means principal value integral.)⁷ The matrix element of the operator $t(E)$ appearing in Eq. (A.15) is often written as $t(\vec{p}, \vec{k}; E)$ and is said to be 'half off-shell', because $|\vec{p}| \neq |\vec{k}| = (2\mu_{12} E)^{1/2}$. The close relationship between the short range behaviour of the two-body wave function (i.e. Fourier components $p \neq k$), and the off-shell behaviour of the two-body t -matrix is made explicit through (A.15).

In practice, it is more common to write an integral equation for the t -matrix itself, rather than working through the wave function. In fact, multiplying (A.13) to the left by V , and [from Eqs. (A.5) and (A.13)] identifying

$$V |\psi_k^{(+)}\rangle = t(E^+) |\vec{k}\rangle, \quad (A.17)$$

one finds

$$t(E^+) = V + V G_0(E^+) t(E^+) . \quad (A.18)$$

This is often represented diagrammatically as in Fig. 1.

Because of the simple form taken by the kinetic energy operator in momentum space, this is by far the most common representation of Eq. (A.18) in use—that is

$$t(\vec{k}; \vec{k}; E) = V(\vec{k}', \vec{k}) + \int d\vec{k}'' V(\vec{k}', \vec{k}'') (E^+ - k''^2/2\mu_{12})^{-1} \times t(\vec{k}'', \vec{k}; E) . \quad (A.19)$$

Equation (A.19) defines elements of the t -matrix for arbitrary values of $(\vec{k}', \vec{k}; E)$, or in the usual jargon 'fully off-shell'. Unfortunately experiment only provides a constraint on one line in that three-dimensional space, namely $|\vec{k}| = |\vec{k}'| = (2\mu_{12} E)^{1/2}$. In fact, comparing Eqs. (A.3) and (A.17) we see that the experimental scattering amplitude is simply

$$f(k, 0) = -4\pi^2 \mu_{12} \quad t(\vec{k}; \vec{k}; k^2/2\mu_{12}) : k = k' , \quad \equiv -4\pi^2 \mu_{12} \quad t(\text{on})(k, \theta) . \quad (A.20)$$

Using the partial wave decomposition

$$t(\vec{k}', \vec{k}; E) = \sum_{l,m} Y_{lm}(\vec{k}') \quad t_l(k', k; E) \quad Y_{lm}^*(\vec{k}) , \quad (A.21)$$

and the usual relation of a scattering amplitude to phase shifts implies

$$t_l(k, k; k^2/2\mu_{12}) \equiv t_l^n(k) = -e^{i\delta_l} \sin \delta_l / (\pi \mu_{12} k) . \quad (A.22)$$

(ii) Unitarity of the two-body t -matrix. In the introduction we mentioned that the concept of unitarity was fundamental to these lectures. Perhaps the simplest example of a unitarity relation is provided by the two-body t -matrix. In fact, multiplying Eq. (A.18) to the left and right by V^{-1} and t^{-1} , respectively, implies

$$t^{-1}(Z) = V^{-1} - G_0(Z) , \quad (A.23)$$

for Z an arbitrary complex parameter. Setting $Z = E \pm i0$ and subtracting leads to the formula for the discontinuity of t across the real axis

$$t^{-1}(E^-) - t^{-1}(E^+) = 2\pi i \delta(E - H_0) , \quad (A.24)$$

Fig. 1 Graphical representation of the two-body Lippmann-Schwinger equation (A.18). The circle represents the t -matrix, and the wiggly line a potential.

Fig. 2 Graphical representation of the unitarity relation (A.25) for the two-body t -matrix. The dashed line represents an on-energy-shell intermediate state.

and hence

$$\text{disc } t(E) \equiv t(E^+) - t(E^-) = -2\pi i \, t(E^+) \, \delta(E-H_0) \, t(E^-). \quad (\text{A.25})$$

This is represented diagrammatically in Fig. 2, where the dashed line represents $-2\pi i \, \delta(E-H_0)$ —that is, an on-energy-shell intermediate state.

The most commonly seen form of Eq. (A.25) is the *optical theorem*, which relates the imaginary part of the forward scattering amplitude to the total cross section. Indeed, taking the on-shell matrix element of (A.25) leads to $(k'=k)$

$$\text{Im } t(\text{on})(k, 0^0) = [\text{disc } t(\text{on})(k, 0^0)/2i] =$$

$$-\pi \mu_{12} k \int |t(\vec{k}', \vec{k}; k^2/2\mu_{12})|^2 d\vec{k}', \quad (\text{A.26})$$

and using (A.20)

$$\text{Im } f(k, 0^0) = \left(\frac{k}{4\pi}\right) \sigma_{\text{total}}. \quad (\text{A.27})$$

We shall see very soon that a similar relation holds even in the far more complicated three-body problem, where many inelastic processes are included in σ_{total} .

While the unitarity of the scattering amplitude is an extremely significant observation, equivalent to conservation of probability, the *operator relation* (A.25) is actually far more powerful. It places constraints on the analytic structure, in the energy variable, of the completely off-shell t-matrix. In a given partial wave, this is simply an algebraic constraint, namely

$$\begin{aligned} \text{disc } t_{\ell}(k', k; E) &= -2\pi i \mu_{12} k_E t_{\ell}(k', k_E; E) t_{\ell}^*(k_E, k; E), \\ k_E &= (2\mu_{12} E)^{1/2}. \end{aligned} \quad (\text{A.28})$$

Clearly then, knowledge of the half-shell t-matrix is sufficient to construct the fully off-shell t-matrix by dispersion relation. Lack of space prevents us from pursuing the consequences of (A.28) further, but the interested reader is referred to the work of Lovelace,³ which is still one of the best discussions available.

For later discussion it is useful to record one more result here concerning the half-shell t-matrix. In fact, it can be shown that in any partial wave, the half off-shell t-matrix is simply related to the fully on-shell t-matrix by

$$t_{\ell}(k', k_E; E) = t_{\ell}^{\text{on}}(k_E) f_{\ell}(k', k_E), \quad (\text{A.29})$$

where $f_{\ell}(k', k_E)$ is unity for $k' = k_E$, and for a real potential is purely real. The function $f_{\ell}(k', k_E)$ is called the Kowalski-Noyes half-shell function.^{5,8}

(iii) Two-body t-matrix in the three-body system. A minimal knowledge of the usual notational conventions in three-body theory is necessary at this stage:

(i) It saves on subscripts if any two-body property is labelled by the spectator particle's number. Thus (V_3, t_3) are the potential and t-matrix for the pair (12).

(ii) Unless otherwise stated we work in the centre of mass (c.m.) of the three particles, so that

$$\vec{p}_1 + \vec{p}_2 + \vec{p}_3 = 0. \quad (\text{A.30})$$

Therefore only two momenta are needed to specify any state. These are usually the relative momentum of any pair [with (i, j, k) cyclic]

$$\vec{q}_i = (m_k \vec{p}_j - m_j \vec{p}_k) / (m_j + m_k), \quad (\text{A.31})$$

and the momentum of the third particle relative to that pair (which is simply $\vec{p}_i$ in the three-body c.m.).

(iii) The kinetic energy operator is then

$$H_0 = p_1^2/2\mu_1 + q_1^2/2\mu_{jk}, \quad (\text{A.32})$$

with

$$\begin{aligned} \mu_1 &= m_i m_{jk}/M; \quad \mu_{jk} = m_j m_k/m_{jk}, \\ m_{jk} &= m_j + m_k; \quad M = m_1 + m_j + m_k. \end{aligned} \quad (\text{A.33})$$

(iv) It is convenient to define the channel Hamiltonian and Green's function

$$H_i = H_0 + V_i; \quad G_i(E) = (E - H_i)^{-1} \quad i \in (0, 1, 2, 3), \quad (\text{A.34})$$

corresponding to just two particles interacting in the presence of a spectator. The fully interacting Green's function is

$$G(E) = (E - H)^{-1} \equiv (E - H_0 - \sum_i V_i)^{-1}. \quad (\text{A.35})$$

Consider now the scattering process shown in Fig. 1, with a third particle line disconnected from the other two—representing the interaction of particles 1 and 2 in the presence of a non-interacting spectator (3). The appropriate Lippmann-Schwinger equation is

$$t_3(E^+) = V_3 + V_3 G_0(E^+) t_3(E^+), \quad (\text{A.36})$$

in analogy with (A.17), but the meaning has changed a little. In fact, t_3 and V_3 are operators in a space of three particles, so that the matrix elements are to be taken between states like $|\vec{p}_3 \vec{q}_3 >$ and $|\vec{p}_3' \vec{q}_3' >$. Clearly the two-body potential in three-body space is simply

$$< \vec{p}_3' \vec{q}_3' | V_3 | \vec{p}_3 \vec{q}_3 > = \delta(\vec{p}_3' - \vec{p}_3) V_3(\vec{q}_3', \vec{q}_3). \quad (\text{A.37})$$

Note that we have not bothered to make the formally necessary discrimination between the operators V_3 on the left and right of (A.37) which act in different spaces. Equation (A.37) says simply what we expect, namely that a pair-wise potential only changes the relative momentum of the pair, not their c.m. motion.

Let us take the momentum representation of Eq. (A.36), and use (A.37). This leads to the equation

$$t_3(\vec{p}_3' \vec{q}_3'; \vec{p}_3 \vec{q}_3; E) = \delta(\vec{p}_3' - \vec{p}_3) V_3(\vec{q}_3', \vec{q}_3) + \int d\vec{p}_3'' d\vec{q}_3'' \frac{\delta(\vec{p}_3' - \vec{p}_3'') V_3(\vec{q}_3', \vec{q}_3'') t_3(\vec{p}_3'' \vec{q}_3''; \vec{p}_3 \vec{q}_3; E)}{\{E - p_3'^2/2\mu_3\} - q_3''^2/2\mu_3}, \quad (\text{A.38})$$

which clearly [c.f. Eq. (A.19)] has the solution

$$t_3(\vec{p}_3' \vec{q}_3'; \vec{p}_3 \vec{q}_3; E) = \delta(\vec{p}_3' - \vec{p}_3) t_3(\vec{q}_3', \vec{q}_3; E - p_3'^2/2\mu_3). \quad (\text{A.39})$$

The δ -function in (A.39) simply says that the non-interacting particle goes through with its momentum unchanged, and that the two-body t -matrix should be evaluated at the two-body sub-energy. (That is, the total three-body energy is E , and the energy of relative motion of the pair and spectator is $p_3^2/2\mu_3$, leaving $(E - p_3^2/2\mu_3)$ in the two-body sub-system. This is a direct consequence of the form of g_0).

While Eq. (A.39) as derived above is almost obvious, its consequences are quite profound! In any three-body scattering problem one needs the t -matrix for arbitrarily high spectator momentum p_3 , and thus the two-body energy variable covers the range $(-\infty, E)$. The first consequence of this is that any two-body pole (bound state or resonance) becomes a cut in the three-body energy variable. Secondly one needs the two-body t -matrix at negative values of the energy parameter. Since Eq. (A.19) defines the t -matrix for arbitrary (complex) values of the parameter ' E ', negative values obviously present no difficulty of principle. Indeed, for a real potential, the negative energy t -matrix defined by (A.19) is obviously real.

We shall see very soon that the form of (A.39) is essential to the proof of three-body unitarity. However, the concept of a

negative energy scattering amplitude seems to disturb many people. Perhaps a word on how it relates to more familiar objects in scattering theory will be worth while. As we mentioned in part (ii) (above), the unitarity relation (A.28) which gives the discontinuity across the only cut the two-body t -matrix has [from $(0, \infty)$], can be used to write a dispersion relation for the t -matrix at arbitrary energy. In fact, in the absence of a bound state, one finds (using the contour of Fig. 3)

$$t_k(k', k; Z) = V_k(k', k) + (2\pi i)^{-1} \int_0^\infty dE \text{disc} \{t_k(k', k; E)\} / (Z - E), \quad (\text{A.40})$$

with ' $V_k(k', k)$ ' the limit of $t_k(k', k; Z)$ as $Z \rightarrow \infty$ (it can be removed by a subtraction). Now using Eq. (A.28) one finds the fully off-shell t -matrix for arbitrary ' Z ' (e.g. on the negative real axis) is directly related to the half off-shell t -matrices at positive energy (i.e. in the scattering region)

$$t_k(k', k; Z) = V_k(k', k) - \int_0^\infty dk_E k_E^2 \frac{t_k(k', k_E; E) t_k^*(k_E, k; E)}{(Z - E)}. \quad (\text{A.41})$$

But we have seen in (A.15) that knowledge of the half-shell t -matrix at positive energy, is equivalent to knowing the scattering wave function at that energy. Clearly then, one could rewrite any system of equations which require negative energy t -matrices, entirely in terms of more familiar scattering wave functions. Such a system of three-body equations has indeed been developed,⁹ but will not be considered here.

Fig. 3 Contour used in the proof of Eqs. (A.40) and (A.41).

(iv) The Necessity for Faddeev's Equations

The standard technique for the numerical solution of an equation like (A.19), after partial wave expansion, is to approximate the integral by some quadrature formula e.g.

$$t_k(k_i, k_j; E) = V_k(k_i, k_j) + \sum_{n=1}^N W_n \frac{V_k(k_i, k_n) \cdot t_k(k_n, k_j; E)}{E^2 - k_n^2 / 2\mu_{12}}. \quad (\text{A.42})$$

Equation (A.42) can then be solved as a set of coupled linear equations, with the accuracy improving as N is increased. A sufficient condition that this numerical solution is possible (leading to a unique solution), is that the kernel ($G_0 V$) of the integral equation should be square integrable (or L^2 in the usual terminology). The L^2 condition means in fact that

$$\text{Tr} [(G_0 V)^\dagger G_0 V] < \infty. \quad (\text{A.43})$$

(For a much more complete discussion of these ideas we refer to the work of Lovelace.)^{3,10} For the usual short range potentials, and $E^\dagger$ not exactly on the real energy axis, it is relatively easy to show that the kernel of the two-body scattering equation (A.19) satisfies Eq. (A.43). The proof that a numerical solution is still possible in the limit where $E^\dagger$ becomes real is more difficult,¹⁰ but the result is not surprising.

However, the kernel of Eq. (A.38) which describes two-body scattering in the three-body system contains a δ -function, representing the non-interacting spectator. Now when one constructs the trace (A.43) one finds the combination

$$\text{Tr} \{ (G_0 V_3)^\dagger (G_0 V_3) \} \sim \int d\vec{p}_3 \int d\vec{p}_3' \delta(\vec{p}_3 - \vec{p}_3') \delta(\vec{p}_3' - \vec{p}_3), \quad (\text{A.44})$$

which is infinite! This may seem very much like making a mountain out of a mole-hill, since we could obviously factor out $\delta(\vec{p}_3 - \vec{p}_3')$ and solve the purely two-body problem as before. Unfortunately this is not so in the true three-body system with more than one interaction. Indeed " $G_0 V$ " in that case is $G_0(V_1 + V_2 + V_3)$ which contains three distinct δ -functions. The trace of this " $G_0 V$ " inevitably contains combinations like $((G_0 V_1)^\dagger G_0 V_1)$, and is therefore not L^2 . While this fact is not sufficient to show that the three-body Lippmann-Schwinger equation

$$T = V + V G_0 T \quad (\text{A.45})$$

does not have unique solutions, it does indicate that there may be severe difficulties. The problem which we have discussed is known as the "disconnectedness problem" (because the δ -functions which

lead to problems are represented diagrammatically by a third line not connected to either of the lines representing the interacting nucleons).

The failure of the Lippmann-Schwinger equation for three (or more) particle scattering was a real formal difficulty in the late 50's. However like many formal problems in physics its existence was ignored for practical purposes. Indeed the formal manipulations which we shall carry out in deriving the optical model in Lecture 2 are in this class—i.e. mathematically meaningless, but the end result happens to be correct! The correct solution to this problem by the Russian mathematician Faddeev not only solved the formal problem but opened the door to considerable new physical understanding.

(v) Faddeev's Equations

The solution given by Faddeev had essentially two steps. First all the series of purely two-body, disconnected potential diagrams which could be summed, were. This is most clearly seen in a perturbation expansion of Eq. (A.45). (While this is not always guaranteed to converge, the reader can rest assured that a more formal operator proof along the lines of the next section is possible.) That is,

$$T = V_1 + V_2 + V_3 + (V_1 + V_2 + V_3) G_0 (V_1 + V_2 + V_3) + \dots \quad (\text{A.46})$$

and the subseries like

$$V_1 + V_1 G_0 V_1 + V_1 G_0 V_1 G_0 V_1 + \dots \quad (\text{A.47})$$

can be identified (through the perturbative expansion of Eq. (A.36) as t_i . This leads to the series

$$T = (t_1 + t_2 + t_3) + t_1 G_0 t_2 + t_1 G_0 t_3 + t_2 G_0 t_1 + \dots \quad (\text{A.48})$$

where all the disconnected diagrams which can be trivially summed have been. Thus we see that the present day emphasis on the two-body t -matrix takes considerable impetus from Faddeev.

Equation (A.48) describes the rather unlikely process where three free particles scatter to three free particles (the so-called $3 \rightarrow 3$ amplitude). The more realistic case of scattering from a bound state is considered in the next section. The second step in Faddeev's solution was to break T into three parts T^i , where T^i is defined so that the last pair to interact were the pair (jk) . Thus, we find

$$T^i = t_i + t_i G_0 (T^j + T^k), \quad (\text{A.49})$$

which is a set of three coupled equations with kernel

$$\tilde{K} = \begin{pmatrix} 0 & t_1 & t_1 \\ t_2 & 0 & t_2 \\ t_3 & t_3 & 0 \end{pmatrix} G_0. \quad (A.50)$$

Let $\tilde{T}$ and $\tilde{t}$ be the obvious column vectors, then Eq. (A.49) can be rewritten as

$$\tilde{T} = \tilde{t} + \tilde{K} \tilde{T}, \quad (A.51)$$

and when iterated once

$$\tilde{T} = \tilde{t} + \tilde{K} \tilde{t} + \tilde{K}^2 \tilde{T}, \quad (A.52)$$

the new kernel $\tilde{K}^2$ is *completely connected*. That is all its elements have the form $t_i G_0 t_j$ ($i \neq j$). This condition (namely $i \neq j$) is sufficient to avoid the infinity caused by two identical δ -functions coming together [as in Eq. (A.44)].

