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High Energy Physics and Cosmic Rays

QUANTUM GRAVITY NEAR TWO DIMENSIONS.

R.Gastmans, R.Kallosh and G.Truffin

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ABSTRACT

Quantum gravity is shown to be renormalizable near two dimensions. The charge renormalisation constant is calculated in the one-loop approximation in the framework of the background functional method. At positive ϵ a non-trivial ultraviolet stable fixed point is present.

On leave from University of Leuven, Belgium

-Justin ³, Bardeen, Lee and Shrock ⁴ have studied a
 er of theories near two dimensions, at $n-2 = \epsilon$. The
 ry was represented as a double series expansion in the
 ling constant and in ϵ . It turned out ¹⁻⁴ that a non-
 ar chiral dynamics, which is nonrenormalizable in the
 ical 4-dimensional space, is renormalizable near two
 mensions and is asymptotically free at $n=2$. Calculation of
 one-loop approximation in this theory has shown that at
 $\epsilon > 0$ there exists the nontrivial uv stable fixed point

$$g = \frac{n-2}{K-2} + O((n-2)^2)$$
 where K is the number of the
 ees of freedom of a scalar field. The existence of an UV
 le point has led the authors of ref. [3] to the statement
 : the theory is renormalizable not only at small ϵ but
 so $n = 4$ as well.

Since nonlinear chiral dynamics has many properties that
 close to quantum gravity, it is natural to try to carry
 an analysis analogous to that of refs. [1-4] for quan-
 gravity.

An essential difference between gravity and chiral di-
 nics(except purely technical complications) is that at
 $\epsilon = 0$, i.e. in a strict two-dimensional case, chiral
 nics does exist and is renormalizable (in power counting)
 e it possesses a dimensionless coupling constant. In
 atum gravity at $n = 2$ the action is an integral of the *total*

the action is an expansion in $h_{\mu\nu}$, where $g_{\mu\nu} = \delta_{\mu\nu} + h_{\mu\nu}$, i.e. the metric is expanded near the flat one, the propagator (or vertices) contains poles $1/n-2$, see refs. [5,6].

At a sufficiently small $n - 2 = \epsilon$ both nonlinear chiral dynamics and quantum gravity are renormalizable, i.e. there exists a limited number of counter terms, which are some invariant local functions, containing no more than two derivatives. But as distinct from the up to now studied theories where the poles in ϵ come only from the internal loop momenta integration, in quantum gravity at $n-2 = \epsilon$, as was mentioned above, there are extra poles in ϵ . Therefore the problem of the representation of the theory as a series in ϵ even after renormalization should be analysed separately. We shall see that these extra poles disappear from the gauge invariant part of the one-loop background functional, and we shall argue that this will be the case also in higher orders.

One more essential difference between quantum gravity and chiral dynamics is the following. In two dimensions both the theories have not only ultra-violet but infra-red divergencies as well, since both are massless. In chiral dynamics infra-red difficulties are eliminated by introducing to the Lagrangian of an additional term $H\sqrt{-\vec{\pi}^2} = H\phi$ that gives mass to the field $\vec{\pi}$ but breaks chiral symmetry. In quan-

eliminated by introducing the graviton mass . we shall investigate local counter terms corresponding to ultra-violet divergencies, as it is done for the case of the massless Jang-Mills field theory ⁷ at $n = 4$.

The question of the number of the physical degrees of freedom is also very important. In chiral dynamics there exist κ scalar fields, the number κ being not at all connected with the dimensionality of the space. In gravity, however, there exist $\kappa = \frac{n(n+1)}{2} - 2n$ physical degrees of freedom, i.e. at $n = 2$, $\kappa = -1$ (!), at $n = 3$, $\kappa = 0$ (!) and, finally, at $n = 4$, $\kappa = 2$. This counting of the number of physical degrees of freedom arouses some scepticism towards the attempt to consider quantum gravity (as distinct from chiral dynamics) near two dimensions. To avoid this difficulty we shall consider the background functional $W[g_{\mu\nu}]$ near small $\epsilon = n-2$, and only after continuation (if possible) to larger ϵ ($\epsilon \rightarrow 2$) we shall construct from $W[g_{\mu\nu}]$ the S-matrix in the physical space according to the usual rule. We shall further on understand expressions of the type $\frac{\sqrt{R}}{n-2}$ as a power expansion in $h_{\mu\nu}$, the tensor indices changing in all the sums from 1 to n .

