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ALGEBRA OF OBSERVABLES AND TRIBES OF REGGE POLES ⁺)G. Domokos ^{*)} and S. Kövesi-Domokos ^{*)}

CERN--Geneva

ABSTRACT

We show that the zero mass trajectory generating algebra $SL(2, \mathbb{C})$ can be identified with the algebra spanned by space integrals of appropriate components of vector and axial vector currents. The algebra is then generalized to include internal symmetries, and is found to be a chiral $SU(6) \times SU(6)$ algebra. After an analysis of the concept of massless composite particle states, we saturate the commutation relations by five Lorentz families of states of baryon number one; the families include the known baryon octet and decuplet families. The symmetry breaking mechanism is studied and trajectory and mass formulae are derived in various approximations.

⁺) A preliminary version of this work under the same title was submitted as a contributed paper to the XIVth International Conference on High Energy Physics and Elementary Particles, Vienna, 1968. (G. Domokos and S. Kövesi-Domokos, CERN preprint 68/856/5 - TH 913, 1968.)

^{*)} On leave of absence from, and address after September 1, 1968 : Central Research Institute for Physics, Budapest, Hungary.

In a series of works ¹⁾⁻¹¹⁾ we investigated the theory and experimental consequences of the classification of hadron states according to representations of an algebra $SL(2, \mathbb{C})$. While at zero mass this classification is exact ⁴⁾ [in the sense that hadron states span an irreducible representation - IR - of $SL(2, \mathbb{C})$], as soon as one follows a hadronic Regge trajectory to non-vanishing values of the invariant mass, one finds it necessary to work with reducible representations of the algebra. Inclusion of representation mixing turned out to be necessary both from the theoretical and experimental points of view. Theoretically, the necessity of representation mixing is indicated both by model calculations ^{1), 2)}, and considerations of analyticity and factorization properties of scattering amplitudes ¹²⁾; experimentally it is necessary since a successful classification of physical hadron levels of non-vanishing mass can be achieved only after the inclusion of mixing terms. In a well-known way, the inclusion of representation mixing leads to a "mass formula" ^{8), 9)} describing the behaviour of Regge trajectories which span an IR of $SL(2, \mathbb{C})$ at zero mass (i.e., belong to the same Lorentz family). It is interesting and important to observe at this point that experimentally the mixing terms turn out to be quite small ⁹⁾ and therefore it is apparently meaningful to speak about a Lorentz family of Regge poles in the $SL(2, \mathbb{C})$ limit, i.e., when all trajectories in the family run parallel to each other for every value of the mass. This is precisely the situation considered recently by Cosenza, Sciarrino and Toller ¹³⁾, henceforth we shall refer to it as the CST limit.

In all the treatments of the " $SL(2, \mathbb{C})$ - conspiracy" of Regge poles given so far, the possible connections with internal symmetries were left out of consideration; in other words, hadronic Regge trajectories were classified according to the direct product $SL(2, \mathbb{C}) \times SU(3)$.

The present paper describes an attempt to unify the treatment of the $SL(2, \mathbb{C})$ trajectory generating algebra (TGA) at zero mass with the internal symmetry algebra $SU(3)$.

Let us summarize briefly the evidence indicating that such a unified treatment may indeed be possible. The mass formula referred to earlier describes the nucleon and delta families of Regge trajectories remarkably well. If one writes the power series expansion of the trajectories in the now-standard form :

$$\alpha_{\kappa}(W) \approx \sigma - \kappa + A_1 W + [A_0 + A_1^2(\sigma - \kappa + \frac{1}{2}) + A_2(\sigma - \kappa)(\sigma - \kappa + 1)] W^2$$

one finds the following values of the parameters ⁹⁾ (Table I).

The best value of the parameter A_1 for the delta is zero ; it was already suggested earlier ^{7),10)} that this may be explained by putting the delta on a trajectory which in the CST limit has the Lorentz quantum number $j_0 = \frac{3}{2}$, while in the same limit the nucleon must necessarily have $j_0 = \frac{1}{2}$, since the lowest lying physical state in it (the nucleon) has spin one-half ¹⁴⁾. Observe that the values of the parameter A_0 [which is the $SL(2,C)$ invariant part of the slope] are quite close to each other for the nucleon and delta families, and the same observation can be made about the $SL(2,C)$ breaking parameter, A_2 . Similar results are found if one determines the parameters from strange baryon resonances ¹⁵⁾. [Table I also brings out the point mentioned before, namely that the CST limit is apparently physically meaningful ; indeed the average magnitude of the $SL(2,C)$ breaking terms is some 5 ÷ 10% of those which preserve $SL(2,C)$, i.e., survive in the CST limit.]