The formal proof that for reasonable short range potentials Eq. (A.52) has a compact kernel in the limit $\epsilon \rightarrow 0^+$, is very difficult, and need not concern us here [c.f. Refs. (2, 11)]. It is enough that such a proof exists, and was made possible by the *separation of T into three parts according to a proper choice of boundary conditions*. This realization that the formal problems of three-body theory arose from the ambiguity of boundary conditions [e.g. particle 1 hitting 2 and 3 can lead to any of the three possible combinations 1 (23), 2 (31) or 3 (12)] was Faddeev's second crucial contribution.

(vi) The Amplitudes U_{ji}

The equations derived immediately above were not very practical as they applied only to the $3 \rightarrow 3$ scattering process. By far the most commonly examined experimental situation is, however, the scattering of a projectile 1 from a bound state "u" of the pair (jk) leading to either elastic scattering or re-arrangement

$$\begin{aligned} i + (jk)_c &\rightarrow i + (jk)_c \\ &\rightarrow j + (ki)_d. \end{aligned} \quad (A.53)$$

The t-matrix for such a process can be proven by formal scattering theory^{3,12} to be

$$\begin{aligned} T_j(d); i(c) (E^+) &= \langle X_{j;d} | \psi_j^{(+)} | \psi_i^{(+)} \rangle \langle U_{ji}^{(+)}(E^+) | X_{i;c} \rangle \\ &= \langle \psi_{j;d}^{(-)} | v^i | X_{i;c} \rangle = \langle X_{j;d} | U_{ji}^{(-)}(E^+) | X_{i;c} \rangle, \end{aligned} \quad (A.54)$$

where $|X_{i;c}\rangle$ represents the asymptotic state of (jk) in the bound state c with a non-interacting spectator i, satisfying the equation

$$H_i X_{i;c} = E X_{i;c}, \quad (A.55)$$

and

$$v^i \equiv V - V_i = V_j + V_k. \quad (A.56)$$

The wave functions $\psi^{(\pm)}$ are the full three-body scattering wave functions, with ingoing or outgoing boundary conditions, which we shall not discuss further. Instead we concentrate on $U_{ji}^{(\pm)}$ which are formally

$$U_{ji}^{(\pm)}(E) = V(ji) + V_j G(E) v^i. \quad (A.57)$$

These operators were first introduced by Lovelace who derived Faddeev-like equations for them.

To prove this we need the operator identity

$$G = G_k + G v^k G_k, \quad (A.58a)$$

$$= G_k + G_k v^k G, \quad (A.58b)$$

which is easily proven using Eqs. (A.8), (A.34) and (A.56). Now, inserting this in the definition (A.57) for $U_{ji}^{(+)}$ and substituting for v^i implies

$$\begin{aligned} U_{ji}^{(+)} &= V^j + \sum_{k \neq i} \{ v^j G_k V_k + v^j G v^k G_k V_k \} \\ &= V^j + \sum_{k \neq i} U_{jk}^{(+)} G_0 t_k, \end{aligned} \quad (A.59)$$

which is of the Faddeev form. [The last step follows from the definition of $U_{jk}^{(+)}$, and the fact that [from Eqs. (A.10) and (A.18) —c.f. also (A.17)].

$$G_k V_k = G_0 t_k. \quad (A.60)$$

Equation (A.59) is clearly of the Faddeev form. Similarly one can show

$$U_{ji}^{(-)} = v^i + \sum_{k \neq j} t_k G_0 U_{ki}^{(-)}. \quad (A.61)$$

While Eqs. (A.59) and (A.61) are of the Faddeev form it does seem somewhat superfluous to have two different transition

operators which give identical on-shell results. Therefore we follow Alt *et al.* in introducing the operator U_{ji} ,¹³ defined by

$$G(E) = G_j(E) \delta_{ij} + G_j(E) U_{ji}(E) G_i(E), \quad (A.62)$$

[with G_i the channel Green's function (A.34)]. The relationship of U_{ji} to $U_{ji}(+)$ is best shown by substituting (A.58a) into (A.58b)

$$\begin{aligned} G &= G_j + G_j V^j G_i + G_j V^j G V^i G_i \\ &= G_j + G_j U_{ji}(+) G_i, \end{aligned} \quad (A.63)$$

and comparing (A.63) with (A.62) we find

$$U_{ji}(E) = \bar{\delta}_{ji}(E-H_i) + U_{ji}(+) G_i, \quad (A.64)$$

[where $\bar{\delta}_{ij} \equiv (1-\delta_{ij})$]. Similarly one can show

$$U_{ji}(E) = \bar{\delta}_{ji}(E-H_j) + U_{ji}(-) G_j. \quad (A.65)$$

An integral equation for U_{ji} is now obtained by substituting (A.64) into (A.59), with the result (after some algebra)

$$\begin{aligned} U_{ji}(E) &= \bar{\delta}_{ji}(E-H_0) + \sum_{k \neq j} U_{jk}(E) G_0(E) t_k(E), \\ &= \bar{\delta}_{ji}(E-H_0) + \sum_{k \neq j} t_k(E) G_0(E) U_{ki}(E). \end{aligned} \quad (A.66)$$

Equation (66) satisfies both of the criteria emphasised by Faddeev. The potentials have been completely eliminated in favour of t -matrices, and the kernel of the three-body equations is completely connected after one iteration [c.f. Eq. (A.52) above]. Finally we stress that the on-shell (physical) transition amplitudes $T_i(q); i(c)$ calculated with U_{ji} are equal to those calculated from $U_{ji}(E)$ [the proof requires only Eq. (A.55)].

(vii) The Break-Up Amplitude and Cross Section

If the kinetic energy of the projectile is greater than the binding energy of bound state 'c', the additional possibility of a three-body final state must be considered in addition to those shown in Eq. (A.53). [This is the 2+3 amplitude $T_{0i}(c)(E)$.] In fact, the general equations (A.54) and (A.57) are quite valid even in this case, and

$$\begin{aligned} T_{0i}(c)(E) &= \langle 0 | U_{0i}(+) (E) | \chi_{1;c} \rangle \\ &= \langle 0 | \sum_k t_k G_0 U_{ki} | \chi_{1;c} \rangle \text{ on-shell} \end{aligned} \quad (A.67)$$

(We have substituted for $U_{0i}(+)$ using Eq. (A.59) with $V^0 \equiv 0$.)

For completeness we record here how to calculate the scattering cross section using the operators described above. In fact our transition operators have precisely the normalizations of Goldberger and Watson¹⁴ so that, if $V_{in} (\equiv \pi i/\mu v)$ is the relative velocity in the initial state, and the final state contains N_f particles with momenta $\{\vec{p}_1, \dots, \vec{p}_{N_f}\}$, the c.m. differential cross section is

$$\begin{aligned} d\sigma_{fi} &= (2\pi)^4 V_{in}^{-1} |T_{fi}|^2 \delta(E_f - E_i) \delta \left[\sum_{j=1}^{N_f} \vec{p}_j \right] \times \\ &\quad \times \prod_{j=1}^{N_f} d\vec{p}_j. \end{aligned} \quad (A.68)$$

For a detailed discussion of the case of identical particles we refer to §1.7 of Ref. 4 and to Ref. 14.

(viii) Unitarity of the Three-Body Equations

In the section above dealing with the two-body t -matrix we discussed its singularity structure in considerable detail—particularly that structure associated with two-body unitarity. As part of our discussion of pion-nucleus optical potentials will be concerned with their unitarity content (in a three-body model), it is important to clearly establish here the unitarity structure of the three-body equations.

The neatest derivation is that of Schmid and Ziegelmann¹⁵ which we shall follow. Let us write Eq. (A.66) as

$$\bar{U} = \bar{U}^0 + \bar{U}^0 \bar{r} \bar{U}, \quad (A.69)$$

in an obvious matrix notation. Then multiplying to the left by $\bar{U}^0 - 1$ and to the right by $\bar{U} - 1$ we find

$$\bar{U}^{-1}(E^+) = \bar{U}^0 - 1(E^+) - \bar{r}(E^+). \quad (A.70)$$

Subtracting the similar expression for E^- yields a relation very similar to (A.24), except that the driving term is no longer hermitian, viz:

$$\bar{U}^{-1}(E^-) - \bar{U}^{-1}(E^+) = [\bar{U}^0 - 1(E^-) - \bar{U}^0(E^+) - 1] + [\bar{r}(E^+) - \bar{r}(E^-)]. \quad (A.71)$$

The inverse of the operator $\bar{U}^0$ can be readily found by writing out the appropriate 3×3 matrix.

$$[(\bar{U}^0)^{-1}]_{ij} = \left(\frac{1}{2} - \delta_{ij} \right) G_0 \quad (A.72)$$

Thus we find the discontinuity of $\tilde{U}^{-1}$ is

$$\text{disc } [U^{-1}(E)]_{ij} = \frac{1}{2} \text{disc } G_0(E) - \delta_{ij} \text{disc } G_i(E) \quad (\text{A.73})$$

(where we used Eq. (A.10) to replace G_0 t_i G_0 in $\tilde{r}$ by the channel Green's function G_i).

From Eq. (A.34) one sees that

$$\text{disc } G_i(E) = -2\pi i \delta(E-H_i), \quad (\text{A.74})$$

which may mean a little more if we multiply by a complete set of eigenstates of H_i —including bound states ($\chi_{i;c}$ with binding energy ϵ_c), and scattering states (ψ_i), for two particles in the presence of a non-interacting spectator.

$$\text{disc } G_i(E) = \sum_c \int d\vec{p}_1 (-2\pi i) |\chi_{i;c}(\vec{p}_1)\rangle \delta(E-\vec{p}_1^2/2u_i-\epsilon_c) \langle \chi_{i;c}(\vec{p}_1) |$$

$$\times \langle \chi_{i;c}(\vec{p}_1) |$$

$$+ \int d\vec{p}_1 d\vec{q}_1 (-2\pi i) |\psi_i(\vec{p}_1\vec{q}_1)\rangle \delta(E-\vec{p}_1^2/2u_i-\vec{q}_1^2/2u_jk) \langle \psi_i(\vec{p}_1\vec{q}_1) | \quad (\text{A.75})$$

Now, with the definitions

$$\Delta_{i;c} = \int d\vec{p}_1 |\chi_{i;c}(\vec{p}_1)\rangle \delta(E-\vec{p}_1^2/2u_i-\epsilon_c) \langle \chi_{i;c}(\vec{p}_1) |,$$

$$\Delta_0 = \int d\vec{p}_1 d\vec{q}_1 |\vec{p}_1\vec{q}_1\rangle \delta(E-\vec{p}_1^2/2u_i-\vec{q}_1^2/2u_jk) \langle \vec{p}_1\vec{q}_1 |, \quad (\text{A.76})$$

and identifying [from Eq. (A.13)]

$$|\psi_i(\vec{p}_1\vec{q}_1)\rangle = (1+G_0 t_i) |p_i q_i\rangle, \quad (\text{A.77})$$

equation (A.71) can be written as

$$\begin{aligned} \text{disc } [\tilde{U}^{-1}(E)]_{ij} &= -\pi i \Delta_0(E) + 2\pi i \delta_{ij} \left\{ \sum_c \Delta_{i;c}(E) \right. \\ &\quad \left. + [1+G_0(E^+)t_i(E^+)] \Delta_0(E) [1+t_i(E^-)G_0(E^-)] \right\}. \end{aligned} \quad (\text{A.78})$$

To obtain the conventional expression for the right-hand discontinuity we next multiply to the left and right by $\tilde{U}(E^+)$ and $\tilde{U}(E^-)$ respectively.

$$\begin{aligned} \text{disc } U_{ji}(E) &= \pi i \sum_{k,l} U_{j,k}(E^+) \Delta_0 U_{k,i}(E^-) \\ &\quad - 2\pi i \sum_{k;c} U_{jk}(E^+) \Delta_{k;c} U_{ki}(E^-) \\ &\quad + \sum_k U_{jk}(E^+) \left(1+G_0(E^+)t_k(E^+) \right) \Delta_0(E) \left(1+t_k(E^-)G_0(E^-) \right) U_{ki}(E^-). \end{aligned} \quad (\text{A.79})$$

Also summing Eq. (A.66b) over j implies that

$$\sum_k U_{ki} = 2 U_{oi}, \quad (\text{A.80})$$

and adding $t_j G_0 U_{ji}$ to both sides of (A.66b) implies

$$(1+t_k G_0) U_{ki} = (\bar{\kappa}_{ki}-1)(E-H_0) + U_{oi}. \quad (\text{A.81})$$

Equations (A.80) and (A.81) can be used to eliminate U_{ki} from all but the bound-state part (second term) of (A.79). Finally, using the fact that $(E-H_0)\Delta_0$ is zero one finds the complete expression for the unitarity of the Alt, Grassberger Sandhas operators U_{ji}

$$\begin{aligned} \text{disc } \{U_{ji}(E)\} &= -2\pi i \sum_{k,c} U_{jk}(E^+) \Delta_{k;c}(E) U_{ki}(E^-) \\ &\quad + U_{jo}(E^+) \Delta_0 U_{oi}(E^-). \end{aligned} \quad (\text{A.82})$$

This result allows us to write down the optical theorem for particle i scattering from a bound state c . Indeed using (A.54) and (A.67) we obtain

$$\begin{aligned} \text{disc } \{T_i(c); i(c) \langle \vec{p}_i(E), \vec{p}_i(E); E \rangle\} &= -2\pi i \sum_{j,d} \int d\vec{p}_j' \\ &\quad |T_j(d); i(c) \langle \vec{p}_j', \vec{p}_j(E); E^+ \rangle|^2 \delta(E-p_j'^2/2u_j-\epsilon_j; d) \\ &\quad + \int d\vec{p}_j' d\vec{q}_j' |T_{oj}; i(c) \langle \vec{p}_j', \vec{q}_j'; i\vec{p}_i(E); E^+ \rangle|^2 \delta(E-p_j'^2/2u_j-\vec{q}_j'^2/2u_jk) \}. \end{aligned} \quad (\text{A.83})$$

which together with Eq. (A.68) gives the promised generalisation of (A.26,27), viz:

$$\begin{aligned} \text{Im} \{T_i(c); i(c) \langle E, \theta = 0^\circ \rangle\} &= -\pi (2\pi)^4 V_{in} \sum_{j,d} \sigma_j(d) + i(c) \\ &\quad + \sigma_{\text{break-up}} \}. \end{aligned} \quad (\text{A.84})$$

We have examined the unitarity relations for the three-body equations in considerable detail. It has been shown that the right-hand unitarity cut is actually a series of cuts, one starting at each e_{ijc} —corresponding to each allowed re-arrangement channel—in addition to the break-up cut beginning at $E = 0$. We stress that even though one usually only calculates explicitly the elastic (or re-arrangement) amplitude for comparison with data, the information about all other open channels is implicitly contained in it, and gives important constraints on its analytic behaviour. This point will be stressed again when in Lecture 3 we examine the reaction content of theories of pion-nucleus scattering.

(ix) Three-Body Interactions for a Separable Two-Body t-Matrix

Historically the acceptance of separable interactions was very closely connected with the development of three-body scattering theory. Over many years of dealing with local Yukawa potentials, based on one pion exchange, the nuclear physics community developed a prejudice which ruled the highly non-local separable N-N interactions unrealistic. Even the demonstration by Lovelace that the *t-matrix* for any potential is separable in the vicinity of a bound state or resonance³ would probably have had little impact were it not for the incredible agreement with low energy (14.1 MeV) n-d elastic data obtained at about the same time by Aaron, Amado and Yam.¹⁶

In general, a potential is *separable* in the l th partial wave if it can be written in the form

$$V_l(\text{Sep})(p', p) = \lambda_l \bar{g}_l(p') \bar{g}_l(p), \quad (\text{A.85})$$

where $\bar{g}_l$ is a linear array of form-factors. A rank-one potential, of the type used by Aaron *et al.*, has just one element in $\bar{g}_l$ —for example in the $3S_1$ N-N state

$$g_0(p) = C/(p^2 + \mu^2). \quad (\text{A.86})$$

While (A.85, 86) may look quite different from a Yukawa potential

$$V_0^{\text{(Yukawa)}}(p', p) \sim \lambda n \left(\frac{p'^2 + p^2 + 2pp' + \mu^2}{(p'^2 + p^2 - 2pp' + \mu^2)} \right) / 4\mu p p', \quad (\text{A.87})$$

the key point is that at the deuteron pole the *t-matrices* given by both have the form³

$$t(p', p; E) \xrightarrow{E \rightarrow -\epsilon_d} \frac{(p'^2 + \mu^2) \psi_d(p') \psi_d(p) (p^2 + \mu^2)}{E + \epsilon_d}, \quad (\text{A.88})$$

and for $p \leq 1 \text{ fm}^{-1}$, Eq. (A.86) with $\beta \sim 1.4 \text{ fm}^{-1}$ gives a reasonable representation of the deuteron wave function $[\psi_d(p)]$.

Thus, even when a separable potential may seem quite unrealistic, it is possible that the resulting t-matrix can be a good approximation to the real thing. However the crucial factor in favour of separability (and this is what motivated Lovelace) is the simplification it gives in the Faddeev equations. To see this, we need to re-write the equations of Alt *et al.* for separable t-matrices. In general, in the three-body system a separable two-body potential has the form

$$V_l = \sum_a \lambda_a \int d\vec{p}_l |l\vec{p}_l; a\rangle \langle l\vec{p}_l; a|, \quad (\text{A.89})$$

where "a" labels possible quantum numbers, such as

$$\langle \vec{p}_l, \vec{q}_l | l\vec{p}_l; a \rangle = \delta(\vec{p}_l - \vec{p}_l') g_l(q_l) Y_{lm}(\hat{q}_l'), \quad (\text{A.90})$$

in the spinless-isospinless case where $a \equiv (lm)$.

Substituting (A.89) into the two-body Lippmann-Schwinger equation, the reader should be able to prove in a couple of lines that the solution is

$$t_l(E^+) = \sum_a \int d\vec{p}_l |l\vec{p}_l; a\rangle \frac{1}{0} \frac{1}{a} (E^+ - p_l^2/2\mu_l) \langle l\vec{p}_l; a|, \quad (\text{A.91})$$

with

$$D_{la}(E^+) = \lambda_a^{-1} - \int d\vec{q}', \frac{|a|\vec{q}'|^2|}{E^+ - q'^2/2\mu_{jk}}. \quad (\text{A.92})$$

Equation (A.91) demonstrates the essence of the separable t-matrix in relation to resonance formation ($\langle a | \vec{q}_l \rangle$), propagation (D_{la}^{-1}) and decay ($\langle \vec{q}_l' | a \rangle$) all independent of each other.