If our approach will prove applicable for rather small

x) It may be that the analogue of the term H_5 in quantum gravity will be the introduction of a cosmological term $\lambda \sqrt{g}$ and the investigation of the corresponding phase transition near $n = 2$, but in this case the metric can not be asymptotically flat.

the renormalization the theory can be extended to large ϵ , it will be an interesting physical problem. Remember that in phase transition theory near the critical point ϵ -expansion is an effective technique⁸. We use 't Hooft's⁹ background functional. In recent years this functional has been shown^{9,5,10} to have some advantages over more familiar functionals that depend on external sources. The paper is organized as follows. Section 2 presents main properties of the W-functional. The use of the equations of motion and the general structure of the counterterms is investigated. In Section 3 we investigate the one-loop theory. The 't Hooft-Veltman⁵ algorithm is fit to the $\epsilon=2$. In Section 4 the one-loop counterterms are calculated and the charge renormalization and β -function^x of the renormalization group are found for a nonzero ϵ . In conclusion we discuss briefly the limits $\epsilon \rightarrow 0$ and $\epsilon \rightarrow 2$ and also the possibility to apply the results obtained. In Appendix we show how to perform the limiting procedure $\epsilon \rightarrow 0$ when investigating the two-dimensional model of black hole evaporation.

the problem of asymptotical freedom in quantum gravity at $d=4$ was investigated by Fradkin and Vilkovisky¹¹.

2. W-FUNCTIONAL IN QUANTUM GRAVITY AT $\epsilon = n-2$

The quantum gravity Lagrangian is ^{x)}

$$\mathcal{L} = -\frac{1}{2\epsilon^2} \left\{ \sqrt{g} R - (\Box \tilde{h}_{\mu\nu} - \partial_\mu \partial_\nu \tilde{h}_{\mu\nu}) \right\} \quad (2.1)$$

where

$$g_{\mu\nu} = \delta_{\mu\nu} + \tilde{h}_{\mu\nu} \quad (2.2)$$

i.e. $\mathcal{L}$ begins from quadratical in $\tilde{h}$ terms. The action, corresponding to $\mathcal{L}$ in (2.1) is invariant under general covariance gauge transformations

$$\delta g_{\mu\nu} = \delta \tilde{h}_{\mu\nu} = D_\mu [\xi_\nu(x)] + D_\nu [\xi_\mu(x)] \quad (2.3)$$

To construct the W-functional we expand the metric $\bar{g}_{\mu\nu} = g_{\mu\nu} + h_{\mu\nu}$ near some metric $g_{\mu\nu}$. The nonrenormalized W-functional is [9,5,10]

$$e^{iW[g_{\mu\nu}]} = \int d^4x d\psi d\psi^* \exp i \left\{ S[g+h] - S[g] - \right. \\ \left. - \frac{iS}{2} h_{\mu\nu} + \sqrt{g} \frac{1}{2} (D^\mu h_{\mu\nu} + \partial_\nu h) \right\} + \quad (2.4)$$

$$+ \sqrt{g} \psi^{*\alpha} Q_{\alpha\beta} [g, g+h] \psi^\beta \}$$

where

$$S[g] = \int d^4x \mathcal{L} \quad (2.5)$$

and $\mathcal{L}$ is defined in eq. (2.1)

^{x)} Let us recall that expansion of $\sqrt{g} R$ begins from the linear term $\Box \tilde{h}_{\mu\nu} - \partial_\mu \partial_\nu \tilde{h}_{\mu\nu}$.

and $\Psi^* Q_{\gamma\beta} \Psi$ is the action of the ghost particles.

The main properties of W-functional [9,5,10] are

1) G-invariance

$$W[g_{\mu\nu}]_{\alpha,\beta} = W[g_{\mu\nu} + \delta g_{\mu\nu}]_{\alpha,\beta} \quad (2.7)$$

with $\delta g_{\mu\nu}$ from eq. (2.3)

2) Mass shell gauge invariance

$$W[g_{\mu\nu}]_{\alpha,\beta} = W[g_{\mu\nu}]_{\alpha+\delta\alpha, \beta+\delta\beta} - i \frac{\delta S}{\delta g_{\mu\nu}} < \delta h_{\mu\nu} > \quad (2.8)$$

In eq. (2.8) $< >$ means the averaging in the sense of the functional integral (2.4). It follows from (2.8) that W-functional is gauge invariant for the fields $g_{\mu\nu}$ satisfying the classical equation of motion (2.6)

$$\frac{\delta S}{\delta g_{\mu\nu}} = 0 \quad (2.9)$$

It follows from dimensional grounds that at $n-2 = \varepsilon$

$< \delta h_{\mu\nu}^{local} > \sim g_{\mu\nu}$ and therefore the gauge noninvariant local counter terms are $\sim \frac{\delta S}{\delta g_{\mu\nu}} g_{\mu\nu} \sim \sqrt{g} R$.

