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AND EXCITATION OF N - DIMENSIONAL
NONSTATIONARY FORCED OSCILLATOR

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COHERENT STATES AND EXCITATION OF N - DIMENSIONAL
NONSTATIONARY FORCED OSCILLATOR.

I.A. Malkin V.I. Man'ko

Abstract.

New integrals of motion, coherent states, transition amplitudes, and dynamical group $U(N, 1)$ are constructed for N - dimensional nonstationary forced oscillator.
Frank-Condon factor is exactly calculated. The path integrals method by Feynman [1] provides to be the simplest method for describing any nonstationary quantum system with quadratic hamiltonian. The coherent states representation [2] is known to be very convenient for considering a quantum system in terms of classical mechanics. The coherent states for a charge moving in a time-dependent magnetic field were introduced in ref. [3]. The aim of this article is to introduce the coherent states for N - dimensional nonstationary forced oscillator. Using the coherent states the transition amplitudes connecting the energy eigenstates in the case when in a remote past and future the forces are equal to zero and the frequencies are constants are calculated in terms of Hermite polynomials of two variables. First for one-dimensional nonstationary forced oscillator the exact solution, Green function and transition amplitudes generating function have been obtained by Huzimi [4].

Let the Schrödinger equation be

$$i\hbar \frac{\partial \Psi}{\partial t} = \left[\sum_{k=1}^N \frac{p_k^2}{2} + \frac{\omega_k^2(t) x_k^2}{2} - f_k(t) x_k \right] \Psi ; \quad p_k = -i\hbar \frac{\partial}{\partial x_k} ; \quad (1)$$

where $f_k(t)$, the impressed forces, may be arbitrary functions of time variable t . The circular frequencies $\omega_k(t)$ can also be variable in time. For the sake of simplicity we shall adopt natural units: $\hbar = c = m_k = 1$. It can be checked that the operators

$$A_k = i 2^{-1/2} [\epsilon_k p_k - \dot{\epsilon}_k x_k] + \delta_k(t) \quad (2)$$

where the complex function $\epsilon_k(t) = |\epsilon_k| \exp i \int_0^t d\tau / |\epsilon_k(\tau)|^2$ satisfies the equation $\ddot{\epsilon}_k + \omega_k^2(t) \epsilon_k = 0$, the complex functions $\delta_k(t) = -i 2^{-1/2} \int_0^t f_k(\tau) \epsilon_k(\tau) d\tau$, are integrals of motion (commute with the operator $i\partial/\partial t - \hat{H}$). The commutation relations are

$$[A_k, A_m^\dagger] = \delta_{km} \quad (3)$$

The nonhermitian linear in respect to coordinates and momenta invariants A_k give a whole set of independent integrals of motion. They have a physical sense of initial coordinates of classical motion in phase space. For one-dimensional oscillator with varying frequency the quadratic invariant $A_k^\dagger A_k$ was constructed and used in ref. [5]. Using the invariant one can find the coherent state $|\vec{\alpha}\rangle$, where $\vec{\alpha}$ is the complex vector with coordinates $\alpha_1, \alpha_2, \dots, \alpha_N$.

$$|\vec{\alpha}\rangle = \prod_{k=1}^N \pi^{-N/4} \bar{E}_k^{-1/2} \exp \left\{ i \bar{E}_k x_k^2 / 2 \bar{E}_k + \sqrt{2} (\alpha_k - \delta_k) x_k / \bar{E}_k - |\alpha_k|^2 / 2 + \right. \\ \left. + \alpha_k (\delta_k \bar{E}_k + \bar{E}_k \delta_k^*) / \bar{E}_k - \alpha_k^2 \bar{E}_k^* / 2 \bar{E}_k - \bar{E}_k^* \delta_k^2 / 2 \bar{E}_k - |\delta_k|^2 / 2 + i P_k \right\}; \quad (4)$$

$$P_k = i \int_0^t (\delta_k \dot{\delta}_k^* - \delta_k^* \dot{\delta}_k) dx / 2;$$

which satisfies the equation (1) being the eigenvector of the invariants (2) $A_k |\vec{\alpha}\rangle = \alpha_k |\vec{\alpha}\rangle$. These coherent states describe the most classical states for the oscillator under consideration. These nonstationary states correspond to motion of classical oscillator. The coherent states (4) form the whole system of functions and give the generating function for the solution of equation (1) which are eigenvectors of the invariants $A_k^+ A_k$ expressed in terms of Hermite polynomials [4].