If one substitutes (A.91) into Eq. (A.66b) for U_{ji} and multiplies to the left and right by $\langle j\vec{p}_j'; b | G_0$ and $G_0 | i\vec{p}_i; a \rangle$ respectively one finds

$$\begin{aligned} X_{jb; ia}(\vec{p}_j', \vec{p}_i; E) &= Z_{jb; ia}(\vec{p}_j', \vec{p}_i; E) \\ &+ \sum_{j', c} \int d\vec{p}_l'' Z_{jb; lc}(\vec{p}_j', \vec{p}_l''; E) D_{lc}^{-1}(E^+ - p_l''^2/2\mu_l) X_{lc; ia}(\vec{p}_l'', \vec{p}_i; E), \end{aligned} \quad (\text{A.93})$$

where we have defined

$$\begin{aligned} X_{jb; ia}(\vec{p}_j', \vec{p}_i; E) &\equiv \langle j\vec{p}_j'; b | G_0(E) U_{ji}(E) G_0(E) | i\vec{p}_i; a \rangle, \\ Z_{jb; ia}(\vec{p}_j', \vec{p}_i; E) &= \delta_{ij} \langle j\vec{p}_j'; b | G_0(E) | i\vec{p}_i; a \rangle. \end{aligned} \quad (\text{A.94})$$

Although we did not write out Eqs. (A.66) in detail in momentum representation, a little reflection will show that in general they are coupled integral equations in *two* momentum variables. These are extremely difficult to solve in practice, although several heroic efforts using Padé techniques seem to have succeeded for the three-nucleon problem.¹⁷ On the other hand, Eq. (A.93) is a set of coupled equations in only one variable, and therefore presents little more difficulty than a coupled channels two-body problem. [The numerical difficulties associated with the more complicated (three-body) singularity structure of the driving term can overcome by contour rotation]¹¹—see particularly Ref. 15]. This is an extremely important practical simplification of the formal three-body theory!

For on-shell momenta

$$|\vec{p}_1'\rangle = |\vec{p}_1\rangle = 2\mu_i(E - \epsilon_i; c), \quad (\text{A.95})$$

the X-amplitude is the elastic (or re-arrangement) t-matrix, up to a normalisation constant. In fact, the form-factor in any two-body channel with bound state is intimately related to the bound-state wave function [in analogy with (A.86, 88)], i.e.

$$\begin{aligned} \langle \vec{p}_1' \vec{q}_1' | G_0(E) | i \vec{p}_1; a \rangle &= -\delta(\vec{p}_1 - \vec{p}_1') (\epsilon_i; a^+ q_1'^2 / 2\mu_j k)^{-1} g_a(\vec{q}_1') \\ &\equiv \langle \vec{p}_1' \vec{q}_1' | X_i; a(\vec{p}_1) \rangle, \end{aligned} \quad (\text{A.96})$$

up to a normalisation constant. [To verify this, the reader should consider the two-body Schrödinger equation $(H_0 + V_1)\psi_B = \epsilon_i; B \psi_B$ with the separable interaction (A.89).] Now, assuming the bound-state normalisation has been included in the definition of the two-body form-factor we obtain

$$X_{j;b} i a(\vec{p}_j', \vec{p}_j; E) = \langle X_{j;b}(\vec{p}_j') | U_j(E) | X_i; a(\vec{p}_i) \rangle_{\text{on-shell}}, \quad (\text{A.97})$$

which is exactly the (elastic or) re-arrangement scattering amplitude $T_i(b); i(a)(E)$. The break-up amplitude is readily shown [from Eqs. (A.67) and (A.91)] to be

$$\begin{aligned} T_{0+i}(a)(\vec{p}', \vec{q}', \vec{p}_i) &= \sum_{k,c} \langle \vec{q}_k' | k; c \rangle D_{0c}^{-1}(E^+ - p_k'^2 / 2\mu_k) \\ &\quad X_{0c}; i a(\vec{p}_k', \vec{p}_i; E). \end{aligned} \quad (\text{A.98})$$

The diagrammatic representation of both Eqs. (A.93) and (A.98) will be considered in the next section.

5B SCATTERING THEORY FOR THE πNN SYSTEM

At first sight the πNN system in the energy region where the (3,3) resonance can play a significant role seems to be ideally suited to a description using the Faddeev equations. Particularly πd scattering involves a bound state in the NN sub-system, and a resonance in each πN sub-system, so that the separable potential equations [5A.(ix) above] should be suitable. This considerably reduces the computational difficulties over the general case, as we have already indicated.

For the moment we choose to ignore the additional complications caused by the pion being a boson, and therefore strictly requiring a field theoretic description. [Some of the implications of standard field theories for the type of t-matrix necessary in a multiple scattering formalism will be considered in part C (below)—see also Dr. Kisslinger's lectures.] Even without this problem, the very low mass of the pion unavoidably presents us with a relativistic scattering problem. Thus pion physics immediately takes us one step beyond standard nuclear physics.

In the early days of three-body theory much of the optimism surrounding the discovery of the Faddeev equations was related to the fact that because the essentially non-relativistic concept of potential had been replaced by the t-matrix, or scattering amplitude, the generalisation to relativistic systems did not seem so obscure. Those heady days of extremely relativistic calculations (such as generating the pion as a three pion bound state) have long been history. Nevertheless, the equations developed in the late 60's still provide an excellent framework for the far more reasonable πNN problem. (We consider it more reasonable, because in the intermediate energy region only one of the three particles is significantly relativistic—as opposed to all three being far off-mass-shell in the case mentioned above.)

The advantages of working first with a three-body system should be apparent here. There are no nuclear structure complications, and the possibility exists to develop a complete, consistent theory—without the necessity to make many-body approximations. For comparison it is interesting to read Heller's criticisms of the inconsistencies of many pion-nucleus calculations performed so far—consistencies related to the ad hoc introduction of relativity into a non-relativistic formalism.¹⁸ Unfortunately the solution he proposed does not seem practical, because the theory (a relativistic potential theory based on the work of Bakamjian and Thomas)¹⁹ inevitably generates many body forces, which render numerical work essentially impossible. We are not aware, for example, of a complete formulation of the three-body problem in this approach (including a proof of unitarity).

One of the most significant early formulations of the relativistic three-body problem²⁰ dealt with the properties of resonant solutions to the two-body Bethe-Salpeter equation as a first step towards generalising the work of A(ix). This was ambitious work aiming at a complete formulation of a practical technique for solving three-body problems in field theory. More recent work has tended to be much more phenomenological,^{21,22} with the limited objective of finding a practical, unitary, Lorentz invariant generalisation of the two-body Blankenbeller-Sugar equation to three-body systems. Essentially for lack of space, we tend to follow the latter approach.

The content of this section is therefore a presentation of the relativistic two-body problem as it appears for a πN pair. It will be seen that almost all commonly used πN wave equations follow as approximations to the general form. Having established a suitable two-body equation, we derive a practical system of relativistic three-body equations for the πNN system. To conclude, we briefly compare some very recent theoretical results with available πd data.

(i) Relativistic Integral Equations for the πN System

All practical formulations of the relativistic two-body problem begin with the Bethe-Salpeter equation, which can be represented diagrammatically as in Fig. 1 (although the meaning has changed considerably from part A). Now the driving term represents the sum of all irreducible diagrams (in some field theory) which result in pion-nucleon scattering. This set of irreducible diagrams plays the role of a potential in the theory (and hence the notation V for it). Algebraically the scattering equation is*

$$M(K', K; P) = \bar{V}(K', K) + (2\pi)^{-4} \int d^4 K'' \bar{V}(K', K''; P) \bar{G}(P, K'') M(K'', K; P), \quad (B.1)$$

where, if $(p, k), (p', k'), (p'', k'')$ are the initial, intermediate and final nucleon and pion four-momenta, the conserved total momentum is $P = (p+k) = (p'+k') = (p''+k'')$, and the relative four momentum is $K, K'', K' [K \equiv (k-p)/2 \text{ etc.}]$. The invariant mass (squared) of the system is $s = P^2$.

Neglecting the complications of spin, the propagator is simply

$$\bar{G}(+) (P, K'') = +i [(P/2 - K'')^2 - m_N^2 + i\epsilon]^{-1} [(P/2 + K'')^2 - m_\pi^2 + i\epsilon]^{-1}, \quad (B.2)$$

*Note that in this section only we conform to the practice of using momentum states normalised so that $\langle \vec{p}' | \vec{p} \rangle = (2\pi)^3 \delta(\vec{p}' - \vec{p})$, which is conventional in the literature on this subject.

where the factor i is included to match other conventions. (In fact, all our considerations will involve a non-relativistic treatment of the nucleon spin.) Using Eq. (A.16) then implies that the discontinuity of this Green's function across the (right-hand) unitarity cut is

$$\text{disc} \{ \bar{G}(P, K'') \} = -4\pi^2 i \delta^+ [(P/2 - K'')^2 - m_N^2] \delta^+ [(P/2 + K'')^2 - m_\pi^2], \quad (B.3)$$

(where as usual $\delta^+(k^2 - m^2)$ is $\delta[k_0 - (\vec{k}^2 + m^2)^{1/2}/2k_0]$). Note that the unitarity cut involves only physical states so that both the pion and nucleon must be on-mass-shell).

The Blankenbeller-Sugar procedure aims to preserve this crucial property (B.3) in a new Green's function $[G(P, K'')]$, which has a much simpler structure—essentially freezing out one degree of freedom of the real system. The easiest way to preserve (B.3) is to write the following dispersion relation

$$G(P, K'') = \frac{1}{2\pi i} \int_{(m_N + m_\pi)^2}^{\infty} \frac{ds'}{(m_N + m_\pi)^2} \frac{ds'}{s' - s} \text{disc } \bar{G}(\sqrt{s'}/s, P, K'') \quad (B.4)$$

Now, if we define the on-mass-shell values of the nucleon and pion energies

$$E_K = (\vec{K}^2 + m_N^2)^{1/2}; \quad W_K = (\vec{K}^2 + m_\pi^2)^{1/2}, \quad (B.5)$$

and remember the standard relation for the Dirac δ -function

$$\delta(a+b) \delta(a-b) = \delta(a) \delta(b)/2, \quad (B.6)$$

Eq. (B.4) is easily evaluated in the three-body c.m. system, where $P = (\sqrt{s}, \vec{0})$. Then we find

$$\begin{aligned} \text{disc } \bar{G}(\sqrt{s'}/s, P, K'') &= -4\pi^2 i \delta^+ \left((\sqrt{s'}/2 - K_0'')^2 - E_K^2 \right) \\ &\quad \delta^+ \left((\sqrt{s'}/2 + K_0'')^2 - W_K^2 \right) \\ &= - \frac{4\pi^2}{E_K^2 W_K^2} \delta(\sqrt{s'}/2 - K_0'' - E_K'') \delta(\sqrt{s'}/2 + K_0'' - W_K'') \\ &= - \frac{24\pi^2}{E_K^2 W_K^2} \delta(\sqrt{s'} - E_K'' - W_K'') \delta(2K_0'' - W_K'' + E_K''). \end{aligned} \quad (B.7)$$

The first δ -function in (B.7) allows us to perform the dispersion integral (B.4)

$$g(p, K'') = \frac{2\pi(E_{K''}'' + W_{K''}'') \delta(2K_0'' - W_{K''}'' + E_{K''}'')}{E_{K''}'' W_{K''}'' [s + (E_{K''}'' + W_{K''}'')^2]} \quad (B.8)$$

Finally, the δ -function in (B.8) makes it possible to remove the K_0'' -integral in the original Bethe-Salpeter equation (B.1). The new equation which results is called the Blankenbecler-Sugar equation

$$T(\vec{K}', \vec{K}; s) = \bar{V}(\vec{K}', \vec{K}) + (2\pi)^{-3} \int \frac{d\vec{K}''}{2E_{K''}'' W_{K''}''} \bar{V}(\vec{K}', \vec{K}'') \frac{(E_{K''}'' + W_{K''}'')}{[s + (E_{K''}'' + W_{K''}'')^2]} T(\vec{K}'', \vec{K}; s). \quad (B.9)$$

Whereas Eq. (B.1) presents major problems for numerical solution even after partial wave decomposition (because it involves two variables $|\vec{K}''|$ and K_0''), Eq. (B.9) reduces to a one variable integral equation just like the Lippmann-Schwinger equation. Thus we have certainly obtained a more practical relativistic equation, which can be solved using the technique described in A (iv). Unfortunately the equation is not unique, because in Eq. (B.4) we could have multiplied $(s'-s)^{-1}$ by any function $f(s, s')$ which is unity when $s = s'$. Alternatively we can write

$$\bar{G} = G + R, \quad (B.10)$$

where R is a residual function which has no two-particle singularities.

The difference in the analytic behaviour of the solutions to (B.1) and (B.9) for a scalar boson exchange interaction $\bar{V}$ was discussed by Blankenbecler and Sugar. In particular the left-hand structure is only approximately reproduced. More recently the tendency is not to use a potential $\bar{V}$ calculated from field theory, but rather to adjust $\bar{V}$ so that the resulting (on-shell) t -matrix fits the experimental phase shifts over some energy range.

$$t_\delta^{(on)} [K, K; s(K)] = - \frac{32\pi^2 s^{1/2}}{K} e^{i\delta_\delta} \sin \delta_\delta. \quad (B.11)$$

While this is a more phenomenological approach, it seems necessary at the present time, if one wants reliable calculations of pion reactions in more complicated systems of two or more nucleons.

Many πN interactions have been constructed to fit the phase shifts in the intermediate energy region, but very few use Eq. (B.9). The most common form is probably the relativistic Lippmann-Schwinger equation, which is "derived" by simply replacing the

kinetic energy operators in Eq. (A.18) by a relativistic (on-mass-shell) form. That is,

$$t_\delta^{LS}(p', p; \sqrt{s}) = v_\delta^{LS}(p', p) + \int_0^\infty dp'' p''^2 \frac{v_\delta^{LS}(p', p'') t_\delta^{LS}(p'', p; s)}{\sqrt{s} - E_{p''} - W_{p''}} \quad (B.12)$$

with the on-shell relationship

$$t_\delta^{LS}[p, p; \sqrt{s(p)}] = -e^{i\delta_\delta} \sin \delta_\delta / \pi \mu(p) \mu(p) = E_p W_p / (E_p + W_p). \quad (B.13)$$

It is rather instructive to see what approximations must be made to reduce Eq. (B.9) to (B.12).

The factor $(2\pi)^3$ in (B.9) is removed by using δ -function normalised momentum states in (B.12). The linearity of the propagator in (B.12) can be obtained by relying on the fact that the most important part of the integral in Eq. (B.12) corresponds to $(E_{K''} + W_{K''}) \sim \sqrt{s}$, so that

$$s - (E_{K''} + W_{K''})^2 \approx 2(E_{K''} + W_{K''})[\sqrt{s} - E_{K''} - W_{K''}]. \quad (B.14)$$

(Notice that this approximation does not affect unitarity!) Finally, let us make the definitions

$$t_\delta^{LS}(\vec{K}', \vec{K}; \sqrt{s}) = (2W_K, 2E_K)^{-1/2} T(\vec{K}', \vec{K}; s) (2W_K, 2E_K)^{-1/2}, \quad (B.15)$$

which explicitly show the relativistic phase space factors,* which relate the Lorentz invariant quantity T to the frame dependent t_δ^{LS} . If we now multiply Eq. (B.9) to the left and right by $(2W_K, 2E_K)^{-1/2}$ and $(2W_K, 2E_K)^{1/2}$ respectively, Eqs. (B.13) and (B.15) lead directly to (B.12).

To some extent the discussion leading to Eq. (B.12) was slightly misleading since we could have written the original dispersion relation (B.4) using $\sqrt{s}$ instead of s as the approximate variable. (Indeed such an approach has been carried through for pion-nucleus scattering by Shakin *et al.*)²⁴ However, in a relativistic theory certain constraints, such as analytically continued time reversal invariance,²⁵ suggest that s is the most natural variable and we prefer (B.9) on these grounds.

*This is the source of the factors which are usually introduced into pion-nucleus scattering theory in a very ad hoc way—c.f. 5D below.

Probably the most widely used πN interactions based on Eq. (B.12) are the rank-one separable interactions of Landau and Tabakin²⁶ and later Londergan *et al.*²⁷ Including spin and isospin the partial wave decomposition of the potential is [generalizing Eq. (A.21) to include spin and isospin]

$$V(\vec{p}, \vec{p}') = \sum_{ljmI} V_{ljI/2j}(\hat{p}') V_{ljI}(p', p) V_{ljI/2j}(\hat{p}) P_I, \quad (B.16)$$

with " P_I " an isospin projection operator. Let us denote the combination (ljI) as α then

$$V_\alpha(p', p) = \lambda_\alpha g_\alpha(p') g_\alpha(p), \quad (B.17)$$

[with $\lambda_\alpha = +1$ (-1) for a repulsive (attractive) interaction.] By the same reasoning used to prove Eq. (A.92), the solution to Eq. (B.12) using (B.17) is

$$t_\alpha(p', p; \sqrt{s}) = g_\alpha(p') D_\alpha^{-1}(\sqrt{s}) g_\alpha(p), \quad (B.18)$$

$$D_\alpha^{-1}(\sqrt{s}) = \lambda_\alpha^{-1} - \int_0^\infty dp'' \frac{p''^2 g_\alpha^2(p'')}{\sqrt{s} + i\epsilon - E_{p''} - W_{p''}} \quad (B.19)$$

For details of the method used to uniquely determine the phenomenological form-factors $g_\alpha(p)$ we refer to Refs. 26, 27. The main difference between the two approaches is that in the Landau-Tabakin work no account was taken of opening inelastic channels. The complex πN phases in the inverse scattering formalism then lead to form-factors which are complex for all momenta. This does not destroy the unitarity of the on-shell t -matrix (by construction), however the half-shell function [c.f. Section A (ii)] for a separable t -matrix is just $\{g_\alpha(p')/g_\alpha(p_{on})\}$, and will in general be complex even below the first inelastic threshold. This is a violation of off-shell unitarity, and would certainly destroy the unitarity relations in a three-body system. On the other hand, Londergan *et al.* managed to avoid this problem by introducing extra energy dependence in the D -function corresponding to the coupled inelastic channels. The form-factors in their formalism are real.