The foregoing considerations suggest that the Lorentz families of octet and decuplet baryon Regge trajectories may span together an IR of some larger algebra, with - probably - a reasonably small representation mixing. To be sure, the algebra to be found need not generate a symmetry of scattering amplitudes in general ; this is already the case for $SL(2,C)$, which generates a symmetry only at $W = 0$ and if the external masses joining a Regge-vertex are such that $W = 0$ (vanishing invariant momentum transfer in the scattering amplitude) implies that the momentum transfer is a null vector.

Let us now list a few basic physical requirements, which the algebra and its representation have to fulfil :

- a) The algebra must contain $SL(2, \mathbb{C}) \times SU(3)$ as a subalgebra, see the foregoing discussion.
- b) The irreducible representations should be spanned by massless pseudostates ¹⁰⁾ ("Regge intercepts").
- c) The elements of the algebra should have a straightforward physical interpretation.
- d) The "physical" representations should not contain "too many" states ; in particular, high representations of $SU(3)$, like 27, 35, etc. should be absent, since they are probably not observed in nature as "single particle" states. The fulfilment of this requirement depends heavily on the previous point ; given an abstract algebra, one cannot tell which representations of it are physically meaningful, unless one is able to identify the elements of the algebra as observables.
- e) The algebra should be "useful" ; in particular, it should be an algebra of finite dimension. This last requirement is a practical one (at least at the moment). On the one hand, infinite dimensional algebras are very hard to handle in practice ; also, one hopes that the future theory of hadrons will be such that all the observables are expressible in terms of a finite number of them. On the other hand, one expects that the more observables are included in the algebra, the more physical information it contains ; hence in choosing the algebra we want to study, we have to find a reasonable balance between the somewhat contradictory requirements of maximum physical information and practical tractability of the algebra. (Clearly, the requirements outlined in this point reflect the present, confused situation of the theory of hadrons ; as soon as somebody constructs a consistent theory, it will dictate by itself which observables to choose - or abandon the whole approach altogether.)
- f) The algebra must obviously (?) respect the special theory of relativity.

The requirements we formulated here are more or less the same in spirit as those outlined in Gell-Mann's program ¹⁶⁾. What appears to be a new element is the emphasis laid on zero mass pseudostates, since the special properties of the latter ¹⁾⁻¹¹⁾ seem nowadays essential in formulating a consistent theory. Also, as we shall see later on, by putting the masses of the hadrons equal to zero, one achieves a tremendous simplification of the technical problems encountered.

Connection with the "real world", i.e., with physical hadron states of non-zero mass and finite spin can be established afterwards by using analyticity arguments, i.e., extrapolating our results to non-zero mass by using the fact that hadrons seem to be composite systems, hence there is an analytic relation between their masses and spins.

The program of the paper is as follows. In Section 2, we find an algebra satisfying the six requirements described above. The algebra we arrive at is a chiral $SU(6) \times SU(6)$, spanned by the space integrals of the currents, $\frac{1}{2}(V_{\mu}^i \pm A_{\mu}^i)$, with $\mu = 0, \dots, 3$, $i = 0, \dots, 8$. After this we begin to study the physically meaningful representations of the algebra obtained. Section 3 contains general comments on them to the effect that in order to produce infinite families of Regge trajectories, one has to saturate the commutation relations with states which span a representation of the entire "conspiracy algebra" and not just its spin part (which is used to construct the supermultiplet algebra). The construction described in Section 2 suggests that the pseudostates which saturate the algebra obtained and correspond to massless composite baryons, have the same quantum numbers as those obtained from a symmetric quark model with orbital excitations at zero total energy momentum. Since this system has particular symmetry properties, Section 4 is devoted to their description. To a certain extent, the treatment given in Section 4 parallels the familiar, non-relativistic treatments of three-fermion systems ^{17), 18)}. However, since the little group belonging to a null-momentum is the entire homogeneous Lorentz group, our results differ considerably from those obtained from non-relativistic treatments.