Let us consider the divergences of the theory (2.4). For

$\varepsilon = n-2$ small enough, the theory can be renormalized by

ter terms are arbitrary local functions of $g_{\mu\nu}$ with at most two derivatives. In dimensional regularization these divergences give the poles in ϵ . It follows from (2.7), i.e. from general covariance, that only two counter-terms, depending on background $g_{\mu\nu}$ are possible

$$a \mathcal{L} \quad , \quad b \sqrt{g} R \quad (2.10)$$

with $\mathcal{L}$ defined in eq. (2.1). The coefficient a is gauge independent and corresponds to the charge renormalization, b is gauge dependent, being proportional to equation of motion, and corresponds to the wave function renormalization.

In the renormalized theory we should take

$$\frac{1}{x_0^2} S[g+h] \rightarrow \frac{\mu^{n-2}}{z_1 \kappa^2} S[z(g+h)] \quad (2.11)$$

μ is an arbitrary normalization momentum and κ is dimensionless renormalized coupling constant. In the one-loop approximation we have the counter terms

$$\begin{aligned} & \mu^{n-2} (1-z_1) \frac{1}{\kappa^2} S[g] + \mu^{n-2} (S[zg] - S[g]) \quad (2.12) \\ & = \mu^{n-2} \left\{ (1-z_1) S[g] + \frac{\delta S}{\delta g_{\mu\nu}} g_{\mu\nu} (z-1) \right\} \end{aligned}$$

It will be shown in a separate publication [12] that z depends on the gauge parameter β in (2.4) in such a way that there exists some number β for which $z=1$. Therefore the statement, that we can use strict classical equation of motion

and traced eq.

$$(n-2)\sqrt{g} R = 0, \quad (2.14)$$

When calculating the S-matrix [5,9,10], can be made in our case more clear. We have the possibility to take a gauge condition where the counterterms, proportional to equations of motion, are absent. Therefore if we look for gauge invariant quantities working even in other gauges it is sufficient to find only such counterterms which are not zero on classical equations of motion. In our case the action $S[g]$ defined in 2.1) is not zero on equations of motion. Let us explain this by noting that the tree approximation in quantum gravity in terms of the background field has the form ^{x)} (up to gauge condition)

$$W_{tree} = S[g] - \frac{\delta S}{\delta g_{\mu\nu}} g_{\mu\nu} = \int d^4x, \quad (2.15)$$

$$+ \frac{4-n}{2} \sqrt{g} R - (\Box \tilde{h}_{\mu\mu} - \partial_\mu \partial_\nu \tilde{h}_{\mu\nu}).$$

It follows from (2.15) that e.g. at $n=4$ all the tree graphs come from the term $\int d^4x (\Box \tilde{h}_{\mu\mu} - \partial_\mu \partial_\nu \tilde{h}_{\mu\nu})$, where

$$\tilde{h}_{\mu\nu} = \int d^4x' \mathcal{D}_{\mu\nu,\mu'\nu'}^c(x,x') \left(\mathcal{T}_{\mu'\nu'} + \frac{\delta \mathcal{L}_{int}}{\delta g_{\mu'\nu'}} \right)_{x'}. \quad (2.16)$$

and $\mathcal{T}_{\mu\nu}$ is the source of $h_{\mu\nu}$. From (2.16) it follows that the last term in (2.15) is not zero despite that it is an integral ∂_μ

Extra poles in δ . Quantum gravity at $n=2$ is conformal invariant, i.e.

$$2g_{\mu\nu} \frac{\delta S}{\delta g_{\mu\nu}} \sim (n-2) \quad (2.18)$$

Therefore at $n=2$ $\frac{\delta^2 S}{\delta g_{\mu\nu} \delta g_{\mu'\nu'}}$ is a singular operator. Indeed, it follows from (2.18) that

$$2g_{\mu\nu} \frac{\delta^2 S}{\delta g_{\mu\nu}(x) \delta g_{\mu'\nu'}(x')} \sim 2 \frac{\delta S}{\delta g_{\mu'\nu'}(x')} \delta^4(x-x') \sim n-2 \quad (2.19)$$

Eq.(2.19) considered at $g_{\mu\nu} = \delta_{\mu\nu}$ gives

$$2\delta_{\mu\nu} \frac{\delta^2 S}{\delta g_{\mu\nu}(x) \delta g_{\mu'\nu'}(x')} \Big|_{g=\delta} \sim (n-2). \quad (2.20)$$

Therefore when calculating ¹² the propagator in arbitrary gauge α, β (2.4) we find the poles in $n-2 = \epsilon$

($g_{\mu\nu} = \delta_{\mu\nu}$)

$$\mathcal{D}_{\mu\nu, \lambda\sigma} = -\frac{2}{p^2} \left\{ Q^2 - \left(1 + \frac{1}{\alpha}\right) Q^2 + \frac{1+2\beta}{(n-2)(2+\beta)} Q^3 - \frac{1}{n-2} Q^4 + \frac{1+2\beta}{(1+\beta)^2} \left(1 + \frac{2\beta+\beta^2}{2\alpha} - \frac{1+2\beta}{n-2}\right) Q^5 \right\}, \quad (2.21)$$

where Q_i ^{are some kinematical structures} ~~some of~~ ^{some of} the extra poles depend on the gauge condition, therefore it seems ~~very~~ likely that the extra poles give no contribution to the gauge invariant quantities. In the one-loop approximation this will be demonstrated explicitly.