The shapes of the form-factors produced in both formulations were very similar. If mapped into co-ordinate space, the apparent range of the (highly non-local) interaction is of the order of 0.6 fm, which is quite a long range! This finding has important implications for pion-nucleus scattering theory, and we shall see later that there is by no means unanimous agreement on this result. However, as explained briefly in part C (Lecture 2), recent theoretical studies in the πNN system have tended to confirm that the πN interaction should be quite long range.

The relativistic form of nucleon kinetic energy was retained in Eq. (B.12). However, this may lead to inconsistencies when calculating pion-nucleus interactions where the nucleons are usually treated non-relativistically. For this reason, an alternative to Eq. (B.12) in which E_p goes to $(m_N + p^2/2m_N)$ has also found practical use *via*:

$$t_\alpha(p', p; E) = V_\alpha(p', p) + \int_0^\infty dp'' \frac{p''^2 V_\alpha(p', p'') t_\alpha(p'', p; E)}{E + i\epsilon - p''^2/2m_N - W_{p''}}. \quad (B.20)$$

In fact, Thomas has found analytic form-factors

$$g_\ell(p) = p^\ell \left[S_1 / (p^2 + \beta_1^2)^{\ell+1} + S_2 p^{2\ell} / (p^2 + \beta_2^2)^{\ell+1} \right], \quad (B.21)$$

which reproduce the experimental phase shifts quite well up to 300 MeV²⁸ when Eq. (B.20) is used. Although we have not discussed the solution of the three-body scattering equations, it should be mentioned that this involves continuation of the form-factors into the complex momentum plane, so that analytic expressions like (B.21) are essential.

(ii) Relativistic Three-Body Equations for the πNN System

In view of the discussion immediately above, the simplest semi-relativistic three-body theory could be obtained from Eqs. (A.93) by replacing $p^2/2m_N$ everywhere by W_p . These equations (called RPK for "relativistic pion kinematics") have been used quite successfully in a calculation of low energy (50 MeV) $\pi^+ d$ scattering as shown in Fig. 4. However, in order to test such an approximate set of equations it is necessary to construct something better.

The main purpose of this section is to explain how that is done, and what numerical effects the improvements have. Perhaps the most significant fact about the RPK calculation now, is that two quite independent groups^{28,30} have obtained agreement on the partial wave πD amplitudes in this model (using the separable πN interactions of Thomas) to better than 1/2%. This system of equations can therefore serve as the test case requested in Ref. 31.

We have already mentioned that the solution of the full two-body Bethe-Salpeter equation (BSE) is very difficult. Clearly then, the full three-body BSE is not a good starting point for practical three-body equations. Instead, the experience gained in part A with separable interactions is used to guess a suitable set of quasi-two-body relativistic equations. Finally the Blankenbecler-Sugar techniques are used to preserve unitarity in a new (calculable) set of one-dimensional integral equations.

Figure 5 illustrates the three-body equations derived in part A for separable two-body t -matrices. For simplicity of notation,

Fig. 4 Comparison of the 50 MeV π^+d data of Axen *et al.*, with the results of a semi-relativistic three-body calculation—corresponding to the two-body equation (B.20)—see Ref. [28].

we consider only the case of *three identical spinless particles* in the formal discussion, and only show the n NN case explicitly at the end of this section. The approach of Aaron *et al.*_{32,33} is to make the *ansatz* that a similar equation can represent the relativistic case of separable pair-wise interactions. That is, one writes the integral equation for scattering of a projectile from a two-body bound (or resonant) state as

$$T(p', p; P) = B(p', p; P) + (2\pi)^{-4} \int d^4 p'' B(p', p''; P) \tau[(P-p'')^2] T(p'', p; P). \quad (B.22)$$

With a slightly different mathematical interpretation, this can also be represented by Fig. 5. As in the non-relativistic case, the driving term "ig" is an effective potential generated by the exchange of one particle between the initial and final pairs.

After partial wave decomposition, Eq. (B.22) leads to coupled two-dimensional integral equations, of complexity similar to the two-body Bethe-Salpeter equation. To obtain more practical one-dimensional equations we apply the Blankenbecler-Sugar procedure to Eq. (B.22). In order to apply unitarity constraints we need the break-up $(2+3)$ amplitude, and consistent with the isobar idea [c.f. Eq. (A.98)], that is given by

$$T_{3+2}(p|p_1 p_3; p; P) \sim g(p_1', p_j') S[(P-p_k')^2] T(p_k', p; P), \quad (B.23)$$

where S describes the isobar propagation, and g the dissociation of the isobar (see Fig. 6).

It is now possible to write down two expressions for the discontinuity of the scattering amplitude T across its right hand cut. The first is simply the general expression of two- and three-body unitarity which is *assumed* to be obeyed. That is_{32,33}

$$T(p', p; s^+) - T(p', p; s^-) = i \sum_{n=2,3} d\rho_n T_{n+2}(n, p'; s^-) T_{n+2}(n, p; s^+), \quad (B.24)$$

where $d\rho_n$ is the two- or three-body relativistic phase space

$$d\rho_n = (2\pi)^4 \delta^4(P - \sum_{i=1}^n p_i) \prod_{i=1}^n [d^4 p_i (2\pi)^{-4} 2\pi \delta^+(p_i^2 - m_i^2)]. \quad (B.25)$$

Fig. 5 Graphical representation of either the non-relativistic three-body equations (A.93), or the relativistic equations (B.22) or (B.35), in the case of separable interactions. In all cases the driving term represents the free exchange of one particle between the initial and final quasi-two-body states.

Fig. 6 Illustration of the isobar assumption (B.23) for the 2+3 (break-up) amplitude.

The second expression for the discontinuity is given directly by the integral equation (B.22)—as first proven by Freedman *et al.* 20 (see also Appendix A of Ref. 33). When the momenta (p', p) are on-energy-shell (that is correspond to physical momenta for scattering from a bound state at energy s), this discontinuity is

$$\begin{aligned} T(p', p; s^+) - T(p', p; s^-) &= (2\pi)^{-4} \int d^4 p'' T(p', p'', s^+) [\tau(\sigma_p^+) \\ &\quad - \tau(\sigma_p^-)] T(p'', p; s^-) \\ &\quad + (2\pi)^{-8} \int d^4 p'' d^4 p''' T(p', p'', s^+) \tau(\sigma_p^n) \\ &\quad [B(p'', p''', s^+) - B(p'', p''', s^-)] \tau(\sigma_p^m) T(p''', p; s^-), \end{aligned} \quad (\text{B.26})$$

where [from Eq. (B.22)] we have defined

$$\sigma_p = (p - p')^2, \quad (\text{B.27})$$

the invariant mass of an interacting pair. Physically the first term on the right has two parts. First the bound state pole in τ (if any) gives the contribution of elastic and re-arrangement scattering to the unitarity relation. The continuum discontinuity of τ , and the second term on the right of (B.26), both contribute to the break-up part of the unitarity relation.

Equating the r.h.s. of (B.26) and (B.24) [using (B.23)]—that is demanding exact two and three-body unitarity—leads to equations for the right hand discontinuities of $B(s)$ and $\tau(s)$. With one more assumption, namely that the isobar propagators in the two and three-body equations are essentially the same

$$\tau(\sigma_p) = 2\pi \delta^+(p^2 - m^2) S(\sigma_p), \quad (\text{B.28})$$

this gives (for details we refer to Ref. 33)

$$\begin{aligned} B(p', p; s^+) - B(p', p; s^-) &= i g[(p - p' - 2p)^2] 2\pi \delta^+[(p - p' - p)^2 - m^2] \\ &\quad g[(p - p - 2p')^2]: p'^2 = p^2 = m^2, \end{aligned} \quad (\text{B.29})$$

$$\begin{aligned} S(\sigma^+) - S(\sigma^-) &= 2\pi i \delta^+(\sigma - \mu^2) \\ &\quad + \frac{S(\sigma^+) S(\sigma^-)}{2(2\pi)^4} i \int d^4 q_i g^2(q_i^2) (2\pi)^2 \delta^+(p_j^2 - m^2) \delta^+(p_k^2 - m^2). \end{aligned} \quad (\text{B.30})$$

We stress that the assumption (B.28) means that Eq. (B.29) applies only for on-mass-shell particles ($p^2 = p'^2 = m^2$). Thus we can

construct only the on-mass-shell born term by writing a dispersion relation with (B.29) as the right-hand discontinuity.

As in the two-body case, this dispersion relation is most easily evaluated in the three-body c.m., where $|\mathbf{k}| = (k^2 + m^2)^{1/2} = w_{\mathbf{k}}^2$

$$\delta \left[\left(\sqrt{s}/s \right) P \cdot k - k'^2 - m^2 \right] = \left(2e^2/k^2 \right)^{-1} \delta \left(\sqrt{s}^2 - e_{\mathbf{k}}^2 - e_{\mathbf{k}'}^2 - e_{\mathbf{k}+\mathbf{k}'}^2 \right) \quad (\text{B.31})$$

(remember that $k^0 = e_{\mathbf{k}}, k'^0 = e_{\mathbf{k}'}$, because of the on-mass-shell condition). Performing the dispersion integral analogous to Eq. (B.4) leads to the result

$$B(p', p; s) = \frac{g(Q'^2) (w_p^2 + w_{p'}^2 + w_{p+p'}^2) g(Q^2)}{w_{p+p'}^2 [s + i\epsilon - (w_p^2 + w_{p'}^2 + w_{p+p'}^2)^2]} \quad (\text{B.32})$$

Similarly the isobar propagator can be constructed from Eq. (B.30) (where μ is the bound state mass, when there is a bound state). In the absence of a bound state, it can be proven that the solution of the two-body Blankenbecler-Sugar equation (B.9) with a separable potential like (B.17)

$$D_\alpha(\alpha) = \lambda_\alpha^{-1} - (2\pi)^{-3} \int_0^\infty \frac{dp}{w_p^2} \frac{p^2}{\alpha + i\epsilon - 4w_p^2} \quad (\text{B.33})$$

satisfies Eq. (B.30), provided

$$S_\alpha(\alpha) = D_\alpha^{-1}(\alpha). \quad (\text{B.34})$$

[We have specialised Eq. (B.33) to the equal mass case.] This is a very satisfying consistency test on the procedure we have used. The key assumption which led to this result was implicit in the very first three-body equation (B.22), where the isobar propagator was only allowed to depend on the invariant mass of the two-body sub-system.

If we now put Eqs. (B.34), (B.28) and (B.32) into (B.22), we obtain the following relativistic three-body equation (in the three-body c.m.)

$$T(\vec{p}', \vec{p}; s) = B(\vec{p}', \vec{p}; s) + (2\pi)^{-3} \int \frac{d^3 p''}{2w_{p''}^2} \frac{B(\vec{p}', \vec{p}'', s) T(\vec{p}'', \vec{p}; s)}{D(\alpha_{p''})} \quad (\text{B.35})$$

$$B(\vec{p}', \vec{p}; s) = \frac{g(Q'^2) (w_{\vec{p}'}^2 + w_{\vec{p}}^2 + w_{\vec{p}+\vec{p}'}^2) g(Q^2)}{w_{\vec{p}+\vec{p}'}^2 [s + i\epsilon - (w_{\vec{p}}^2 + w_{\vec{p}'}^2 + w_{\vec{p}+\vec{p}'}^2)^2]} \quad (\text{B.36})$$

Notice that the on-mass-shell assumption for the spectator particle in intermediate states [Eq. (B.28)] results in equations for only the on-mass-shell scattering amplitudes. Fortunately this is what we need for calculating cross sections. (The deeper question of whether this assumption is actually necessary has only recently been asked.³⁴ It may well be that the apparent solution of the clustering problem through (B.28) is not strictly correct. This is certainly a topic for further research.) Thus, the theory which we have described is an on-mass-shell, off-energy-shell theory, analogous to the coupled channels Lippmann-Schwinger equation. The resulting scattering amplitudes are nevertheless Lorentz invariant.

The generalisation of these equations to the πd system (including spin and isospin) has been given by Ring³⁵ and Thomas³⁵ (see also Ref. 36). (N.B. Henceforth $w_{\vec{p}}^2 = (\vec{p}^2 + m_\pi^2)^{1/2}$, $E_{\vec{p}} = (\vec{p}^2 + m_N^2)^{1/2}$.)

$$X_{d,d}(\vec{p}, \vec{p}_0; s) = 2 \int \frac{d^3 p'}{16\pi^3 E_{\vec{p}}^2} Z_{d,\Delta}(\vec{p}, \vec{p}'; s) D_\Delta^{-1}(\alpha_{\vec{p}}^2) X_{d,d}(\vec{p}', \vec{p}_0; s)$$

$$X_{d,d}(\vec{p}, \vec{p}_0; s) = Z_{d,d}(\vec{p}, \vec{p}_0; s) + \int \frac{d^3 p'}{16\pi^3 E_{\vec{p}}^2} Z_{d,\Delta}(\vec{p}, \vec{p}'; s) D_\Delta^{-1}(\alpha_{\vec{p}}^2)$$

$$X_{d,d}(\vec{p}', \vec{p}_0; s)$$

$$+ \int \frac{d^3 p'}{16\pi^3 w_{\vec{p}}^2} Z_{d,d}(\vec{p}, \vec{p}'; s) D_d^{-1}(\alpha_{\vec{p}}^2) X_{d,d}(\vec{p}', \vec{p}_0; s).$$

(B.37)

In addition to indicating whether the bound or resonating pair is a deuteron or P_{33} state, the labels " d " and " Δ " (respectively) include spin and isospin projections. After a partial wave decomposition which is rather more complicated than usual [see Ref. 35 Eqs. (2.21) and (3.7)], this leads to a set of one dimensional coupled integral equations which can be solved by the usual technique of contour rotation and numerical quadrature.

On-shell, the Lorentz invariant πd scattering amplitude $X_{dd}[\vec{p}_0, \vec{p}_0; s(p_0)]$ is related (in a diagonal partial wave " l, l ") to the phase shifts by

$$X_{l,d,l,d}^{\pi\pi} [p_0, p_0; s(p_0)] = - \frac{32\pi^2}{p_0} s^{1/2} e^{i\delta_{l,d}} \sin \delta_{l,d}. \quad (\text{B.38})$$

The driving terms in Eq. (B.37) are specified entirely by kinematic terms and the $d \rightarrow \pi N$ and $\Delta \rightarrow \pi N$ form factors [c.f. (B.36)]

$$Z_{d,d}(\vec{p}, \vec{p}'; s) = \frac{g_\Delta(\vec{Q}) (E_{\vec{p}}^2 + w_{\vec{p}}^2 + E_{\vec{p}+\vec{p}'}^2) g_\Delta(\vec{Q}')}{E_{\vec{p}+\vec{p}'}^2 [s + i\epsilon - (E_{\vec{p}}^2 + w_{\vec{p}}^2 + E_{\vec{p}+\vec{p}'}^2)^2]} \quad (\text{B.39})$$

Fig. 7 Pion-deuteron scattering at 142 MeV with only the P_{33} interaction. This shows the effect of various theoretical approximations (a) RPK means that relativistic kinematics are used for the pion only (as in Fig. 4), (b) RPKI shows the effect of using the correct relativistic relative momenta (B.42), (c) FR is the result of the fully relativistic calculation using Eq. (B.37).³⁵⁾

$$Z_{\Delta, \Delta}(\vec{q}, \vec{q}'; s) = \frac{g_{\Delta}(\vec{q})(E_{\vec{p}} + E_{\vec{p}'} + w_{\vec{p}+\vec{p}'}^2)g_{\Delta}(\vec{q}')}{w_{\vec{p}+\vec{p}'}^2[s + i\epsilon - (E_{\vec{p}} + E_{\vec{p}'} + w_{\vec{p}+\vec{p}'}^2)^2]} \quad (\text{B.40})$$

So far the only quantities not defined are the relative three-momenta $(\vec{q}, \vec{q}')$. The correct expressions for these were first derived by Aaron *et al.*³² showing that with a careful choice the three-vector dot-product $\vec{q} \cdot \vec{q}'$ could be made Lorentz invariant. The answer was later shown³⁷ to be equivalent to applying a Lorentz boost to the c.m. of the pair of particles. That is, if two particles have (on-mass-shell) momenta (k_1, k_2) in some frame, then defining the total momentum $K = k_1 + k_2$, and $L(K)$ the Lorentz transformation which boosts to the frame where K is time-like, one finds

$$\pm \vec{q}_i = L_{1\mu}(K) \begin{cases} (k_1)^\mu \\ (k_2)^\mu \end{cases}, \quad (\text{B.41})$$

which are the momenta of 1 and 2 in their c.m. system. More explicitly this relativistic relative momentum is

$$\vec{q}(k_1, k_2) = \vec{q} - (\vec{q} \cdot \vec{K}) \vec{K} / (K_0 + (K^2)^{1/2}). \quad (\text{B.42})$$

(It is the rather complicated dependence of $\vec{q}$ on the energies of the particles which complicates the partial wave decomposition, as we mentioned earlier.) Note that consistent with the on-mass-shell off-energy-shell nature of these equations, the four-momentum of (say) the exchanged nucleon in $Z_{\Delta, \Delta}$ [Eq. (B.39)], used to calculate $(\vec{q}, \vec{q}')$, is

$$P_{ex} = (E_{\vec{p}+\vec{p}'}, -\vec{p}-\vec{p}'). \quad (\text{B.43})$$

(iii) Applications to πd Scattering

The relativistic three-body equations whose derivation was sketched in the preceding section, represent the minimal equations which guarantee unitarity within an isobar-like theory. We refer to the original paper [35] for details of the deuteron and Δ form-factors used. Fig. 7 illustrates the effect at 142 MeV laboratory energy (the lowest energy studied), of gradually including the relativistic extension of the non-relativistic expression for the pion kinetic energy. (These results have now been confirmed completely independently by the Lyon group.)³⁰⁾ In this model the πN relative momentum was simply

$$\vec{q}_{\pi N}^{RPK} = (m_N \vec{k}_\pi - m_\pi \vec{k}_N) / (m_N + m_\pi). \quad (\text{B.44})$$

The effect of using the relativistic relative momentum (B.42) is shown as RPKI (improved). This is obviously quite a significant effect beyond $\sim 70^\circ$. Unfortunately no check has been made of the expression

Fig. 8 A comparison of the fully relativistic nd calculation of Rinat and Thomas³⁵⁾ with the data of Norem³⁹⁾ at 180 MeV for various deuteron d -state probabilities. (N.B: The deviation at the furthest forward angles occurs because no account is taken of the coulomb interaction.)