In Section 5 we proceed to actually construct the baryonic representation of the algebra out of the symmetric quark states we found in the previous Section. We find that the algebra can be saturated by a finite number of Lorentz families of pseudostates, each family of pseudostates having a definite transformation property under $SU(3)$; the highest representation of $SU(3)$ occurring is of course the ten dimensional one. We shall call these families together a tribe. Section 6 contains the description of the symmetry breaking mechanism. As the simplest possibility, we suggest to break just the $SU(3)$ subalgebra at $W = 0$ and at $W \neq 0$ the subalgebra $SL(2, C)$ as well. This mechanism leads to trajectory formulae which are then subject to a rough experimental test in Section 7. We find that the agreement between the predicted and observed positions of baryon states is quite satisfactory ; in particular, we are able to predict absolute mass values of the decuplet states (as opposed to mass ratios only, as for example, in a static $SU(6)$ classification). Finally, Section 8 is devoted to the discussion of the results.

It should be pointed out at this point that although in the paper we use the quark model as a guide, our results are obviously independent of the existence of quarks as physical particles. There can be little doubt about the heuristic value of a quark model and we want to take advantage of it. Most probably every "good" result obtained on the basis of the quark model could be reformulated without the use of quark fields ; however, we do not want to dwell on this problem here. (The reader might as well recall that models which later on turned out to be unrealistic did quite often serve as a useful guide to construct valid physical theories ; Maxwell's mechanical model which lead him to the fundamental equations of classical electrodynamics ¹⁹⁾ being but one example.)

2. - SEARCH FOR THE SUPERMULTIPLY ALGEBRA

Let us recall that one way to obtain the TGA of massless pseudostates is the following ¹⁰⁾. One notices that at finite "mass" (the "mass" may be imaginary for pseudostates with spacelike momentum), the pseudostates span a representation of the algebra of Wigner operators, W_μ :

$$[W_\mu, W_\nu] = i \varepsilon_{\mu\nu\sigma\tau} P_\sigma W_\tau. \quad (2.1)$$

Here, as usual, $W_\mu = \frac{1}{2} \varepsilon_{\mu\nu\sigma\tau} M_{\nu\sigma} P_\tau$, $P_\mu W_\mu = 0$, $[P_\mu, W_\nu] = 0$. $M_{\mu\nu}$ and P_σ are the generators of homogeneous Lorentz transformations and translations, respectively. When applied to eigenstates of P_μ , the operators W_μ can be replaced by the expression $w_\mu = \frac{1}{2} \varepsilon_{\mu\nu\sigma\tau} M_{\nu\sigma} p_\tau$, where now p_σ stands for the eigenvalue of the operator P_σ ²⁰⁾. In that case the relation (2.1) expresses the commutation relations (CR) of the algebra spanned by the w_μ . In order to secure that a "Regge vertex", i.e., the scalar product ²¹⁾ of a pseudostate with a physical two - (or many) - particle state be an analytic function of the total mass of the former for every external mass configuration, one has to require that the vertex be an analytic function of the four-vector p_μ in some finite neighbourhood of $p_\mu = 0$. (This requirement in turn is sufficient - together with some other conventional assumptions - to reproduce the results of the "conspiracy theory" at zero momentum transfer ²²⁾.) In order to achieve analyticity in p_μ (and not just W or W^2), one has to demand ¹⁰⁾ that the pseudostates at $W = 0$ span a representation of the larger algebra which besides the three independent operators (say w_1, w_2, w_3 for W real) taken from (2.1) contain their derivatives as well with respect to the parameter $\lambda = (1-v)(1+v)^{-1}$ at $\lambda = 0$. (Here v stands for the modulus of the velocity of the pseudostate, $v = |\underline{p}|(W^2 + \underline{p}^2)^{-\frac{1}{2}}$.)

This enlarged algebra was shown in Ref. ¹⁰⁾ to be identical to the algebra $SL(2,0)$ - apart from the choice of the basis - spanned by the operators $M_{\mu\nu}$ ²³⁾. In a free quark model the operator $M_{\mu\nu}$ can be written as the sum of an orbital and a spin part :

$$M_{\mu\nu} = L_{\mu\nu} + S_{\mu\nu} \quad (2.2)$$

with

$$L_{\mu\nu} = \int d^3x (-\bar{\psi} \gamma^0 \partial^\mu \psi x^\nu + \bar{\psi} \gamma^0 \partial^\nu \psi x^\mu), \quad (2.3)$$

$$S_{\mu\nu} = i \int d^3x \bar{\psi} \gamma^0 \sigma_{\mu\nu} \psi. \quad (2.4)$$