Coherent states (4) may be constructed by means of unitary operator $D(\vec{\alpha}) = \prod_{k=1}^N \exp(\alpha_k A_k^+ - \alpha_k^* A_k)$; $|\vec{\alpha}\rangle = D(\vec{\alpha}) |\vec{0}\rangle$.

Let us consider the N -dimensional oscillator which for the times $t \leq 0$ has the forces $f_k(t)$ equal to zero and the frequencies $\omega_k(t)$ equal to the constants ω_k^i . It implies that for these times the functions $E_k(t) = (\omega_k^i)^{-1/2} \exp i \omega_k^i t$ and $\delta_k(t) = 0$, $t \leq 0$. Then the frequencies start to vary and the forces appear. In the remote future the forces disappear and the frequencies become again constants equal to ω_k^f .

The operators A_k ($t < 0$) coincide with the usual annihilation operators $A_k^i = 2^{-1/2} (\sqrt{\omega_k^i} x_k + \partial / \sqrt{\omega_k^i} \partial x_k) \exp i \omega_k^i t$. Because of this one can see that the invariants $A_k^+(t) A_k(t)$ correspond to conserved initial energies of the classical motion

of oscillators. In the remote future one can introduce the corresponding annihilation operators $A_k^f = 2^{-1/2} \exp(i\omega_k^f t) \otimes (\sqrt{\omega_k^f} X_k + \partial/\sqrt{\omega_k^f} \partial X_k)$. The invariants (2) are expressed in terms of these operators

$$A_k = \beta_k(t) A_k^f + \eta_k(t) (A_k^f)^\dagger + \delta_k(t) \quad (4)$$

where the functions $\beta_k(t)$, $\eta_k(t)$ are connected with the function ε_k by the relations $\varepsilon_k = (\omega_k^f)^{-1/2} [\beta_k(t) \exp i\omega_k^f t - \eta_k(t) \exp(-i\omega_k^f t)]$; $\dot{\varepsilon}_k = i(\omega_k^f)^2 [\beta_k \exp i\omega_k^f t + \eta_k \exp(-i\omega_k^f t)]$ and satisfy the condition $|\beta_k|^2 - |\eta_k|^2 = 1$; when the time $t \rightarrow \infty$ the functions $\beta_k(t)$, $\eta_k(t)$, $\delta_k(t)$ become constants β_k , η_k , δ_k . Let us express in

terms of these parameters the transition amplitudes between the initial and final coherent states. The final coherent state $|\vec{\mu}, f\rangle$ is given by the formula (4) where $\delta_k = 0$, $\omega_k = \omega_k^f$, $\alpha_k = \mu_k$, $\varepsilon_k = (\omega_k^f)^{-1/2} \exp i\omega_k^f t$.

The direct calculation of integral gives

$$\langle \vec{\mu}, f | \vec{\alpha}, i \rangle = \prod_{k=1}^N \beta_k^{-1/2} \exp \left[-(\mu_k^2 + |\mu_k|^2)/2 + \alpha_k^2 \eta_k^* / 2\beta_k - \mu_k^* \eta_k / 2\beta_k + \alpha_k \mu_k^* / \beta_k + \alpha_k (\delta_k^* - \eta_k^* \delta_k / \beta_k) - \frac{\mu_k^* \delta_k}{\beta_k} + \frac{\delta_k^2 \eta_k^*}{2\beta_k} - \frac{|\delta_k|^2}{2} + i p_k \right] \quad (5)$$

Since the coherent states give the generating function for the energy eigenstates the amplitude (5) gives the generating function for the transition amplitudes between the energy eigenstates.