[5] with extra attention to dependence on n . The gauge condition is defined in eq.(2.4) with $\alpha = -1$, $\beta = -\frac{1}{2}$. The one-loop effective Lagrangian is

$$\underline{\mathcal{L}} - \frac{1}{2} C_\mu^2 + \mathcal{L}_\psi \quad (3.1)$$

where $\underline{\mathcal{L}}$ is the quadratical in $h_{\mu\nu}$ part of the expansion $\mathcal{L}[g+h]$ in $h_{\mu\nu}$. C_μ is the gauge condition, and $\mathcal{L}_\psi$ is the ghost Lagrangian.

$$\underline{\mathcal{L}} - \frac{1}{2} C_\mu^2 = \sqrt{g} \left\{ -\frac{1}{4} h_{\alpha\beta}^\rho h_{\rho}^{\alpha,\nu} + \frac{1}{8} h_{\alpha,\nu}^\rho h_{\rho}^{\beta,\nu} + \frac{1}{2} h_{\rho}^{\alpha\mu} X_{\alpha\nu}^{\beta\mu} h_{\mu}^{\nu} \right\}, \quad (3.2)$$

$$X_{\alpha\nu}^{\beta\mu} = 2 \left(-\frac{1}{8} \delta_{\alpha}^{\rho} \delta_{\nu}^{\mu} R + \frac{1}{4} \delta_{\nu}^{\rho} \delta_{\alpha}^{\mu} R - \frac{1}{2} \delta_{\nu}^{\rho} R_{\alpha}^{\mu} + \frac{1}{2} \delta_{\alpha}^{\rho} R_{\nu}^{\mu} + \frac{1}{2} R^{\rho\mu}{}_{\alpha\nu} \right), \quad (3.3)$$

$$\mathcal{L}_\psi = \sqrt{g} \psi^{*\mu} \{ D_{\mu} D^{\alpha} g_{\mu\nu} - R_{\mu\nu} \} \psi^{\nu}. \quad (3.4)$$

After the doubling trick^[5] we get

$$\underline{\mathcal{L}} = \sqrt{g} \{ h_{\alpha\beta}^* P^{\alpha\beta\mu\nu} D_{\mu} D^{\nu} h_{\mu\nu} + \frac{1}{2} h_{\alpha\beta}^* \cdot (X^{\alpha\beta\mu\nu} + X^{\mu\nu\alpha\beta}) h_{\mu\nu} \} \quad (3.5)$$

$$P^{\alpha\beta\mu\nu} = \frac{1}{2} g^{\alpha\mu} g^{\beta\nu} - \frac{1}{4} g^{\alpha\beta} g^{\mu\nu} \quad (3.6)$$

The theory (3.5) can be reduced to a simpler form by replacing the integration variable x)

$$h_{\alpha\beta}^* \rightarrow h_{\mu\nu}^* P_{\mu\nu\alpha\beta}^{-1}, \quad (3.7)$$

where

$$P_{\mu\nu\alpha\beta}^{-1} = g_{\mu\alpha} g_{\nu\beta} + g_{\mu\beta} g_{\nu\alpha} - \frac{2}{n-2} g_{\mu\nu} g_{\alpha\beta}, \quad (3.8)$$

then instead of (3.5) we get

$$\begin{aligned} \underline{\mathcal{L}}_2 = & \sqrt{g} \{ h_{\alpha\beta}^* D_\gamma D^\gamma h^{\alpha\beta} + \frac{1}{2} h_{\mu\nu}^* \cdot \\ & \cdot P_{\mu\nu\alpha\beta}^{-1} (X^{\alpha\beta\mu'\nu'} + X^{\mu'\nu'\alpha\beta}) h_{\mu'\nu'} \}. \end{aligned} \quad (3.9)$$

Notice that some terms in (3.9) contain the pole $1/\varepsilon$ connected with the last term in (3.8). The corresponding terms take the form

$$\frac{2}{n-2} h_{\alpha\beta}^* \left(R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R \right) h_{\mu\nu}, \quad (3.10)$$

i.e. the terms containing $1/\varepsilon$ are proportional to eq. of motion, as was predicted in Sec.2.