Let us now assume that the operator (2.2) acts on a state composed of two quarks (or a quark and an antiquark). If the momenta of the quarks in a particular configuration are $p_{1\mu}$ and $p_{2\mu}$ respectively, one can introduce their total and relative momenta by writing

$$P_{1\mu} = \frac{1}{2} E_\mu + p_\mu,$$

$$P_{2\mu} = \frac{1}{2} E_\mu - p_\mu.$$

Correspondingly, the orbital part (2.3) acts on the $q\bar{q}$ wave function as

$$L_{\mu\nu} \rightarrow i \left(E_\mu \frac{\partial}{\partial E_\nu} - E_\nu \frac{\partial}{\partial E_\mu} \right) + i \left(p_\mu \frac{\partial}{\partial p_\nu} - p_\nu \frac{\partial}{\partial p_\mu} \right).$$

If the total four-momentum of the $q\bar{q}$ system goes to zero, only the second term remains, since the dependence of the $q\bar{q}$ wave function of the total four-momentum is supposed to be analytical near $E_\mu = 0$. Hence the "conspiracy algebra", $SL(2,0)$ is generated on a $q\bar{q}$ wave function with vanishing total four-momentum by the operators

$$m_{\mu\nu} \equiv l_{\mu\nu} + S_{\mu\nu}, \quad (2.5)$$

8.

where
$$\ell_{\mu\nu} = i \left(p_\mu \frac{\partial}{\partial p_\nu} - p_\nu \frac{\partial}{\partial p_\mu} \right).$$

(A three-quark composite state can be handled in a similar way.) The orbital part $\ell_{\mu\nu}$ is not useful from our point of view, since, when combined with internal symmetries, it leads to an algebra of infinite dimension. Hence we decide to study the spin part further. On introducing the operators M_K, N_K in the usual way, i.e. :

$$\begin{aligned} M_K &= \frac{1}{2} \epsilon_{K+5} S_{+5}, \\ N_K &= S_{0K}, \end{aligned} \quad (2.6)$$

we obtain immediately :

$$\left. \begin{aligned} M_K &= i \int d^3x \bar{\Psi} \gamma_5 \gamma_K \Psi, \\ N_K &= i \left(i \int d^3x \bar{\Psi} \gamma_K \Psi \right), \end{aligned} \right\} \quad (2.7)$$

i.e., the spin part of the generators of the conspiracy algebra can be identified with the space integrals of the space components of the axial - and (i) times the vector currents. Following the usual procedure, one then introduces the operators which span the algebra in its real form :

$$F_a^{(\pm)} = \frac{1}{2} (M_a \mp i N_a) = i \int d^3x \bar{\Psi} \frac{1 \pm \gamma_5}{2} \gamma_a \Psi \quad (2.8)$$

which then generate a chiral algebra $SU(2) \times SU(2)$. Having made this observation, we can proceed by including the internal symmetry algebra $SU(3)$. A moment of thinking shows that the most natural form of this is to take chiral $SU(3) \times SU(3)$ with the generators

$$F_A^{(\pm)} = \frac{i}{2} \int d^3x \bar{\Psi} \frac{1 \pm \gamma_5}{2} \gamma_0 \lambda^A \Psi \quad (2.9)$$

(where the λ^A are the matrices as defined by Gell-Mann ²⁴).

So far we have the direct product $[SU(2) \times SU(3)]_{\text{LEFT}} \times [SU(2) \times SU(3)]_{\text{RIGHT}}$ as an algebra of which the massless pseudostates presumably span a representation. In view of what was said in the Introduction, we try and embed this algebra into a larger one.

This can be done in a most natural way by including space integrals of the space components of octet chiral currents :

$$F_{aA}^{(\pm)} = \frac{i}{2} \int d^3x \bar{\psi} \frac{1 \pm \gamma_5}{2} \gamma_a \lambda^A \psi. \quad (2.10)$$