Fig. 9 Comparison of the Glauber calculations of Hoenig and Rinat⁴¹⁾ (solid curve), with data and the relativistic three-body calculations of Rinat *et al.*⁴⁰⁾ (dashed curve), for nd scattering at several energies through the resonance region.

$$\tilde{Q}(\text{semi-relativistic}) = (m_N \vec{k}_N - w_N^2 \vec{k}_N) / (m_N + w_N^2), \quad (8.45)$$

which can be shown analytically to be closer to Eq. (8.42), and which has found common use in pion-nucleus calculations. Finally the curve labelled FR (fully relativistic) shows the effect of using the non-linear form of propagator (B.39) [c.f. $(\sqrt{s} - w)^{-1} - E_N^2 - E_N^2 - 1$ for RPK], as well as retaining the relativistic phase space factors (w_N^2 , etc.). This improvement clearly produces significant differences at all angles.

The comparison of theory with experiment is still in a very early stage. There is no available polarisation data, and while the general trends of the differential cross section have been mapped out by a number of groups, there is as yet no high precision data (i.e. errors of order 5%). Hopefully this will soon be remedied by the group using the pion spectrometer SUSIE at SIN—at least through the resonance region.

If one includes only the d and Δ two-body interactions, the agreement with existing data³⁹ is quite good at 180 MeV (see Fig. 8). This is because the πN interaction has almost no real part at this energy, so that the interference effect of other non-resonant πN channels is negligible. Away from resonance, however, the interference effects of other partial waves are very important. Fig. 9 compares the recent relativistic three-body calculations of Rinat *et al.*⁴⁰ with Glauber calculations⁴¹ and the available data at 142, 180, 234 and 256 MeV. The agreement at 142 MeV is very good, and the non-resonant background is essential to this. At 180 and 234 MeV the situation is also quite good. However, the CERN data at 256 MeV⁴² is very much below the theory in the (80°-160°) region. This experimental trend seems to have been confirmed by recent LAMPF data.⁴³

For more detail of the results of recent calculations, and the latest experiments we refer to the latest review articles.⁴⁴ To summarize briefly, however, the theory seems good (as far as it has been tested) below about 230 MeV. Above this energy some new process seems to set in which is not contained within the present model.

LECTURE 2

The first lecture was devoted to the development of a very solid basis in the non-relativistic theory of three-body scattering, and its relativistic generalisation. If we have succeeded in communicating some understanding of the two-body t -matrix, the advantages of the momentum representation, and the fundamental role of unitarity in few-body problems our aims have been met.

In this section we first describe (in Section C) a very recent application of three-body scattering theory to the πN system which has thrown considerable new light on the question of the range of the πN interaction. As we shall see later, this is a very important question in developing a theory of pion-nucleus scattering. The major part of this lecture will then be used to explain some aspects of the treatment of pion-nucleus scattering by an optical model. As well as deriving the conventional optical potentials from the general formalism, we will explain the very recent three-body model of the first-order optical potential in some detail.

C. The Range of the πN Interaction

The field theory of the πN system has been discussed in considerable detail by Dr. Kisslinger, so we will only recall the aspects directly relevant to the present discussion. In the static Chew-Wick field theory the $N \rightarrow N\pi$ vertex is schematically

$$H_I \sim (f/m_\pi) v(k) \vec{\sigma} \cdot \vec{k}, \quad (C.1)$$

where $\vec{k}$ is the momentum of the pion produced, and $v(k)$ is a form-factor [$v(k) \sim \theta(m_N - k)$] for the vertex. The lowest order contribution to πN scattering is therefore ($|\vec{k}| = |\vec{k}'|$)

$$t_{\pi N}^{(\text{Born})}(\vec{k}', \vec{k}) \sim \left(\frac{f}{m_\pi}\right)^2 \frac{v(k') v(k)}{(4w_k w_{k'})^2} \left\{ \frac{\vec{\sigma} \cdot \vec{k}'}{w_k} \frac{\vec{\sigma} \cdot \vec{k}}{w_{k'}} - \frac{\vec{\sigma} \cdot \vec{k} \vec{\sigma} \cdot \vec{k}'}{w_k} \right\}. \quad (C.2)$$

This is shown diagrammatically in Fig. 10, where the direct (s-channel) nucleon pole term (Fig. 10a) is the first term on the r.h.s. of Eq. (C.2), and the crossed term (Fig. 10b) is the second. In a theory with no anti-nucleons, Fig. 10a can only contribute in a channel with the quantum numbers of the nucleon, namely ($J^\pi = \frac{1}{2}^+$, $I = \frac{1}{2}$). In the standard notation of πN scattering this is the P_{11} channel.

The crossed term, represented by Fig. 10b, contributes in all πN P -waves. In fact, the amplitude (C.2) (including isospin) is crossing symmetric, and one may look at the crossed term as being a necessary addition to the nucleon pole [clearly Eq. (C.2) has a pole

Fig. 10 The Born approximation diagrams for πN scattering in the Chew-Low model.

at $w=0$, which is the nucleon mass] to guarantee crossing symmetry—this is its 'raison d'être'. Using the conventional normalisation, and including isospin, the correct expression for the (on-shell) πN Born amplitude is

$$t_{\alpha}^{(\text{Born})}(\vec{k}', \vec{k}; w) = \frac{4\pi v(k') v(k)}{(4w_k' w_k)^{1/2}} \sum_{\alpha} P_{\alpha}(\lambda_{\alpha}/w), \quad (\text{C.3})$$

where P_{α} is a projection operator for channel α [c.f. Eq. (B.15)], and $\lambda_{\alpha} = 2/3 (f^2/m_{\pi}^2) (-4, -1, -2)$ for $\alpha = (P_{11}, P_{13} = P_{31}, P_{33})$.

It is a fundamental, but often neglected, problem of multiple scattering theory, how to incorporate the pole at $w=0$ evident in Eqs. (C.2) and (C.3). The direct pole in the P_{11} channel (with strength $\lambda_{11}(\text{Dir}) = -3 f^2/m_{\pi}^2$ —see Fig. 10a) is associated with pion absorption in a system with more than one nucleon, as we shall mention again in Lecture 3. However, all the other πN P -waves also have a pole at $w=0$, or at the nucleon mass. This is a little more difficult to interpret because it seems to be merely a reflection of the direct P_{11} pole into the other channels as a result of crossing symmetry—through the relationship^{4,6)}

$$h_{\alpha}(-w) = A_{\alpha\beta} h_{\beta}(+w). \quad (\text{C.4})$$

In Eq. (C.4) $h_{\alpha}(w)$ is related to the full Chew-Low scattering amplitude by

$$t(\vec{k}', \vec{k}; w) = \frac{4\pi v(k') v(k)}{(4w_k' w_k)^{1/2}} \sum_{\alpha} P_{\alpha} h_{\alpha}(w), \quad (\text{C.5})$$

and in Born approximation h_{α} equals (λ_{α}/w) .

In fact, the correct fully off-shell energy denominator corresponding to this pole in the on-shell P_{33} interaction is

$$G_0 \sim (w + i\epsilon - w_k^2 - w_{k'}^2)^{-1} \quad (\text{C.6})$$

which for real momenta $\vec{k}$ and $\vec{k}'$ cannot vanish for $w < 2m_{\pi}$. That is, the first unitarity cut associated with this diagram begins at the threshold for pion production. Therefore, in potential language the pole in the on-shell P_{33} amplitude is left-hand structure, and properly incorporated with the potential singularities (i.e. in the N -function in an N/D decomposition).

This is where the Chew-Low representation of the πN scattering amplitude is therefore a little misleading. Equation (C.5) is identical to the form of a separable t -matrix [c.f. Eq. (B.17)], so that one is tempted to identify $D_{\alpha}^{-1} \equiv h_{\alpha}$. However, as we have shown, h_{α} contains a 'nucleon pole' even for $\alpha \neq P_{11}$. An additional problem, which we have not mentioned, is that h_{α} also has a left-hand cut $w \in (-\infty, -m_{\pi})$, resulting from the application of the crossing relation (C.4) to the standard right-hand (unitarity) cut. This too is properly incorporated amongst the potential singularities in a potential model.

In case this argument sounds like an irrelevant exercise in changing the name associated with a singularity, we remind the reader that in any many nucleon system, and particularly in a three-body system, the energy argument of each two-body t -matrix is needed over the range $(-\infty, E_{C.m.})$, with $E_{C.m.}$ the three-body c.m. energy [c.f. part (iii) of Section A]. In a separable model, this means performing an integral over the D -function for this energy range. The naive identification of D_{α}^{-1} with h_{α} would then mean that the pole at $w_N=0$, and the cut for $w_N \in (-\infty, -m_{\pi})$, would both contribute to any positive energy (actually any $E > -m_{\pi}$ and $E > -2m_{\pi}$ respectively). This would destroy the three-body unitarity relation (or optical theorem), because as we discussed near Eq. (C.6), Fig. 10b cannot describe a real intermediate state unless the two body kinetic energy is greater than m_{π} . This observation was first made by Nijmmer and Thomas,⁵¹⁾ in connection with the triple scattering contribution to πd elastic scattering shown in Fig. 11.

Fig. 11 One triple scattering contribution to πd scattering involving the πN resonant interaction (Δ) and the $N-N$ P_1 interaction. The model dependence of results obtained in Ref. [51] for this graph, led to the consideration of the πN range problem based on three-body unitarity.

To solve this problem we could divide h_α ($\alpha \neq P_{11}$) into two parts h_α^L and h_α^R (for left and right), so that

$$h_\alpha^L(w) = \frac{\lambda_\alpha}{w} + (2\pi i)^{-1} \int_{m_\pi}^{\infty} \frac{dw'}{w' + w} \frac{\text{disc } \{h_\alpha(-w')\}}{w' + w},$$

$$h_\alpha^R(w) = (2\pi i)^{-1} \int_{m_\pi}^{\infty} \frac{dw'}{w' - w} \frac{\text{disc } \{h_\alpha(+w')\}}{w' - w},$$

$$h_\alpha = h_\alpha^L + h_\alpha^R; \quad \alpha \neq P_{11}. \quad (C.7)$$

Here, h_α^R contains only the 'right-hand' (s-channel unitarity) cut and is properly identified with the D-function of a separable t-matrix. On the other hand, the full 'potential structure' in the Chew-Low model is $\{h_\alpha^L(w_k) v^2(k)\}$, which is approximated by the form-factors of a separable model $g_\alpha^2(k)$. Now we are in a position to understand why Ernst and Johnson find $v(k)$ to have a very short range of order m_π^{-1} [actually $v(k) \sim \exp(-30 k^2)$ with k in units of m_π], whereas all separable models of the πN t-matrix have form factors with quite a long range [$\sim \frac{1}{2}$ fm - see the discussion after Eq. (B.18)].

What the separable models simulate is the nearby left-hand structure in $h_\alpha^L(w_k)$, namely a pole at $k = \pm i m_\pi$ and a cut starting at $k = \pm 2i m_\pi$. With an analytic form like Eq. (B.21), for example, one gets poles in the on-shell t-matrix at k equals $i\beta_1$ and $i\beta_2$. Clearly if $h_\alpha^L(w_k)$ has its most significant strength at small values of w_k (e.g. at $w_k=0$ from the nucleon pole), at least one of the fitted ranges β_1 and β_2 could be quite small—of order one or two times m_π . Thomas, for example, using the semi-relativistic scattering Eq. (B.20), found that (β_1, β_2) equals (1.475, 3.4) fm $^{-1}$ gave a good representation of the P_{33} phase shifts below 300 MeV.²⁸

Our conclusion therefore, is that the range of a πN t-matrix suitable for use in a multiple scattering treatment of pion scattering from more than one nucleon, is quite long (in co-ordinate space). The implications for pion-nucleus scattering will be discussed in the following sections, but essentially it reduces the importance of N-N short-range correlations, and implies that Bèg's theorem^{47, 48} is not applicable. (That is, one cannot calculate pion-nucleus scattering from purely on-shell πN information.) Our conclusion concerning the relationship of the πN range to the singularity structure of Chew-Low theory is closely related to other recent work.^{49, 50} Nevertheless we feel that the clear statement that unitarity is violated in pion scattering from more than one nucleon if the separation (C.7) is not made,⁵¹ is the most forceful demonstration yet presented. It indicates the power of considerations based on few body systems where exact unitarity is a fundamental consideration.

D. Pion-Nucleus Scattering Theory

In recent times a considerable use has been made of the fixed scatterer approximation (f.s.a.) as an approach for calculating pion-nucleus elastic scattering. Indeed the f.s.a. is an ideal pedagogical tool,⁵² and may be especially useful in obtaining a qualitative understanding of previously unexplored density regions.⁵³ We have also seen in Dr. Gibbs lectures that, at least for light nuclei ($A \leq 12$), once the f.s.a. has been made the complete pion-nucleus multiple scattering series can be summed exactly.

The alternative approach, which we prefer, is to retain the nucleon motion, while expanding the scattering amplitude in series depending on the degree of nuclear excitation. Formally such an approach reduces the full $(A+1)$ body problem to an effective two-body problem, where the optical potential contains all the many-body complications. Although the formulation of the optical model which we follow was first given in 1959 by Kerman *et al.* (KMT),⁵⁴ we shall see that three-body ideas have recently deepened our understanding of it (and helped to explain recent low energy data).

(i) Three-body model of the first-order optical potential. The intimate connection of the equations derived here with three-body scattering theory was first emphasised by Tandy *et al.*⁵⁵⁻⁵⁷ (for proton-nucleus scattering), but similar equations had been found earlier by Reval.⁵⁸ The starting point is non-relativistic scattering theory for a system of a pion (particle α) and A nucleons, with only pairwise potentials $[V_{0j} \text{ and } V_{ij}, (i,j) \in (1 \dots A)]$. Then the channel Hamiltonian for a free pion incident on A bound nucleons is

$$H_0 = K_0 + H_A = K_0 + \sum_{i=1}^A K_i + \sum_{i < j} V_{ij}. \quad (D.1)$$

The interaction which scatters the pion is

$$V^0 = \sum_{i=1}^A V_{oi} , \quad (D.2)$$

and the appropriate scattering equation is the $(A+1)$ -body Lippmann-Schwinger equation

$$T = V^0 + V^0 G_A T , \quad (D.3)$$

$$G_A(E^+) = E^+ - H_0 . \quad (D.4)$$

[Note that we ignore for the moment the formal difficulties of the non-compactness of Eq.(D.3) as discussed below Eq.(A.44).]

The KMT formalism has the advantage that only physically allowed anti-symmetric nuclear intermediate states occur. This is signified by carrying the anti-symmetrization operator ' A ' explicitly for each intermediate state. With the understanding that all the following operator equations are to be evaluated between anti-symmetric nuclear states one can replace $\sum_i V_{i0}$ by AV . Equation (D.3) then becomes

$$T = AV(1 + AG_A T) . \quad (D.5)$$

Let us define the operator ' τ ', which we shall see below is closely related to the free nN t -matrix

$$\tau = V(1 + AG_A \tau) . \quad (D.6)$$

Equation (D.6) may be inverted to obtain

$$V = \tau(1 + AG_A \tau)^{-1} . \quad (D.7)$$

Similarly Eq.(D.5) becomes

$$T = AV(1 - AG_A AV)^{-1} , \quad (D.8)$$

and substituting for V from (D.7) implies after a little algebra

$$T = A[1 - (A-1)AG_A \tau]^{-1} . \quad (D.9)$$

This can be converted to the integral equation

$$T' = U^{(0)} + U^{(0)}AG_A T' , \quad (D.10)$$

with the definitions

$$T' = [(A-1)/A] T , \quad (D.11)$$

$$U^{(0)} = (A-1)\tau . \quad (D.12)$$

Unfortunately, Eq.(D.10) still involves nuclear excited states explicitly. As KMT observed, one expects that in general matrix elements of U leading to an excited state should be smaller ($\sim 1/A$) than diagonal ones. One therefore introduces the ground and excited state projection operators

$$P = |0\rangle\langle 0| ; Q = \sum_{n \neq 0} |n\rangle\langle n|$$

$$= A - P . \quad (D.13)$$

Inserting $(P + Q)$ into Eq.(D.10) we find

$$T' = U^{(0)} + U^{(0)}(P + Q)G_A T' = U + UPG_A T' , \quad (D.14)$$

provided that the new operator U satisfies the equation

$$U = U^{(0)} + U^{(0)}QG_A U . \quad (D.15)$$

Equation (D.14) is ideally suited for the description of pion-nucleus elastic scattering. Indeed taking the ground state expectation value of Eq.(D.14) we obtain a two-body scattering equation

$$T'_{00} = U_{00} + U_{00}G_A^{(0)}T'_{00} , \quad (D.16)$$

where of course $G_A^{(0)}$ is the ground state expectation value of the channel Green's function (D.4). The effective potential U_{00} , which is non-local and energy dependent in general, is called the optical potential. Clearly all the difficulties associated with virtual nuclear excitation have been hidden in it [c.f. the Q operator in Eq.(D.15)].

It is unfortunate that there is no unique definition of the *first order optical potential* ' $U(1)$ '. Such freedom of choice is one of the wonderful things in nuclear physics! The definition which we propose is designed to incorporate as far as possible only the one-body properties of the nucleus. First we discard all but the first term in Eq.(D.15), so that there is no explicit inclusion of nuclear excitation. Second, we approximate H_A by the shell model form $\frac{1}{2}(K_i + u_i)$, neglecting residual N - N interactions. Third, all the nuclear wave functions are assumed to be given by the independent particle shell model (IPSM).

With these approximations Eq.(D.6) is very close to the scattering equation for a pion from one nucleon in a binding potential ' u '. This is only spoiled by the anti-symmetrisation operator which can

involve more than one nucleon in fourth and higher order through exchange effects. Such terms involve higher nuclear excitation, that is larger energy denominators and will be neglected. With this last approximation the operator τ_1 , now labelled $\tau(1)$, to indicate that it is the first order approximation, obeys the equation

$$\tau(1) = V + V \frac{Q(P)}{E^+ - K_0 - K_{A-1} - K - u - \epsilon_B} \tau(1) \quad (D.17)$$

(We have assumed that only the ground state expectation value of $\tau(1)$ is needed.) Here E is the c.m. kinetic energy in the pion-nucleus system, K_{A-1} the kinetic energy operator for the c.m. motion of the core, K and u the nucleon kinetic energy operator and binding potential respectively, and $\epsilon_B(>0)$ the single particle binding energy g_{π_1} . (We set the energy scale so that $H_A \psi_A(g.s.) = 0$, which means $H_{A-1}^{int} \psi_{A-1} = \epsilon_B \psi_{A-1}$.) Finally, $Q(P)$ is a Pauli operator which does not allow the nucleon into an occupied level of the $(A-1)$ system—this is equivalent to the operator A in the approximation outlined above.