The further treatment of (3.9) is passing over from the covariant derivatives D_γ acting on $h_{\mu\nu}$ as a second-rank tensor to the derivatives, which do not "see" the indices μ, ν , i.e. acting on $h_{\mu\nu}$ as a scalar. In equation (3.9) we designate the fields $h_{\alpha\beta} = \{h_{11}, h_{12}, \dots\}$ as $\varphi_i = \{\varphi_1, \dots, \varphi_{n(n-1)/2}\}$. Then (3.9) can be presented as follows

$$\sqrt{g} \{ (\nabla_\mu \varphi)^\dagger (\nabla^\mu \varphi) + \varphi^\dagger X \varphi \}, \quad (3.11)$$

where

$$\nabla_\mu = \partial_\mu \delta_{ij} + \partial_\mu \varepsilon_{ij}, \quad (3.12)$$

$$X_{ij} = \frac{1}{2} \rho_{\alpha\beta}^{-1} (X^{\mu\nu\alpha\beta'} + X^{\alpha'\beta\mu\nu}) \bigg|_{\substack{i=\alpha, \beta \\ j=\alpha', \beta'}} \quad (3.14)$$

ghost Lagrangian (3.4) can also be represented in the form (3.11) with $\varphi_i = \{\varphi_1, \dots, \varphi_n\}$ and

$$\mathcal{Q}_{\mu ij} = - \Gamma_{\mu}^{\alpha'} \quad \begin{matrix} i = \alpha \\ j = \alpha' \end{matrix} \quad (3.15)$$

$$X_{ij} = - R_{\alpha}^{\alpha'} \quad \begin{matrix} i = \alpha \\ j = \alpha' \end{matrix} \quad (3.16)$$

Thus, the original theory is reduced to a simpler form (3.11), where $\frac{n(n+1)}{2}$ of scalar particles φ interact with the external fields $g_{\mu\nu}$, $\mathcal{Q}_{\mu}$ (3.13) and X (3.14).

n complex scalar particles of Fermi-statistics (fermions) interact with $g_{\mu\nu}$ and with $\mathcal{Q}_{\mu}$. X determined in (3.15) and (3.16), respectively.

We are investigating now the theory of the type (3.11) at the arbitrary external fields $\mathcal{Q}_{\mu}$ and X and in the arbitrary external metric $g_{\mu\nu}$. We want to find local ultraviolet divergencies at small ε . Consider the invariant properties of the one-loop functional.

- I. The one-loop functional of the classical fields $g_{\mu\nu}$, $\mathcal{Q}_{\mu}$ and X is invariant under a gauge group of general cov-

2. The following transformation ¹³ can be performed in
(3.11)

$$\begin{aligned}\delta\varphi &= \Lambda(x)\varphi, \\ \delta X &= \Lambda X - X\Lambda, \\ \delta\varphi_\mu &= -D_\mu X + \Lambda\varphi_\mu - \varphi_\mu\Lambda,\end{aligned}\quad (3.17)$$

where $\Lambda(x)$ is some anti-Hermitian matrix

$$\Lambda(x) = -\Lambda^\dagger(x). \quad (3.18)$$

The Lagrangian (3.11) is invariant under transformations (3.17)-(3.18). Thus, the structure of counter terms near $n=2$ is determined by general covariance, invariance under transformations (3.17-3.18) and considerations of dimensionality. The counter terms to be added to the initial Lagrangian to eliminate the divergences of the theory have the form (in dimensional regularization)

$$\Delta\mathcal{L} = a_1\mathcal{L} + b_1\sqrt{g}\text{Sp}X, \quad (3.19)$$

$\mathcal{L}$ being defined in (2.1). Note, that in (3.19) there is no φ_μ -dependence. Remember ⁵ that at $n=4$ $\Delta\mathcal{L}$ contains a φ_μ -dependent term of the form $\frac{\sqrt{g}}{n-4}\text{Sp}\varphi_{\mu\nu}^2$, where

$$\varphi_{\mu\nu} = D_\mu\varphi_\nu - D_\nu\varphi_\mu + \varphi_\mu\varphi_\nu - \varphi_\nu\varphi_\mu. \quad (3.20)$$

Near $n = 2$ the corresponding counter term would have the form

it is equal to zero, since $\gamma_{\mu\nu}$ is antisymmetric.

4. The calculation of the one-loop charge renormalization.

Before determining the coefficients in (3.19) we would like to make sure that calculations will confirm the structure (3.19), where there is no γ_μ -dependence. Consider diagrams with two external γ_μ -fields, Figs. 1,2. The self-energy diagram of Fig.1 is calculated in dimensional regularization and gives a nonzero result for the counter term

$$\sim \frac{\sqrt{g}}{n-2} \gamma_\mu \gamma^\mu. \quad (4.1)$$

Diagram in Fig.2 determined by the vertex $\sqrt{g} (\gamma_\mu \psi)^\dagger \gamma_\mu \psi$ gives also a counter term of the form (4.1), which should compensate the corresponding term in (4.1) of the diagram Fig.1. In this case the formal property of the symmetry under transformations (3.17)-(3.18), leading to the independence of the UV counter terms on γ_μ near $n=2$, will be confirmed in calculations. A massless tadpole in the dimensional regularization is commonly thought to be zero, but this result $\int \frac{d^2 q}{q^2} = 0$ stems from a cancellation of UV and IR divergences at $n = 2$. If, however, we turn to Ref. [14], we