This function $\prod_{k=1}^N \exp [(\mu_k^2 + |\mu_k|^2)/2] \cdot \langle \vec{\mu}, f | \vec{\alpha}, i \rangle$ is the

generating function for Hermite polynomials of two variables [6].

Thus one can see that transition amplitude $\langle \vec{m}, f | \vec{n}, i \rangle$ between the initial excited state $|\vec{n}, i\rangle$ and the final one $|\vec{m}, f\rangle$ has a form

$$\langle \vec{m}, f | \vec{n}, i \rangle = \langle \vec{0}, f | \vec{0}, i \rangle \prod_{k=1}^N (m_k! n_k!)^{-1/2} H_{n_k m_k}(x_1^k, x_2^k) \quad (6)$$

where the function $H_{n_k m_k}(x_1^k, x_2^k)$ is the Hermite polynomial of two variables $x_1^k = \delta_k - \eta_k \delta_k^* / \gamma_k^*$, $x_2^k = -\delta_k^* / \gamma_k^*$, and the parameters a_k , b_k , c_k of the quadratic form, which determines these polynomials, have the following values $a_k = -\eta_k^* / \gamma_k$, $b_k = -\gamma_k^{-1}$, $c_k = \eta_k / \gamma_k$. The factor $\langle \vec{0}, f | \vec{0}, i \rangle$ is given by the formula (5) where $\mu_k = \alpha_k = 0$. The polynomials $H_{n_k m_k}(x_1^k, x_2^k)$ may be expressed in terms of the Hermite polynomials of one variable

$$\langle \vec{m}, f | \vec{n}, i \rangle = \langle \vec{0}, f | \vec{0}, i \rangle \prod_{k=1}^N (\eta_k^* / \sqrt{-2\gamma_k \eta_k^*})^{n_k} (\eta_k / \sqrt{2\gamma_k \eta_k^*})^{m_k} \otimes \sum_{p=0}^{\min(m_k, n_k)} 2^p (\sqrt{-2\gamma_k \eta_k^*} / \eta_k^* \sqrt{2\gamma_k \eta_k^*})^p (p! m_k! n_k! p!)^{-1/2} H_p(\frac{\delta_k}{\eta_k^* p \sqrt{2\gamma_k \eta_k^*}}) H_p(\frac{\eta_k^* \delta_k^*}{\sqrt{2\gamma_k \eta_k^*}}) \sqrt{m_k! n_k!} \quad (7a)$$

Using the results, obtained in ref. [3], for the transition amplitude in the case $f_k = 0$; ($\delta_k = 0$, $x_1^k = x_2^k = 0$) we have

$$\text{the relation } H_{n_k m_k}(0, 0) = n_k! \sqrt{\frac{m_k - n_k}{m_k + n_k}} e^{i(\frac{m_k - n_k}{2} \varphi_{\eta_k} - \frac{m_k + n_k}{2} \varphi_{\gamma_k})} \delta_{m_k, n_k} \quad (7b)$$

Using the property of generating function (5) one can obtain the formula

$$\langle \vec{\mu}, i | \vec{n}, j \rangle = \langle \vec{\mu}, i | \vec{0}, i \rangle \prod_{k=1}^N \left(\frac{\mu_k}{\sqrt{-2\lambda_k \eta_k^*}} \right)^{n_k} \left(n_k! \right)^{-1/2} H_{n_k} \left(\frac{\mu_k^* \delta_k - \lambda_k \delta_k^* - \mu_k^*}{\sqrt{-2\lambda_k \eta_k^*}} \right); \quad (7c)$$

The formula (6) gives the Frank-Condon factor [7] for the harmonic potential. The corresponding overlap integral was considered in many papers (see for example [8,9]). The limit cases of the formula (5)-(7) the connection with density matrix and applications, the adiabatic invariants evolution will be published elsewhere. The formula (7) one can generalise for the Hermite polynomials of many variables (to be published elsewhere). It should be noted that the nonstationary equation (1) has the dynamical symmetry $U(N, 1)$ the generators of which one can construct from the invariants A_k by the usual way. The space of solutions (4) forms the space of irreducible ladder representation of this group. This is the same representation as for the stationary case. Because of this the amplitudes (6), (7) may be considered from group-theoretical point of view. This implies that Hermite polynomials of many variables are connected with noncompact group representation.