If we omit the Pauli operator for the moment, Eq. (D.17) represents the simple three-body problem shown in Fig. 12. Notice particularly that *there is no projectile-core interaction in the first-order potential*. Let us define the πN and N -core t -matrices as usual

$$t_{\pi N}(z) = V + V G_0(z) t_{\pi N}(z), \quad (D.18)$$

$$t_{NC}(z) = u + u G_0(z) t_{NC}(z), \quad (D.19)$$

$$G_0(z^+) = (z + i\epsilon - K_0 - K - K_{A-1})^{-1}. \quad (D.20)$$

Then solving (D.18) for V we find

$$V = (1 + t_{\pi N} G_0)^{-1} t_{\pi N}. \quad (D.21)$$

Substituting (D.21) into Eq. (D.17) (without $Q(P)$) implies that

$$\begin{aligned} \tau(1)(E^+) &= t_{\pi N}(E^+ - \epsilon_B) + t_{\pi N}(E^+ - \epsilon_B) [G_u(E^+ - \epsilon_B) - G_0(E^+ - \epsilon_B)] \\ &\quad \times \tau(1)(E^+), \end{aligned} \quad (D.22)$$

$$G_u(z) = (z^+ - K_0 - K - K_{A-1} - u)^{-1}. \quad (D.23)$$

Now Eq. (A.10) allows us to replace $(G_u - G_0)$ by $G_0 t_{NC} G_0$. The resulting form of Eq. (D.22) is equivalent to the following coupled integral equations

$$\begin{aligned} \tau(1)(E^+) &= t_{\pi N}(E^+ - \epsilon_B) G_0(E^+ - \epsilon_B) \tau_C(E^+), \\ \tau_C(E^+) &= G_0^{-1}(E^+ - \epsilon_B) + t_{NC}(E^+ - \epsilon_B) G_0(E^+ - \epsilon_B) \tau(1)(E^+). \end{aligned} \quad (D.24)$$

Fig. 12 Illustration of the simple three-body problem (in which only two of the three pairs interact) whose solution defines $\tau(1)$.

Fig. 13 A comparison of various first order optical potential calculations^{61, 64} with the 51 MeV ${}^4\text{He}$ data of Croue *et al.*. The difference between the short and long dashed curves illustrates the sensitivity to poorly known low energy πN phase shifts, while the long dashed to solid comparison shows the importance of the three-body choice of πN interaction energy.

Comparison with Eq. (A.65) shows that Eq. (D.24) has exactly the form of the three-body equations of Alt *et al.* for the problem shown in Fig. 12, where there are only two non-zero interactions [e.g.

$$U_{11}(E - \epsilon_B) \equiv \tau(1)(E), \quad U_{21}(E - \epsilon_B) \equiv \tau_C(E), \quad t_1 = t_{\pi N}, \quad t_2 = t_{NC}, \quad t_3 = 0].$$

While we have omitted $Q(P)$ from equations (D.18) through (D.24) they can all be corrected simply by re-inserting it wherever there is a Green's function. The effect is merely to restrict the range of momenta the nucleon can have in each intermediate state.

The most obvious approximation to Eq. (D.24) is to iterate it once, so that $\tau(1)$ is approximately the πN t -matrix in the three-body system evaluated at $(E^+ - \epsilon_B)$. From the discussion in section A (iii) we know that the πN interaction energy therefore covers the range $(-\infty, E^+ - \epsilon_B)$ because of the recoil energy of the core and the πN c.m. motion. In terms of $t_{\pi N}$, the πN t -matrix in the two-body c.m. the first order optical potential is therefore

$$U_{00}^{(1)}(E^+) \approx (A-1) \langle 0 | \tilde{t}_{\pi N}(E^+ - \epsilon_B - K_{A-1} - K_{\pi N}^{c.m.}) | 0 \rangle. \quad (D.25)$$

(The corrections to (D.25) arising from the pion-nucleus to πN c.m. transformation are discussed in the next section.) Since typical nucleon binding energies are of the order of (10-20) MeV and so are the kinetic energy terms, there is a considerable downward shift of the πN interaction implied by Eq. (D.25). This effect was first observed in a non-relativistic model of pion-nucleus scattering, 59) but essentially the same result has also been derived in a fully relativistic theory. 60) A very important result of the shift is that this optical potential is purely real for $E < \epsilon_B$. This is sufficient to guarantee elastic unitarity for T_{00} (to order $1/A$), if there is not sufficient energy to knock-out a nucleon.

The difficulty with this naive use of Eq. (D.25) is of course that t_{NC} is not weak—indeed it supports a bound state. From the fact that u is attractive, the original equation (D.17) suggests that the effect of the N -core interaction might be to compensate to some extent the full energy shift in (D.25). For this reason Landau and Thomas61) treated ϵ_B as a parameter ' E_B ' to vary between the real binding energy and zero, in an attempt to explain the dramatic failure of conventional optical models52) for 50 MeV $\pi^{\pm}$ ^4He and π^+ ^{12}C scattering. (The conventional models are briefly discussed below, but all ignore the three-body shift completely.) Figure 13 shows the improvement of a factor of (2-3) obtained in the fit to the 50 MeV ^4He data when this shift was introduced (with $E_B = 5$ MeV) into a momentum space calculation using separable πN interactions. This was the first strong empirical evidence for the theory described here (see Refs. 63, 64 for a more detailed presentation, including Pauli effects and absorption).

Clearly the present situation regarding the three-body nature

of $\tau(1)$ is unsatisfactory. The assumption in most calculations that a shift of the energy in the two-body t -matrix can approximate the full solution of the three-body problem is untested. Given such an assumption, the work of Landau and Thomas61) ($E_B \sim 5$ MeV) represents an intermediate result—intermediate between Liu and Shakin60) who ignore binding effects completely (and thus get an effective downward shift of order 30 MeV), and Schmidt *et al.*65) who find no shift at all using an approximate Padé technique to solve Eq. (D.17).

The one firm theoretical result on this matter was obtained very recently by Amado *et al.*, 66) who used essentially three-body techniques to extend a theorem of Fuda for n -d scattering to the general projectile - nucleus optical model problem. They established unambiguously that the longest range (peripheral) part of any optical potential can be written entirely in terms of the on-shell projectile-nucleon t -matrix at an energy equivalent to that used in Eq. (D.25). In conclusion, while some hard theoretical work remains to be done, the importance of the three-body picture of the first-order potential, and particularly the choice of πN sub-energy has been established beyond doubt.

(ii) Relativistic aspects of the first-order potential. Almost all theoretical descriptions of pion-nucleus scattering start with non-relativistic scattering theory as we have just done. The relativistic aspects of the problem are then added in at the end (as we do below). A much more satisfying alternative is to follow the same sort of procedure we used for the relativistic three-body problem in part B. That is, one writes down effectively a Bethe-Salpeter equation for the pion-nucleus system and uses the Blankenbeller - Sugar techniques to make it calculable. The latter approach has been carried through by Shakin *et al.*24) Fortunately both these approaches lead to essentially the same first-order potential.

The first correction arises because the pion-nucleon t -matrix is usually calculated in the πN c.m. system using a relativistic Lipmann-Schwinger equation [e.g. Eq. (B.12)]. This t -matrix is how-ever needed in the pion-nucleus c.m. in Eq. (D.25). To make this transformation we use Eq. (B.14) to relate the t -matrix in the πN c.m. [for scattering from $(\vec{Q}, -\vec{Q})$ to $(\vec{Q}', -\vec{Q}')$], to a Lorentz invariant amplitude T . A similar expression yields the t -matrix in the pion-nucleus system [for scattering $(\vec{k}, \vec{p})$ to $(\vec{k}', \vec{p}')$] in terms of T . To summarise then, the first order optical potential including relativistic effects is approximately64)

$$U(1)(\vec{k}', \vec{k}; E^+) = \frac{A-1}{A} F_{c.m.}(\vec{q}) \sum_{\vec{a}} \int d^3p_a^* (\vec{p}-\vec{q})^{\dagger} \times \left(\frac{W_{\vec{Q}'}^{\dagger} E_{\vec{Q}'}^{\dagger} W_{\vec{Q}}^{\dagger} E_{\vec{Q}}^{\dagger}}{W_{\vec{k}'}^{\dagger} E_{\vec{p}-\vec{q}}^{\dagger} W_{\vec{k}}^{\dagger} E_{\vec{p}}^{\dagger}} \right)^{1/2} t_{\pi N}^{LS}(\vec{Q}', \vec{Q}; E^+ - E_B(\vec{a}) - \frac{(\vec{p}+\vec{k})^2}{2M}) \times \psi_a(\vec{p}). \quad (D.26)$$

(Here M is the reduced mass of the (πN) + core system, and $F_{c.m.}(\vec{q})$ is the usual correction factor, based on a harmonic oscillator model, which corrects for the omission of a c.m. constraint in the IPSM. 67) Here the sum is over all the nucleons in their IPSM states (ψ_a) , the momentum transfer is $\vec{q} = \vec{k}' - \vec{k}$, $t_{\pi N}$ is calculated in the πN c.m. system, and we have allowed E_B to depend on the state of the struck nucleon. As in the relativistic πNN case we choose to keep all particles on-mass-shell, so that the relative momenta $(\vec{0}, \vec{q}')$ are given by Eq.(B.41) again.

(iii) Comparison with conventional potential models
a) Landau - Phatak - Tabakin (LPT):

Our entire development of two and three-body scattering theory, as well as the optical model (D.26) has been illustrated using the momentum representation. Because of the simple representation of particle energies this is the natural representation for scattering problems. The reader should try to write the transformation factor in Eq.(D.26) in co-ordinate representation to appreciate the problem! It therefore comes as something of a shock to realize that the first momentum space calculation of pion-nucleus scattering was not published until 1973. 68) One reason is that the price for the extra sophistication is a considerable increase in computational difficulty which does require a large modern computer.

The LPT potential was derived using the theory of Kerman *et al.*, 54) and is equivalent to Eq.(D.26) with two approximations. First the choice of interaction energy was not derived from any theoretical considerations, such as those leading to the three-body choice above. Instead the on-mass-shell four-momenta for the pion and nucleon were used to construct the conventional relativistic s and t for the pair as

$$\begin{aligned} s &= (k + p)^2 = (w_k^2 + E_p^2) - (\vec{p} + \vec{k})^2 \\ t &= (k' - k)^2 = (w_{k'}^2 - w_k^2)^2 - (\vec{k} - \vec{k}')^2 \end{aligned} \quad (D.27)$$

The πN interaction energy (in the two-body c.m. system) is then $(\sqrt{s} - m_N)$ - including the pion mass—and runs above and below the incident pion energy as we integrate over $\vec{p}$. This choice of energy will be called E_2 -body from now on. It is not unique, of course, because s' defined as $(k' + p')^2$ will not be equal to s in general (i.e. for an arbitrary πN collision)!

The second difficulty in the LPT approach is also related to the non-uniqueness of the interaction energy. Their πN t -matrix in the c.m. was parameterised in terms of a scattering angle which was calculated in the standard way from s and t . Unfortunately in an on-mass-shell, off-energy-shell theory (i.e. $s \neq s'$) of this type (c.f. Section B), this does not always lead to a physical scattering angle $[\cos \theta \in (-1, +1)]$.

While these two approximations sound rather drastic, in practice the nuclear form-factor [which is contained in (D.26) i.e. $U(\vec{k}', \vec{k}) \sim F(\vec{k}', -\vec{k})$] restricts $|\vec{k}'| \sim |\vec{k}|$ so that the degree by which $s \neq s'$ is not so great. On the positive side, the transformation factor in (D.26) was included for the first time in any pion-nucleus calculation. In addition, the definition of πN scattering angle while not strictly correct, incorporated an extremely important physical effect—later dubbed the angle transformation—into pion-nucleus scattering theory for the first time. The results of the calculations were in excellent agreement with the available data on ^{12}C and ^{16}O in the resonance region. 68, 69)

b) Δ - propagation:

The full sophistication of the work by Lenz and others has been described at this meeting by Dr. Moniz. In its simplest form, their approach represents an alternative calculational technique to the first order optical model. Their basic πN interaction in the P_{33} channel is separable, but greater use is made of the separability to identify the ' Δ ' as an extra degree of freedom. It should be born in mind that their Δ has exactly the same meaning as that in the relativistic three-body equations (B.36).

The 'propagation of the Δ ' refers to the independent realization by Lenz 70) (in the context of pion scattering in the resonance region), that the three-body energy $\sim (E^2 - E_B - T_\Delta)$ is the natural argument of the πN D -function in a many nucleon system. Without the kinetic energy term T_Δ the c.m. of the πN sub-system does not move through the nucleus. This clear inclusion of E_3 -body is a little hidden when higher order connections are introduced through a complex Δ - core interaction, but the essential idea is there.

The scattering angle in the πN c.m. system is calculated through $\vec{Q} \cdot \vec{Q}' / |\vec{Q}| |\vec{Q}'|$, with the πN relative momentum

$$\vec{Q} = (m_N \vec{k} - w_k^2 \vec{p}) / (m_N + w_k^2) \quad (D.28)$$

The reader should confirm that this is the semi-relativistic limit of Eq.(B.41), and is probably a rather good approximation to it. For details of the results of this approach I defer to the lectures of Dr. Moniz.

c) Co-ordinate space calculations:

The co-ordinate space potentials of Kisslinger, 71) and later (and more sophisticated) of Ericson and Ericson, 72) are still being very widely used to understand pion-nucleus data. Undoubtedly a major reason is the familiarity of the co-ordinate space picture to nuclear physicists, and the existence of many scattering programs which could easily be modified to accept the velocity dependence in this model.

To be fair, the model also provided first an explanation of the role of the nuclear surface in pion scattering, leading to strong back-scattering,⁷¹ and then (with some parameter adjustment) to an impressive explanation of the strong interaction shifts in pionic atoms.⁷³

The first step in deriving the Kisslinger-type potential is to factorise the πN t-matrix out of Eq. (D.26). As we soon show, this is no problem in the usual co-ordinate space picture as the dependence of $t_{\pi N}$ on p is usually ignored. However, factorisation has also been used in almost all momentum space calculations.* For a detailed description of the steps involved we refer to Refs. [64] (33B) and [70]. The crudest factorisation of (D.26) is simply,

$$u(1)(\vec{k}', \vec{k}; E) \simeq (A-1) F(\vec{q}) t_{\pi N} [\vec{Q}'(\vec{k}', \vec{p}_0 - \vec{q}), \vec{Q}(\vec{k}, \vec{p}_0)];$$

$$E^+ - E_B - (\vec{p}_0 + \vec{k})^2/2M \quad (D.29)$$

where γ is the relativistic transformation factor and $\vec{p}_0$ is the nucleon momentum which optimises the approximation, and $F(\vec{q})$ is the nuclear form-factor, as measured (say) by electron scattering.

At this stage, if we drop the factor γ (which we noted above cannot be calculated in the co-ordinate space representation), and approximate the p-wave πN t-matrix as

$$t_{\pi N}(\vec{Q}', \vec{Q}; E^+ - E_B - (\vec{p}_0 + \vec{k})^2/2M) \simeq c(E) \vec{k}' \cdot \vec{k}, \quad (D.30)$$

we obtain the Kisslinger potential

$$u^{Kiss}(\vec{k}', \vec{k}; E) = (A-1)c F(\vec{q}) \vec{k}' \cdot \vec{k}. \quad (D.31)$$

When Fourier transformed this leads to the velocity dependent potential

$$u^{Kiss}(\vec{r}) = (A-1)c \vec{\nabla} [\rho(r) \vec{\nabla}], \quad (D.32)$$

and it is the emphasis on the gradient of the single particle density $[\rho(r)]$ which gives the nuclear surface its importance in pion scattering. Note that in order to derive Eq. (D.31, 32), on top of all the other approximations the three-body energy shift, and the very important dependence of $(\vec{Q}, \vec{Q}')$ on the nucleon momentum were ignored.

*The exceptions are calculations by Lee,⁷⁴ and Liu and Shakin,⁷⁵ both of whom find that the approximation gets worse at lower energy. Nevertheless the agreement between Liu and Shakin,⁷⁵ and the approximation of Landau and Thomas⁶⁴ is quite good even at 50 MeV.

The latter approximation has been shown to be so poor,⁶⁸ that recent calculations have attempted to make some improvement on it.^{76, 77} Using Eq. (D.28) one obtains⁷⁶ $[\pi_N/(\pi_N + \pi_K^2)] \sim 1$, $\pi_K^2/(\pi_N + \pi_K^2) = \xi$

$$\vec{Q}' \cdot \vec{Q} \simeq [\vec{k}' - \xi (\vec{p}_0 - \vec{q})] \cdot [\vec{k} - \xi \vec{p}_0], \quad (D.33)$$

and if p_0 is set to zero, to $0(\xi)$ we find

$$\vec{Q}' \cdot \vec{Q} \simeq (1 + \xi) \vec{k}' \cdot \vec{k} - \xi \vec{k}^2. \quad (D.34)$$

The extra term $\xi \vec{k}^2$ contributes to an effective s-wave πN interaction in the pion-nucleus system. Depending on the way its approximated it gives either a purely local term ($\vec{k}^2 \rho(r)$),⁷⁷ or a mixture of a velocity dependent and a local part.⁷⁶ In either case this recoil correction has a dramatic effect on low energy pion elastic scattering.