In (4.2) $f\left(\frac{n}{2}\right)$ is a function used for an analytical continuation and possessing a number of properties, ^{one} of which is

$$f\left(\frac{n}{2}\right) = 0 \quad \text{at } n = 0, 1, 2, \dots; \quad (4.3)$$

in particular, at $n \rightarrow 2$

$$f\left(\frac{n}{2}\right) \sim (n-2)^2. \quad (4.4)$$

From (4.3) (4.4) it follows that the tadpole determined in (4.2) is zero at all integer n , except $n \rightarrow 2$, where

$$\begin{aligned} I &= \left[(n-2)^2 \right]^{\frac{n-2}{2}} \frac{2\pi}{2-n} + \dots = \\ &= e^{(n-2) \ln(n-2)} \frac{2\pi}{2-n} + \dots = \frac{2\pi}{2-n} + \dots \end{aligned} \quad (4.5)$$

i.e. near two dimensions the tadpole cannot be considered to be zero. It turns out that if one calculates the self-energy diagram (Fig.3) by the dimensional regularization formulae and for the seagull diagram (Fig.2) uses the value from (4.2) (4.5), one does obtain the compensation of the counter terms of the form (4.1). No other ϵ_m -dependence can appear from general covariance and considerations of dimensionality. Thus, dimensional regularization keeps the symmetry (3.17)-(3.18) of the initial functional integral.

Discuss now the term $\theta_{\mu\nu} g_{\mu\nu} S_P X$ in eq. (3.19). It is easy to calculate $S_P X$, where X is determined in (3.14) for gravitons and in (3.16) for fictitious particles. The extra poles $\frac{1}{n-2}$ in X (see eq. (3.10)) will

disappear as soon as the trace is taken, since

$$Tr(R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R) = R(1 - \frac{n}{2}) .$$

For gravitons and fictitious particles $Sp X$ proves proportional to R . The coefficient is determined by the diagram like (4.2) of fig. 3. The result is

$$b_2^{gr} \sqrt{g} Sp X = \frac{\sqrt{g} R}{4\pi(n-2)} \quad (4.6)$$

for gravitons and

$$b_1^{gh} \sqrt{g} Sp X = \frac{\sqrt{g} R}{2\pi(n-2)} \quad (4.6')$$

for ghost particles.

Now we shall pass over to the determination of the coefficient α_2 in (3.19). For this purpose we shall consider (3.11) with $\varphi_n = X = Q$, i.e. a simple theory of a complex scalar field in the external gravitational field $g_{\mu\nu}$

$$\mathcal{L} = -\sqrt{g} g^{\mu\nu} \partial_\mu \varphi^* \partial_\nu \varphi = -\partial_\mu \varphi^* \partial_\mu \varphi + \quad (4.7)$$

where $+ S_{\mu\nu} \partial_\mu \varphi^* \partial_\nu \varphi + \dots$

$$S_{\mu\nu} = h_{\mu\nu} - \frac{1}{2} \delta_{\mu\nu} h^\lambda{}_\lambda \quad (4.8)$$

The term linear in the gravitational field (Fig.4) is zero, since the diagram with two derivatives appears

$$\int \frac{d^n k}{k^2} K_\mu K_\nu = 0 \quad (4.9)$$

which is equal to zero in the dimensional regularization.

The term quadratic in gravitational field ^{arises} due to the self-

energy diagram (Fig.5) alone, since the diagram with two external gravitons (Fig.6) has also the form of (4.9) and is zero. Consider now the diagram in Fig.5. It is equal to

$$\begin{aligned} & \frac{2 \cdot 2}{2! i (2\pi)^n} \int \frac{d^n k}{k^2 (k+p)^2} k_\mu (k+p)_\nu k_{\mu'} (k+p)_{\nu'} \quad (4.10) \\ &= (\delta_{\mu\nu} \delta_{\mu'\nu'} + \delta_{\mu\mu'} \delta_{\nu\nu'} + \delta_{\mu\nu'} \delta_{\mu'\nu}) \frac{4\pi^{\frac{n}{2}}}{2(2\pi)^n 16 (n^2-1)} \times \\ & \times [\Gamma(n-2)]^{-1} \Gamma\left(\frac{n}{2}-\frac{n}{2}\right) \Gamma\left(\frac{n}{2}-1\right) \Gamma\left(\frac{n}{2}-1\right) p^{2n} + \dots \end{aligned}$$