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Appendix

We will write some new formula with Hermite polynomials of two variables. First using the generating function (5) we have

$$\langle \vec{m} | \vec{\alpha}, i \rangle = \langle \vec{0} | \vec{\alpha}, i \rangle \prod_{k=1}^N \frac{1}{\sqrt{m_k!}} \left(\frac{\eta_k}{\sqrt{2\gamma_k\eta_k}} \right)^{m_k} H_{m_k} \left(\frac{\alpha_k - \delta_k}{\sqrt{2\gamma_k\eta_k}} \right) \quad (A 1)$$

The formula (7a) may be obtained by expansion of this formula. Using (5) and the generating function for Legendre polynomials [6] (see also [3]) we have

$$H_{pq}(x_1^k, x_2^k) = p!q! \sum_{n=0}^p \sum_{m=0}^q \frac{\delta_{m+n} \cdot \left(-\frac{\delta_k}{\gamma_k}\right)^{q-m} \left(\delta_k^* - \eta_k^* \frac{\delta_k}{\gamma_k}\right)^{p-m}}{m! p-m! q-m!} \cdot P_{\frac{m-n}{2}}^{\frac{m+n}{2}} \left(\cos \frac{\theta_k}{2} \right) \cdot \exp \left\{ i \frac{m-n}{2} \varphi_{\eta_k} - i \frac{m+n}{2} \varphi_{\gamma_k} \right\} \quad (A 2)$$

$$\cos \theta_k = 1 - 2 \left| \frac{\eta_k}{\gamma_k} \right|^2$$

The formula (7b) is the special case of this relation for $\delta_k = 0$. This relation is very convenient for transition amplitude calculation when δ_k is a small number. For the case of small numbers η_k we can write down the convenient formula following from the generating function for Laguerre polynomials [4] and from (5)

$$\langle \vec{n}, f | \vec{m}, i \rangle = \langle \vec{0}, f | \vec{0}, i \rangle \prod_{k=1}^N \sqrt{m_k! n_k!} \sum_{i=0}^{\lfloor \frac{m_k}{2} \rfloor} \sum_{\ell=0}^{\lfloor \frac{n_k}{2} \rfloor} \left(\frac{q^*}{23} \right)^i \left(-\frac{q}{23} \right)^\ell \cdot \quad (A 3)$$

$$\cdot \frac{1}{2!} \frac{1}{\ell!} C_{n_k-2i, m_k-2\ell}^k$$

where

$$C_{pq}^k = \left[\frac{p+q+|p-q|}{2} \right]^{-1} \left(\delta_k^* - q_k^* \frac{\delta_k}{3_k} \right)^{\frac{p-q+|p-q|}{2}} \left(-\frac{\delta_k}{3_k} \right)^{\frac{q-p+|p-q|}{2}} \cdot$$

$$\cdot \left(1/3_k \right)^{\frac{p+q-|p-q|}{2}} \prod_{k=1}^N \frac{|p-q|}{2} \left[\delta_k (\delta_k^* - q_k^* \frac{\delta_k}{3_k}) \right]$$

Using the unitarity of S - matrix ($SS^\dagger = I$) we have the relations ($N=1$)

$$\sum_{m=0}^{\infty} \frac{1}{m! n!} |H_{nm}(x_1^* x_2^*)|^2 = \frac{1}{|\langle 0, f | 0, i \rangle|^2} \quad (A 4)$$

$$\sum_{m=0}^{\infty} \frac{1}{m!} H_{nm}(x_1^* x_2^*) H_{\ell m}^*(x_1^* x_2^*) = 0; (\ell \neq n)$$

These formulas may be generalized for the Hermite polynomials of many variables.

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