The p-wave part of the Ericson-Ericson potential is also given by Eq. (D.32). In addition they added a term $\{b_0 \rho(r)\}$ corresponding to s-wave πN scattering, plus terms dependent on $\rho^2(r)$ representing a phenomenon which really lies outside the scope of potential theory—namely pion absorption. (We shall return to this question in Lecture 3.) Finally Ericson and Ericson noticed that the divergent off-shell behaviour of the postulated πN interaction (D.30) placed a very strong emphasis on the short-range behaviour of the two nucleon wave function. In particular, the double scattering term at zero energy goes like⁷⁸

$$T^{(2)}(\vec{k}', \vec{k}) \sim c^2 \int d\vec{r}_1 d\vec{r}_2 \rho(\vec{r}_1, \vec{r}_2) \int d\vec{q} \frac{e^{i\vec{q} \cdot (\vec{r}_1 - \vec{r}_2)}}{0^+ - \vec{q}^2} (\vec{k}' \cdot \vec{q})(\vec{q} \cdot \vec{k})$$

$$\sim c^2 \vec{k} \cdot \vec{k}' \int d\vec{r}_1 d\vec{r}_2 \rho(\vec{r}_1, \vec{r}_2) \int d\vec{q} e^{i\vec{q} \cdot (\vec{r}_1 - \vec{r}_2)}$$

$$\sim c^2 \vec{k} \cdot \vec{k}' \int d\vec{r}_1 d\vec{r}_2 \rho(\vec{r}_1, \vec{r}_2) \delta(\vec{r}_1 - \vec{r}_2), \quad (D.35)$$

which should vanish in the presence of nucleon-nucleon correlations! This is the simplest derivation of the Lorentz-Lorentz effect, first derived for pions by the Ericsons.⁷² It says that for a realistic nuclear density the single-scattering approximation is essentially exact for the p-wave potential (D.32). To subtract out the spurious higher order scattering the p-wave term has a non-linear form,^{72, 79} and the Ericson-Ericson potential (without absorption) is

$$E-E$$

$$u(\vec{r}) = -4\pi b_0 \rho(r) + 4\pi c_0 \vec{\nabla} [\rho(r)/(1 + \frac{4\pi}{3} c_0 \rho(r))] \vec{\nabla}, \quad (D.36)$$

where $\lambda=1$ is the 'full Lorentz-Lorentz effect'. For an excellent discussion of the origin of (D.36) and the definition of b_0 and c_0 in terms of πN phase shifts we refer to Ericson's lectures in Ref. [79]. Recent work by Thies⁷⁶ and Stricker *et al.*⁷⁷ has

introduced some improvements, such as the angle transformation (D.34) and some energy dependence for λ , but the essential physics has not changed.

For comparison, we recall the discussion of Section C concerning the range of the πN interaction. Whereas the postulated πN interaction (D.30) has zero range [$\sim \delta(\vec{r}_N - \vec{r}_\pi)$], there we suggested it should be of order 0.6 fm. This results in a tremendous change in emphasis in the theory. Indeed, as Eisenberg *et al.* (80) have shown [and Hirata *et al.* have confirmed⁷⁰], the longer range πN interaction 'smooths over' the effect of correlations, so that in a typical momentum space calculation correlation effects can be ignored in a first approximation. Thus, as we mentioned in part C, the question of the πN range is absolutely fundamental in pion-nucleus scattering theory.

LECTURE 3

'He that riseth late must trot all day.'
Benjamin Franklin.

The difficulty with the topic being discussed here is that by the time enough basic material has been explained, there is little time left for the most interesting questions. Thus our strategy has been to put enough detail into Lecture 1, not all of which could be covered orally, that the serious student can draw on it in places like part E. There we are motivated by the three-body model of $U(1)$, to examine what inelastic reactions are contained in it. Amongst the processes not included is real pion absorption. The final section therefore reviews what recent theoretical understanding the πNN system has provided on this question.

E. The Reaction Content of the First-Order Potential

We have shown throughout our discussion of three-body theory the emphasis placed on exact unitarity. The real advance made by Tandy *et al.*⁵⁵ for proton-nucleus scattering therefore, was not that they were able to write three-body-equations for $U(1)$, but that they used these equations to show what approximations concerning the inelastic channels were implicit in $U(1)$. The first application of these ideas to pion-nucleus scattering was by Thomas and Landau.⁸¹

In this section we illustrate the unitarity relation with a simple example. We then establish the exact unitarity relation for the t -matrix $T(1)$ corresponding to $U(1)$. The practical implications of this for calculations of pion-nucleus cross-sections, as well as calculations involving pion distorted waves, are then discussed. Finally we indulge in a little speculation on 'the way things really are' in the (3,3) resonance region.

(1) The unitarity relation in Born approximation. Consider the Born approximation, $T(1)_{\text{Born}} \equiv U(1)$, to the optical model Eq. (D.16) using the first order potential (D.26). (We drop the relativistic transformation and c.m. correction factors for simplicity.) Using the unitarity relation (A.24) for $t_{\pi N}$ implies

$$\begin{aligned} \text{disc } t_{\pi N}(\vec{Q}', \vec{Q}; \epsilon) &= -2\pi i \int d\vec{Q}'' t(\vec{Q}', \vec{Q}'', \epsilon^+) \delta(\epsilon - H_0(\vec{Q}'')) \\ &\quad \times t^*(\vec{Q}'', \vec{Q}; \epsilon^+) . \end{aligned} \quad (E.1)$$

Therefore, the discontinuity of $T(1)_{\text{Born}}$ in the forward direction ($\vec{k} = \vec{k}'$) is simply

$$\begin{aligned} \text{disc } (T_{\text{Born}}^{(1)}(E, 0^\circ)) &= -2\pi i(A-1)/A \sum_a \int d\vec{p} \, d\vec{q}'' \, \psi_a^*(\vec{p}) \\ &\times t^*[\vec{Q}, \vec{Q}'', E^+ - E_B^-(a) - (\vec{p} + \vec{k})^2/2M] \, t[\vec{Q}'', \vec{Q}, E^+ - E_B^-(a) - (\vec{p} + \vec{k})^2/2M] \psi_a(\vec{p}) \\ &\times \delta[E^+ - E_B^-(a) - (\vec{p} + \vec{k})^2/2M - H_0(\vec{Q}'')]. \end{aligned} \quad (E.2)$$

In terms of diagrams, Eq.(E.2) corresponds simply to cutting the on-energy-shell free nN intermediate states of a nN t -matrix (c.f. Fig.2) in the presence of a non-interacting core. This is the clue to understanding Eq.(E.2). The combination ' $\psi_a t_{nN}$ ' is the matrix element for nucleon knock-out by the incident pion in *plane wave impulse approximation* [denoted $M(a)_{PWIA}(\vec{k}, \vec{p}, \vec{Q}'')$]. Thus, Eq.(E.2) says simply that

$$\begin{aligned} \text{Im } (T_{\text{Born}}^{(1)}(E(\vec{k}), 0^\circ)) &\propto \int d\vec{p} \, d\vec{Q}'' \sum_a |M(a)_{PWIA}(\vec{k}, \vec{p}, \vec{Q}'')|^2 \\ &\times \delta(E_{\text{initial}} - E_{\text{final}}), \\ &\propto \sigma_{PWIA}(\pi, nN). \end{aligned} \quad (E.3)$$

That is, the optical theorem for the Born approximation solution to the first order optical potential (D.26) implies that the imaginary part of the forward amplitude is related to the cross-section for pion quasi-free scattering in PWIA. With a little forethought we could have guessed this result, because in the three-body model the core is inert, and the pion does not interact with it (c.f. Fig. 12). Clearly then, the outgoing pion wave is never distorted in the first-order three-body potential. In the next section we show that using the full solution of the pion-nucleus scattering equation $T^{(1)}$, leads to distortion of the incident pion wave only.

The point which this simple demonstration is meant to stress, is that the only inelastic channel whose presence is incorporated (through unitarity) in a first-order optical model is quasi-elastic scattering (one nucleon knock-out).

(ii) The unitarity relation for the first order potential.

Let us forget the relativistic aspects of the problem for a moment. We recall that the three-body equations (D.24) for the operator $\tau^{(1)}$ are of the AGS form. In fact, identifying $\tau^{(1)} = U_{1,1}$, $t_C = U_{2,1}$, $t_{nN} = t_1$, $t_{nN} = t_2$, $t_3 = 0$, we see that $\tau^{(1)}$ is an elastic scattering operator, and t_C describes a pick-up process, in which the projectile and nucleon stick together. However, in the present case the nN interaction has no bound state, so the discontinuity relation for $U_{1,1}$ [c.f. Eq.(A.82)] has only one bound state contribution $\Delta_{1,a}$ [where ' a ' denotes the IPSM level of the one nucleon involved—this label will be dropped for notational convenience below Eq.(E.4)]. Using Eq.(A.82) we now find the discontinuity of $\tau^{(1)}$ across its right-hand cut

$$\begin{aligned} \text{disc } (\tau_a^{(1)}) &= -2\pi i \{ \tau_a^{(1)} + \Delta_{1,a} \tau_a^{(1)} \\ &+ (t_C^\dagger G_0^\dagger t_{nN}^\dagger + \tau_a^{(1)\dagger} G_0^\dagger t_{nN}^\dagger) \text{disc } (G_0) \\ &\times (t_{nN} G_0 t_C + t_{nN} G_0 \tau_a^{(1)}) \}. \end{aligned} \quad (E.4)$$

However $t_{nN} G_0 \tau^{(1)}$ can be replaced by just t_C using Eq.(D.24), and so the second term on the r.h.s. of Eq.(E.4) contains the combination

$$(1 + t_{nN} G_0) t_C \equiv \Omega_{nN} t_C, \quad (E.5)$$

where Ω_{nN} is the Møller operator converting a plane wave nN state into a full nN scattering wave function [c.f. Eq.(A.13)], in the presence of a non-interacting core. Next we recall the optical model Eq.(D.16). By manipulations similar to those used in Eqs.(A.70,71) the reader should convince him- (her-) self that

$$\begin{aligned} \text{disc } (T_{00}'^{(1)}) &= T_{00}'^\dagger + \text{disc } G_A^{(0)} T_{00}'^{(0)} \\ &= T_{00}'^\dagger \{ U_{00}^{-1}(E^+) - U_{00}^{-1}(E^-) \} T_{00}'^{(0)} \\ &= T_{00}'^\dagger + \text{disc } (G_A^{(0)}) T_{00}'^{(0)} + \Omega_0^\dagger \text{disc } (U_{00}) \Omega_0. \end{aligned} \quad (E.6)$$

In Eq.(E.6) we have defined the elastic distortion operator as usual ($\Omega_0 \equiv U_{00}^{-1} T_{00}'^{(0)}$).

Now the major trick in proving elastic unitarity within the KMT formalism is that to get the physical scattering amplitude ($T_{00}^{(0)}$) we must form $[A/(A-1)] T_{00}'^{(0)}$. Thus, ignoring the second term on the r.h.s. of (E.6) we find

$$\text{disc } T_{00} \sim \frac{A-1}{A} |T_{00}'|^2. \quad (E.7)$$

A comparison with the usual two-body unitarity relation shows that elastic unitarity is not guaranteed to order $(1/A)$ without the first term on the r.h.s. of Eq.(E.4) for disc $\tau^{(1)}$. This in turn requires a full solution of the three-body Eq.(D.24). Therefore the impulse approximation $\tau^{(1)} \rightarrow t_{nN}$ formally violates elastic unitarity [to $O(1/A)$]. Having pointed this out, we ignore it [along with other minor details of $O(1/A)$] for clarity of presentation.

Equation (E.6) then becomes

$$\begin{aligned} \text{disc } T_{00} &= T_{00}^\dagger + \text{disc } (G_A^{(0)}) T_{00} \\ &+ \sum_a \Omega_0^\dagger t_C (a)^\dagger \Omega_{nN}^\dagger + \text{disc } (G_0) \Omega_{nN}^\dagger t_C (a) \Omega_0. \end{aligned} \quad (E.8)$$

That is, the optical theorem relates $T_{00}(0^\circ)$ to the total elastic

cross-section (σ_{elastic}) plus the cross-section for break-up of the bound nuclear system leading to a π -N core three-body final state. The combination $\phi_{\pi N} \tau(a)$ can be rewritten as $\phi_{\pi N} \tau(1) \Omega_0$ (an exercise for the reader), which leads to the partially distorted matrix element

$$T_{\pi N \leftarrow \pi}^{(+)} = \langle \chi_N^{(-)} | \phi_{\pi} | \tau(1) | \chi_{\pi}^{(+)} | \psi_a \rangle. \quad (\text{E.9})$$

Here ϕ_{π} is a plane wave state and $\chi_N^{(-)}$ is the nucleon-core scattering wave function in the real shell-model potential U . Finally, using the impulse approximation $\tau(1) \sim t_{\pi N}$ this is simply

$$T_{\pi N \leftarrow \pi}^{(+)} \text{IA} = \langle \phi_N \phi_{\pi} | t_{\pi N} | \chi_{\pi}^{(+)} | \psi_a \rangle. \quad (\text{E.10})$$

Of course, $\chi_{\pi}^{(+)}$ is the optical model wave function for the incident pion. In comparison with Eq. (E.3) we see that the only effect of using the full solution of the optical model equation, as opposed to Born approximation, is this extra distortion. In summary, the optical theorem for the solution of the first-order optical model is⁸¹

$$\text{disc } T_{00}(0^{\circ}) \sim \sigma_{\text{elastic}} + \sigma(\pi, \pi N), \quad (\text{E.11})$$

where the quotation marks on the quasi-elastic total cross-section imply it is evaluated approximately.

(iii) Comparison with data. Until very recently, the prevailing attitude to the reaction content of the first-order optical model was rather naive. Experimentally it is known that quasi-elastic scattering accounts for only about one-third of the total inelastic cross-section on ^{12}C at 130 MeV.⁸² At lower energies this fraction seems to become even smaller—although to be honest the data are very meagre. This did not prevent (for example) the authors of Ref. [68]—c.f. part (b) of section D(ii) above—making a comparison of the total cross-sections calculated using their first-order potential with the experimental total cross-sections. Surprisingly the agreement was quite good in the resonance region—far better than it had a right to be!

The reason why the theoretical total cross-section can be as high as experiment is obvious from the discussion in part (ii) above. In particular, in the resonance region the pion-nucleus interaction is strongly absorptive, so that the plane wave in the approximate DWIA (E.10) results in a considerable overestimate of $\sigma(\pi, \pi N)$ in Eq. (E.11).^{*} This is apparently sufficient to (fortuitously) compensate for the many other open channels whose presence is ignored in $U_{00}^{(1)}$.

*The omission of absorption in the nucleon wave function also tends to overestimate $\sigma(\pi, \pi N)$.

In view of the importance of the insight which the unitarity relation provides into the content of $U(1)$, it is worth while to state clearly what has and has not been proven, using low energy pion scattering from ^{12}C as an example.⁸¹

a). The purpose of the derivation of Eqs. (E.9-11) was not to obtain a better DWIA for quasi-elastic pion scattering—as suggested recently by Eisenberg.⁸³ Indeed, as we have stressed here, Eq. (E.10) should overestimate the 'true' DWIA matrix element more and more as the pion energy rises.

b). Therefore, if one were interested in quasi-elastic scattering per se, the best agreement with experiment would (hopefully) be obtained by calculating a full DWIA using distorted initial and final pion wave functions calculated by an optical model code. On the other hand, the calculation of $\sigma(\pi, \pi N)$ (implicit in all the approximations necessary to obtain a calculable form of $U(1)$) via the optical theorem [Eq. (E.11)] is an important test of the internal consistency of the theory.

c). One problem in comparing this result with data is of course that Eq. (E.11) is only exact for $U(1)$. If one included an extra term (e.g. proportional to $p^2(r)$ —as we mention in part F) in the potential to describe the effect of real pion absorption, this unitarity relation would become much more complicated. For example, both the initial and final pion waves in Eq. (E.10) would include the effects of absorption, and $\sigma(\pi, \pi N)$ would be lowered. Fortunately, at (say) 50 MeV the pion mean free path for absorption is greater than $\lambda_{\text{abs}} \approx 6$ fm in ^{12}C (a number calculated from the experimental total cross-section which includes real absorption). Since a pion needs to travel only of the order (1-2) fm to escape from carbon, the reduction in $\sigma(\pi, \pi N)$ from this effect is expected to be no more than $O(e^{-4/6})$, or less than a factor of two. Thus the omission of true pion absorption should not alter our qualitative conclusions.

d). Figure 14 shows the comparison of the implicit quasi-elastic cross-sections in several of the first-order potentials discussed in Section D, with appropriate 'data' constructed from the $\pi^{\pm}$ measurements of Dropecky *et al.*⁸⁴—see Ref. [81]. The key point is that the simple first-order potentials with essentially free πN strengths give values of $\sigma(\pi, \pi N)$ which lie between one and two orders of magnitude above the data near 40 MeV. The crude inclusion of Pauli effects (which restrict the range of momenta allowed to the outgoing nucleon) by Stricker *et al.* is a slight improvement. However, only the calculations which include the three-body energy and a more sophisticated treatment of Pauli effects are within a factor of two of the data!

e). We stress that the 'agreement' between the ' E_3 -body+Pauli' calculation and the data in Fig. 14 is excellent (at least below

Fig. 14 A comparison of the total cross-sections for quasi-elastic scattering implicit in several standard first-order optical potentials. Note particularly the discrepancy of one to two orders of magnitude in the region below 60 MeV in models that do not use E_3 -body and Pauli effects.

60 MeV) from our present perspective! Remember we cannot expect that the approximations necessary to obtain a calculable form of $u(1)$ will ever allow an accurate prediction of $\sigma(\pi, \pi N)$ through the optical theorem. It is enough that the $(\pi, \pi N)$ cross-section is 'in the right ball park'—namely a small part of the total inelastic cross section.

f). The reason why the use of E_3 -body lowers $\sigma(\pi, \pi N)$ is very instructive. In fact following through the derivation of Eq. (E.2) again, it will hopefully be obvious that the argument of the energy conserving δ -function is directly related to that of the two-body t -matrix. In particular, *the correct phase space factor in $\sigma(\pi, \pi N)$ is only obtained when E_3 -body is used.* The reader is invited to prove that when E_2 -body (c.f. Eq. (D.27)) is used only the Q'' integral is restricted by the energy-conserving δ -function in Eqs. (E.2,3). That is, the p -integral is unrestricted, and the unitarity relation contains a possibly quite large unphysical contribution! In any case, the use of E_2 -body leads to a very poor representation of the real π -N-core phase space.

The 'Pauli effects' which are referred to in Fig. 14 are calculated as described in Ref. [64]. Their origin is simply that we retain the $Q(p)$ operator (i.e. anti-symmetry) in Eq. (D.17) and the following equations. This effect reduced the quasi-free cross-section because it essentially forbids (occupied) low momentum states to the knocked-out nucleon. The rather severe drop near 30 MeV in Fig. 14 is just an artifact of the nuclear matter approximation for $Q(p)$.