Dots mean other structures containing $p_\mu p_{\mu'}$ etc. It follows from (4.10) that the counterterm, which eliminates the one-loop divergence of one real scalar field in external $g_{\mu\nu}$, is equal to (see also [15])

$$\Delta \mathcal{L} = \frac{x^2 \mathcal{L}}{24\pi(n-2)}, \quad (4.11)$$

$\mathcal{L}$ being defined in (2.1). Notice, that the second term in the action corresponding to (2.1) does not violate the general covariance, since the parameters of the general covariant transformation $\xi_\mu(x)$ in (2.3) are regarded to be well decreasing functions of x , and the integral

$$\int d^n x (\Box \delta_{\mu\nu} - \partial_\mu \partial_\nu) \delta \tilde{h}_{\mu\nu} = 0 \quad (4.12)$$

even if

$$\int d^n x (\Box \delta_{\mu\nu} - \partial_\mu \partial_\nu) \tilde{h}_{\mu\nu} \neq 0 \quad (4.13)$$

as it is the case for the fields $\tilde{h}_{\mu\nu}$, which are solutions of the classical equations of motion.

If we remember now that in the initial gravitational problem there are $\frac{n(n+1)}{2}$ degrees of freedom connected with $\tilde{h}_{\mu\nu}$ and $2n$ (n complex) degrees of freedom connected with fictitious particles, we have

$$a_1 = \left(-2n + \frac{n(n+1)}{2} \right) \frac{x^2}{24\pi(n-2)}, \quad (4.14)$$

and at $n = 2$

$$a_1 \mathcal{L} = - \frac{x^2 \alpha^0}{24\pi(n-2)}; \quad b_1 \sqrt{g} S p X = \frac{18\sqrt{3} R}{24\pi(n-2)}. \quad (4.15)$$

We have performed also a detailed calculation without the use of the t' Hooft-Veltman algorithm. The background field was expanded around the flat space and $\Delta \mathcal{L}$ was calculated in the second order in the background. It appears that general covariance is maintained only with the prescription (4.5). The result is

$$\Delta \mathcal{L} = - \frac{(1+18)}{24\pi(n-2)} x^2 (\mathcal{L})_{\mu\nu, \kappa\lambda}, \quad (4.16)$$

where $(\mathcal{L})_{\mu\nu, \kappa\lambda}$ is the second order expansion of $(-\frac{\sqrt{3}R}{x^2})$.

The first term in (4.16) is from the self energy diagrams like that in Fig.5, and the second term is from the seagull diagrams like that in Fig.6. This result coincides with the second order expansion of both terms in the previously obtained result (4.15).

It will be shown in the next publication ¹² that the variation of the one-loop counter terms in the background functional with the gauge condition β (2.4) is proportional to $\bar{\beta}(1-\bar{\beta})^2 \sqrt{g} R$, where $\bar{\beta} = \frac{1+2\beta}{2(1+\beta)}$. ^{x)}

We conclude that the one-loop charge renormalization in (2.11) is

$$\bar{Z}_1 = 1 + \frac{\kappa^2}{24\pi(n-2)} . \quad (4.17)$$

After the charge renormalization the gauge invariant radiative corrections of the one-loop approximation contain no singularities in ϵ . At small enough ϵ the whole procedure can be repeated also for the next approximations. As it was shown in Sec.2 the only two counter terms (2.10) appear and the theory is renormalizable.

It was shown in the one-loop approximation that the extra poles $1/\epsilon$ disappear from the gauge invariant quantities. This fact and also the gauge noninvariance of the poles in the propagator (2.21) suggest that the extra poles in ϵ disappear from gauge invariant quantities also in higher approximations.

If this is the case, the theory will be represented as a double series in ϵ and in the coupling constant, i.e. in this respect quantum gravity ^{will} repeat the case 1-4 investigated already in nonlinear chiral dynamics at $\epsilon = n-2$. The action will be expressed through a renormalized dimensionless charge as in eq. (2.14), where

B! x) If despite of (2.15) we have taken no attention on the linear term of $k-2,2$, we could conclude that the only general covariant counterterm is $\sqrt{g} R$ and since it is gauge dependent, there exists some gauge condition where there is no uv counterterms at all, i.e. the theory needs no uv renormalization.

we could conclude that the theory is renormalizable

$$z^2 = z_1 k^2 \mu^{2-n} . \quad (4.18)$$

Invariance of a nonrenormalized theory under the variations of μ at a fixed z gives the renormalization group equations

$$\left\{ \mu \frac{\partial}{\partial \mu} + \beta \frac{\partial}{\partial k^2} - \frac{N}{2} \right\} \Gamma^N(\rho, k^2, \mu) = 0 \quad (4.19)$$

$$\beta(k^2) = (n-2)k^2 + k^2 \mu \frac{\partial}{\partial \mu} \ln z,$$

In the one-loop approximation from eq. (4.17) it follows that

$$\beta(k^2) = (n-2)k^2 - \frac{k^4}{24\pi} + O(k^6), \quad (4.20)$$

i.e. there exists an u.v. stable fixed point at

$$k_c^2 = 24\pi \varepsilon . \quad (4.21)$$

From (4.20) it follows also that $\beta(k^2)$ has a limit at $n \rightarrow 2$ and it is negative in this limit, i.e. we deal with asymptotical freedom at $n \rightarrow 2$.