It is an interesting and as yet unanswered question whether the differences between the potential models in Fig. 14 have observable consequences in other reactions. From studies in the resonance region on reactions like (π^+, π^0) , (γ, π^0) , etc. It seems that the conventional optical models may be too absorptive. This energy region will be mentioned again in Section E(iv). However, the low energy region ($E_\pi < 100$ MeV) where we have found the most dramatic (order of magnitude) overestimates of the implicit quasi-elastic cross-section has hardly been investigated at all. We have calculated the pion co-ordinate space wave function using (case I) the pure E_2 -body model (essentially Ref. [68]), and the more sophisticated model (case II) with $U(1)$ including E_3 -body and Pauli, plus a term $[U(4b5)]$ representing the effect of true absorption. The latter term will be described in part F below, but for now we note that it restores σ_{total} to near experiment (~ 120 mb on ^{12}C at 50 MeV), but has a $p_\pi(r)$ dependence, and can therefore produce a different wave function.

Indeed, at 50 MeV the p -wave function in case II is far less strongly distorted by the nuclear surface than case I. Preliminary calculations of the (γ, π^0) reaction⁸⁵ using these wave functions

indicate a significant qualitative difference (of order 300%) in the back-angle cross-sections. Further tests in experimentally more accessible reactions like charge exchange and quasi-free scattering are essential.

(iv). Higher order corrections - self-consistency. A major reason for the overestimate of the implicit $(\pi, \pi N)$ cross section, as we have mentioned, is the omission in $U(1)$ of any pion-core interaction while the 'active' nucleon is in a continuum state. This is the major characteristic of the first-order potential and resulted in the appearance of ϕ_π in Eq.(E.10)—rather than $\chi_\pi(-)$. It is therefore tempting to try to remedy this defect by incorporating some higher order corrections into the theory through a pion-core interaction. Because the interaction of the pion with the core will not be drastically different from its interaction with the original nucleus, this approach is closely related to the question of the 'self-consistency' of the pion-nucleus interaction.

The first investigation of the effect of imposing self-consistency on the pion-nucleus interaction was an investigation by Ericson and Hufner⁸⁶⁾ of total cross-sections in the resonance region. In particular, if a pion is traversing nuclear matter of density ρ , the dispersion relation relating its momentum k in the medium and its energy is ($w^2 = k^2 + \mu^2$)

$$-\nabla^2 + \mu^2 - 2wV + 2wV^2 - w^2 = 0, \quad (E.12)$$

or explicitly for a factorised form of $U(1)$

$$K^2 + \mu^2 - 4\pi\rho f_{\pi N}(k, w) = w^2. \quad (E.13)$$

In Eq.(E.13) $f_{\pi N}(k, w)$ is the forward πN scattering amplitude. Ericson and Hufner suggested that what should really appear as the momentum in the πN scattering amplitude is the local pion momentum K . This leads in essence to a self-consistency relation for $K(w)$, or the refractive index $n(w) \equiv (K(w)/k)$

$$K^2 + \mu^2 - 4\pi\rho f_{\pi N}(K, w) = w^2. \quad (E.14)$$

Using a reasonable extrapolation of the resonant (3,3) scattering amplitude into the near off-shell region

$$f_{\pi N}(k, w) = \frac{C k^2}{w - w_R + i\Gamma/2}, \quad (E.15)$$

they showed that the effect of self-consistency in the form (E.14) is to lower the resonance position [i.e. where $\text{Re}(n(w)-1)$ goes through zero] by (30-40) MeV.

We have mentioned this work in some detail, because it represents a first attempt to introduce this new concept. However, it

should be clear from our discussion of the three-body model of $U(1)$ that we disagree with the approach. The natural place to introduce a pion-core interaction $V_{\pi C}$ is in fact the denominator of Eq.(D.17). In the crude approximation where $V_{\pi C}$ is a constant, the effect is to modify the interaction energy in $\tau(1)$, not the momentum dependence! For the nuclear matter case this would still involve a half-off-shell πN amplitude, but the off-shell extrapolation is different. We leave it as an interesting exercise for the student to carry this simple calculation through.

On the phenomenological level, something very similar to what we have just described has been carried out by Hirata *et al.*⁷⁰⁾ In part (b) of section D(iti) we have already mentioned that their simplest calculation includes the three-body energy shift through the argument of the D-function—namely $D_\Delta(E^+ - E_B - T_\Delta)$, with T_Δ the kinetic energy of the πN c.m. However, as a means of including some higher order effects, in practice they replace T_Δ by $(T_\Delta + V_\Delta)$, where the complex Δ -core potential is adjusted to fit the pion-nucleus scattering data. Clearly then V_Δ is a phenomenological representation of $\{u + V_{\pi C}\}$ in our language (and possibly includes some effects of true pion absorption as well).

To conclude this section I would like to use a little poetic licence in describing the way things really are in the resonance region. A considerable amount of evidence has been amassed showing that optical models are far too absorptive there. Even the qualitative features of single charge exchange, for example, are wrong—with distorted wave calculations producing a dip near the resonance where experiment has a slight bump!⁸⁷⁾

One indication of this problem from the theoretical point of view is the well known observation (see e.g. Ref. [88]) that adding terms representing real absorption has essentially no effect on the pion scattering results in conventional optical models—in spite of the fact that real absorption represents of the order of 50% of the experimental total reaction cross-section.⁸²⁾ Clearly one needs a mechanism to reduce the total cross-section produced by $U(1)$, (i.e. the potential without absorption) to a value of the order of the observed elastic plus quasi-elastic cross-sections.

One possible mechanism suggested by the three-body model is the inclusion of the pion-core interaction. From the point of view of a first principles calculation (rather than phenomenology), it is important to observe that the energy shift means that this potential is needed at an energy considerably lower than the starting potential. Since the optical potential grows rapidly weaker as the energy falls this sort of self-consistency procedure may converge very rapidly....
tum est.

F. The Effect of Pion Absorption

L'urgent est déjà fait,
Le difficile est en train de se faire,
Pour l'impossible on demande plusieurs jours.
Inconnu.

A discussion of the effect of absorption does not fit strictly into our development of multiple scattering theory, starting from the Faddeev equations. Nevertheless, our considerations of unitarity alone for the π -nucleus system have shown that true absorption is a very important inelastic channel. Until recently the inclusion of absorption was very phenomenological. In this final section we shall briefly outline the development of the phenomenology and show that the effects on the differential cross-section can be very large in the low energy region.

The real theoretical advances in understanding this phenomenon have however been made in a few body system (namely $NN\pi$) which has served as an excellent theoretical laboratory. The results obtained in this system and the implications for the pion-nucleus problem will also be mentioned briefly.

(i) Phenomenological treatment of absorption. The new distinct feature of pion-nucleus scattering, which was evident even in the early 50's in the formation of stars in photographic emulsions, is that the pion can disappear giving its rest mass into a violent breaking apart of the nucleus. A single nucleon with 140 MeV kinetic energy has a momentum of greater than 500 MeV/c, which is very difficult to find in a single particle wave function. If on the other hand the pion is absorbed on two nucleons which are emitted in almost opposite directions, little momentum transfer is required from the nuclear wave function. This was the original reason why Ericson advocated the use of the $(\pi, 2N)$ reaction to measure the N - N correlation function.⁸⁹⁾ While distortion effects have made this essentially impossible, there is now considerable evidence that absorption on two nucleons is the basic mechanism underlying pion absorption.^{90, 91)}

This evidence also suggests that absorption is more likely on a deuteron-like pair (i.e. n - p in $I=0$, $S=1$). Thus the original model proposed by the Ericsons to explain the widths of pionic atom levels (remember $U(1)$ is real in their model at threshold)^{72, 73)} is still

*The phrase 'true absorption' is not meant as a religious catch-phrase. This term is used to distinguish processes in which the pion actually disappears, from inelastic processes [e.g. $(\pi, \pi N)$] which result in a loss of flux (absorption) from the elastic channel.

in keeping with current ideas. Using a zero-range two nucleon operator (as an approximation to the real operator, which should have a short range because of the high relative momentum involved), they derived the cross-section for $[\pi d \rightarrow NN]$ and $[\pi A \rightarrow NN (A-2)]$ in terms of two constants—namely the absorption strengths of s - and p -wave pions (with respect to the NN c.m.).

In the πA system this gave rise to a new term in the optical potential proportional to $\rho^2(r)$ (because of the zero-range approximation). This term is supposed to represent a phenomenon not contained at all in the pion-nucleus multiple scattering series—the interaction with a cluster. Incorporating this into the potential means that by solving the pion-nucleus scattering equation we allow for processes where (e.g.) after interaction with a cluster the pion can scatter from other nucleons.

The Ericson-Ericson absorptive potential was

$$U_{E-E}^{(abs)}(\vec{r}) = -(4\pi/2m_\pi) [-B_0 A^2 \rho^2(r) + \vec{V}(C_0 A^2 \rho^2(r))\vec{V}], \quad (F.1)$$

with $\rho(r)$ the single particle density. In practice the calculated values of $\text{Im } B_0$ and $\text{Im } C_0$ were a factor of two too small to fit pionic atom level widths. In retrospect this seems to have been more because of uncertainties in the zero-range approximation [e.g. the need for $\psi_d(\vec{r}=0) \neq 1$], than a failure of the quasi-deuteron model. Indeed, a very recent calculation by Bertsch and Riska gave 80% of the pionic atom value of $\text{Im } B_0$.⁹²⁾

The intensive study of low energy pion elastic scattering in the last two years has shown the necessity for extending Eq. (F.1) above $E_\pi=0$. This is most reasonably done within the quasi-deuteron model by scaling the s - (B_0) and p -wave (C_0) strengths⁵⁴⁾ according to the elementary $\pi d \rightarrow NN$ cross-section.⁹³⁾ In addition, since the conventional absorption mechanism involves scattering of the pion from one nucleon before its absorption on a second, it is reasonable to introduce form-factors (g_0 and g_1) into the momentum space generalisation of (F.1), with a range typical of the πN system (c.f. Section D). This leads to the generalisation of Eq. (F.1)⁶⁴⁾

$$U_{E-E}^{abs}(\vec{k}', \vec{k}) = -4\pi \left[2m_\pi (2\pi)^3 \right]^{-1} A(A-1) \left\{ B_0(k_0) \frac{g_0(k') g_0(k)}{g_0^2(k_0)} + C_0(k_0) \frac{g_1(k') g_1(k)}{g_1^2(k_0)} \cos \theta_{\vec{k}' \vec{k}} \right\} \rho^2(\vec{k}' - \vec{k}), \quad (F.2)$$

where k is the πN c.m. momentum and ρ^2 the Fourier transform of $\rho^2(r)$. The values of $\text{Im } B_0(k_0=0)$ and $\text{Im } C_0(k_0=0)$ were fixed by the appropriate pionic atom values [$\text{Im } B_0(0) = 0.02 m_\pi^{-1}$, $\text{Im } C_0(0) = 0.04 m_\pi^{-1}$ corresponding to a full Lorentz-Lorenz correction—or finite size πN interaction].

$$\text{Im } B_0(k_0) \approx \text{Im } B_0(0) \left\{ \frac{k_0 \sigma_{\ell=0} (\pi^+ d \rightarrow pp; k_0)}{\lim_{k \rightarrow 0} k \sigma_{\ell=0} (\pi^+ d \rightarrow pp; k)} \right\}, \quad (\text{F.3})$$

$$\text{Im } C_0(k_0) \approx \text{Im } C_0(0) \left\{ \frac{k_0 \sigma_{\ell=1} (\pi^+ d \rightarrow pp; k_0)}{\lim_{k \rightarrow 0} k \sigma_{\ell=1} (\pi^+ d \rightarrow pp; k)} \right\}. \quad (\text{F.4})$$

For further discussion of the details of Eqs. (F.3) and (F.4) we refer to Ref. [64]. (Note particularly that the separation into $\ell=0$ and $\ell=1$ parts of the $\pi^+ d \rightarrow pp$ cross-section is quite ambiguous.)

Figure 15 shows the calculation of 30, 40 and 50 MeV $\pi^+ {}^{12}\text{C}$ scattering in comparison with data from TRIUMF (at 30 and 40 MeV),⁹⁴⁾ and old data of Carnegie-Mellon (at 50 MeV). The dominant feature of low energy π^+ scattering in the forward direction is the destructive interference between the repulsive coulomb and attractive strong interactions. Because of the use of E_3 -body in the first order potential of Landau and Thomas,⁶⁴⁾ this leads to a very low cross-section—particularly at 30 MeV. (Both the coulomb, and strong amplitude calculated with $U(1)$, were almost real.) As the figure shows, the inclusion of some imaginary part in the strong amplitude through $U(\text{abs})$ is essential to fit the data! This was the first demonstration of the importance of absorption in elastic scattering.

(ii) Theoretical approach to the effect of absorption. While the use of $U(\text{abs})$ given by Eqs. (F.2-4) is physically reasonable, the need for a deeper theoretical understanding has been felt for many years. Progress has been made recently by considering a simple field theory of the πNN system (where the nuclear physics uncertainties are minimal). The same result has now been derived by four methods.⁹⁵⁻⁹⁸⁾ We shall follow the diagrammatic method, which is the fastest way to establish the correct result. However, the serious student is urged to also read carefully the work of Koltun and Mizutani (using Feshbach projection operators),⁹⁷⁾ and Rinat (using the reduction techniques of Ballot and Becker).⁹⁸⁾

The power of the diagrammatic technique is that we do not have to specify exactly the Hamiltonian involved. It is sufficient that we have somewhere in the (renormalized) theory an operator which produces and absorbs pions. For simplicity we neglect all couplings to negative energy nucleon states. Finally we assume that the full $\pi\text{NN} \rightarrow \pi\text{NN}$ scattering amplitude (denoted $\langle 1|T|1 \rangle$, where $|1 \rangle$ shows the number of pions present) is given by the set of all renormalized diagrams connecting one one-meson two-nucleon state with another.

The set of all such diagrams falls naturally into two (topological) subsets:

Fig. 15 A comparison with available low energy pion-nucleus elastic scattering data, of the results of Landau and Thomas using

$U(1)$ alone (including E_3 -body)—dashed curve—or including the effect of absorption through the phenomenological term $U(\text{abs})$ [c.f. Eq. (F.2)].⁶⁴⁾ Clearly absorption is essential for agreement at 30 and 40 MeV.

(a) those in which every intermediate state has one or more pions (denoted $\langle 1|T|1\rangle_1$, with the subscript giving the minimum number of pions)
 (b) those in which there is at least one intermediate state with no mesons.
 If we cut the diagrams in part (b) at the last two-nucleon-only intermediate state we find

$$\langle 1|T|1\rangle = \langle 1|T|1\rangle_1 + \langle 1|T|0\rangle_1 G_0 \langle 0|T|1\rangle_1. \quad (F.5)$$

(Again the subscript '1' on $\langle 1|T|0\rangle_1$ means at least one pion in every intermediate, and G_0 is the propagator for two non-interacting nucleons.)

Next one may cut $\langle 0|T|1\rangle$ at the first two-nucleon-only intermediate state, so that

$$\langle 1|T|1\rangle = \langle 1|T|1\rangle_1 + \langle 1|T|0\rangle_1 \{G_0 + G_0 \langle 0|T|0\rangle G_0\} \langle 0|T|1\rangle_1. \quad (F.6)$$

Now $\langle 0|T|0\rangle$ is simply the sum of all diagrams which take us from the initial to the final N-N state—i.e. the N-N t-matrix t_{NN} in this theory. From Eq. (A.10) we recognise the term in braces as G_{NN} , the interacting N-N Green's function. Using the spectral representation of G_{NN} we find that (F.6) becomes

$$\langle 1|T|1\rangle = \langle 1|T|1\rangle_1 + \sum_n \langle 1|T|0\rangle_1 \frac{|\psi_n^{(+)}\rangle \langle \psi_n^{(+)}|}{E^+ - \epsilon_n} \langle 0|T|1\rangle_1. \quad (F.7)$$

The combination $\langle f| \langle 1|T|0\rangle_1 |\psi_n^{(+)}\rangle$, where $\langle f|$ is a two-nucleon final state (e.g. the deuteron) is loosely speaking the complete matrix element for pion production.*

Thus, the full elastic amplitude in this model is the sum of a term which could have been generated by some multiple scattering theory—namely $\langle 1|T|1\rangle_1$ —plus a term which involves virtual pion absorption. Of course $\langle 1|T|1\rangle_1$ contains many complicated intermediate states with more than one meson, but just as Chew picked out the one meson diagrams for nN as those relevant to unitarity, here we expect also that the approximation

$$\langle 1|T|1\rangle_1 \approx T(Fadd), \quad (F.8)$$

*The caution here is because strictly to get the NN - nN amplitude one should pick out the residue of the deuteron pole term in $\langle 1|T|0\rangle_1$. Having done this, the condition on $\bar{T}$ in $M_{nd, NN} \equiv \langle \psi_d | \langle 1|\bar{T}|0\rangle_1 |\psi_d^{(+)}\rangle$, is simply that the last interaction be a nN scattering—not N-N scattering!

should be a good starting point.

The second term in Eq. (F.7) also involves $T(Fadd)$, in the same level of approximation, as we see by cutting $\langle 0|T|1\rangle_1$ at the last place where there is one-pion-only intermediate state. That is,

$$\langle 0|T|1\rangle_1 = \langle 0|T|1\rangle_2 [1 + G_0 \langle 1|T|1\rangle_1], \quad (F.9)$$

and we recognise the term in brackets as the Møller operator for the NN system with one pion always present [$Q(Fadd)$ - c.f. Eq. (A.13)]. In such a theory $\langle 0|T|1\rangle_2$ is essentially the elementary pion production vertex, given to lowest order by Eq. (C.1), for example. To summarise therefore, the contributions of multiple scattering and absorption can be separated quite clearly in such a simple model as

$$T(E^+) = T(E^+)^{(Fadd)} + Q(E^-)^{(Fadd)} \langle 1|T|0\rangle_2 G(E^+) \langle 0|T|1\rangle_2 Q(E^+)^{(Fadd)}. \quad (F.10)$$

Equation (F.10) is agreed upon by each of Refs. [95-98]. It is currently being applied to the nd problem between zero and 250 MeV. [99,100] The key to avoiding double counting of absorption somewhere in the Faddeev term is the subscript '1' in $\langle 1|T|1\rangle_1$. That is, every intermediate state in the first term in Eq. (F.10) must contain at least one pion. Therefore the nucleon pole term (c.f. Fig. 10a) in the P_{11} interaction must be excluded from the Faddeev calculation. This is the only caution necessary to avoid double counting problems.

The formal generalisation of Eq. (F.10) to the pion-nucleus case is straightforward. However, when one tries to connect that with the phenomenology discussed in part (i), the picture is very unclear. Hopefully the near future will see some clarification of this very important problem in pion physics.

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