From the renormalization group equations it follows that at large momenta the asymptotic behaviour is determined not by perturbation theory but there exists a power behaviour determined by the fixed point (4.21)

$$\Gamma^N(\lambda \rho, k^2)_{\lambda \rightarrow \infty} \sim \lambda^{n - \frac{N}{2} (n-2 + \kappa_c^2)} . \quad (4.22)$$

The existence of a fixed point may indicate renormalizability of the theory also at not very small ε , the same as it is in chiral dynamics 1-4 . .

5. Conclusion

It is thus shown that it is possible to work with quantum gravity near $n = 2$ and to calculate the background functional. It turns out renormalizable, i.e. the counterterms have the structure coinciding with the initial Lagrangian and therefore can be eliminated by the charge renormalization.

What can be said about the limit $\varepsilon \rightarrow 0$? It seems natural to expect that the theory of gravitons become trivial in this limit. Indeed, the renormalized background functional $W[g]$ is presented as a series in ε and the coefficients, the functions of $g_{\mu\nu}$, may well become trivial at $n=2$ when the gauge invariant part of $W[g_{\mu\nu}]$ is calculated for $g_{\mu\nu}$ satisfying the classical equation of motion. But if we consider e.g. scalar field in the background such as two-dimensional collapsing-shell metric [15], the theory is nontrivial and we obtain by the correct limiting procedure $\varepsilon \rightarrow 0$ the same result (for the trace of the energy-momentum tensor) as in [15] where the calculation is performed directly at $n = 2$ by means of the point splitting method. We found also that the contribution of neutrino equals exactly the scalar field contribution in two-dimensional black-hole evaporation, see Appendix.

In connection with a more interesting limit to the physical space $\varepsilon \rightarrow 2$ we can only hope that due to the presence

of the u.v. stable fixed point at $\varepsilon > 0$ the representation of quantum gravity as a double series in ε and charge may prove useful.

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APPENDIX

Consider the two-dimensional reduced Schwarzschild metric [15]

$$ds^2 = -\left[1 - \frac{2M}{\varepsilon}\right] dt^2 + \left[1 - \frac{2M}{\varepsilon}\right]^{-1} dx^2 \quad (\text{I})$$

In two dimensions every metric satisfies

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \equiv 0$$

but if at first we consider the equation of motion (2.13), (2.14) at $n-2 = \varepsilon \neq 0$, where they are meaningful, and then consider the limit $\varepsilon \rightarrow 0$, we see that R should not be zero at $n = 2$, as is the case for the metric (I). It follows from our eq. (4.11) that for the scalar field in the background we have

$$W_{\text{scalar}}^{\text{div}} [g_{\mu\nu}] = -\Delta \mathcal{L} = \frac{-\mathcal{L}}{24\pi(n-2)} + \dots,$$

dots mean the nonsingular in $(n-2)$ contribution. Therefore

$$T^{\text{div}}_{\mu\nu} = \frac{2}{16} \frac{\delta W^{\text{div}}}{\delta g_{\mu\nu}} = \frac{-2(R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R)}{24\pi(n-2)} + \dots$$

$$T^{\mu}_{\mu} = g_{\mu\nu} T^{\text{div}}{}^{\mu\nu} = \frac{-2R(1 - \frac{n}{2})}{24\pi(n-2)} = \frac{R}{24\pi}$$

It can be shown that [16]

$$T_{\mu}^{\mu} = T_{\mu}^{\text{div}\mu} + T_{\mu}^{\text{finite}} = 0.$$

Therefore

$$T_{\text{scalar}\mu}^{\text{finite}\mu} = -\frac{R}{24\pi} = \frac{1}{6\pi} \frac{M}{\epsilon^3}.$$

We have used the Capper and Duff calculations [16] of the counterterm for the neutrino in the background, and found

$$W_{\text{neut}}^{\text{div}}[g_{\mu\nu}] = \frac{-L}{24\pi(n-2)}, \quad T_{\text{neut}\mu}^{\text{finite}\mu} = +\frac{1}{6\pi} \frac{M}{\epsilon^3}.$$

It follows that when we calculate by means of dimensional regularization and perform the limit $\epsilon \rightarrow 0$, the result is gives identical scalar and neutrino contribution to the two-dimensional black hole evaporation. In the papers [15] this result seems to depend on the details of calculations and is not stated definitely.

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Fig. 1

Fig. 2

Fig. 3

Fig. 4

Fig. 5

Fig. 6

vector field

η_μ

scalar field

φ

scalar field

χ

gravitational field

$g_{\mu\nu}$

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