One immediately verifies that at equal times the seventy observables (2.8) to (2.10) span the chiral algebra $SU(6)_{\text{LEFT}} \times SU(6)_{\text{RIGHT}}$. The non-vanishing CR they satisfy are indeed the following :

$$[F_a^{(X)}, F_b^{(X')}] = i \delta_{XX'} \epsilon_{abc} F_c^{(X)}, \quad (2.11a)$$

$$[F_A^{(X)}, F_B^{(X')}] = i \delta_{XX'} f_{ABC} F_C^{(X)}, \quad (2.11b)$$

$$[F_a^{(X)}, F_{bB}^{(X')}] = i \delta_{XX'} \epsilon_{abc} \delta_{BC} F_c^{(X)}, \quad (2.11c)$$

$$[F_A^{(X)}, F_{bB}^{(X')}] = i \delta_{XX'} f_{ABC} \delta_{bc} F_c^{(X)}, \quad (2.11d)$$

$$[F_{aA}^{(X)}, F_{bB}^{(X')}] = i \delta_{XX'} \left\{ \frac{1}{2} \epsilon_{abc} d_{ABC} F_c^{(X)} + \frac{1}{6} \epsilon_{abc} \delta_{AB} F_c^{(X)} + \right. \\ \left. + \frac{1}{4} \delta_{ab} f_{ABC} F_C^{(X)} \right\}, \quad (2.11e)$$

The Clebsch-Gordan coefficients of $SU(3)$, f_{ABC} , d_{ABC} are normalized according to the convention of Ref. ²⁴⁾; ϵ_{abc} is the Levi-Civita symbol. The superscript χ stands for the chirality, $\chi = \pm 1$.

The CR (2.11) can be recognized to be those of $SU(6)$ for each sign of the chirality separately. The rest of this paper is devoted to the investigation of our fundamental assumption, namely that the massless pseudostates span an IR of the algebra (2.11).

The reader will easily verify that the algebra we constructed satisfies the requirements we described in the Introduction. A word is in order here, however, about the physical interpretation of the operators. In the framework of the quark model at least, the connection of the densities

$$\frac{i}{2} \bar{\Psi} \frac{1 + \gamma_5}{2} \gamma_\mu \lambda_A \Psi$$

with the currents entering - in various linear combinations - the expression of the weak and electromagnetic interactions is evident. Thus the hypothesis is just formulated effectively amounts to say that the spin part of the TGA of massless hadronic pseudostates is generated by the space integrals of the appropriate components of weak and electromagnetic currents.

3. - GENERAL COMMENTS ON REPRESENTATIONS

Before entering a detailed investigation of the physically meaningful representations of the algebra (2.11), we make a few general comments about them.

The construction described in the previous Section strongly suggests to try and saturate the CR with - in general reducible - representations of the algebra

$$O(3,1) \times SL(2,C) \times SU(3). \quad (3.1)$$

The essential observation is that this algebra is not a subalgebra of (2.11), due to the presence of the direct factor $O(3,1)$ in (3.1).

Physically, this choice is motivated by the following reasoning. In the previous Section we arrived at the algebra (2.11) after having separated off a spin part of the generators of the TGA (2.5). Although such a separation is certainly a natural one from the algebraic point of view, it appears to be highly artificial from the point of view of the dynamics. We know experimentally that massless pseudostates in general correspond to infinite dimensional representations of the conspiracy algebra (2.5), since the leading trajectories in a Lorentz family do not pass through integer (resp. half integer) values of the angular momentum at $W = 0$. Also, in the framework of the quark model, one can see very easily that the spin part of (2.5) generates finite dimensional representations only, which cannot be interpolated analytically to give a Regge trajectory. [This follows immediately from the meaning of the operators $F_a^{(\pm)}$: they act on quark spins, i.e., on representations of $SL(2,C)$ of the type $(j,0)$ and $(0,j)$ respectively.]

The impossibility of an analytic interpolation of the quark-spin wave functions can be seen as follows.

One recalls ²⁵⁾ that an IR of $SL(2, \mathbb{C})$ is characterized by a pair of complex numbers, (j_1, j_2) , such that $2(j_1 - j_2) = \text{real integer}$.

The representation is finite dimensional if both j_1 and j_2 are half-integers. It follows that a representation of the type $(j, 0)$ or $(0, j)$ is always finite dimensional.

In order to produce infinite dimensional representations (which we need physically), we have to invoke the orbital momentum of the composite state. In fact, the orbital part, $\ell_{\mu 2r}$ generates self-conjugate representations of $SL(2, \mathbb{C})$ [or of $O(3, 1)$] of the form (n, n) , which can be interpolated analytically. [They continue to be representations of $O(3, 1)$ for any complex n .] Hence the physically meaningful representations of the algebra must be those which can be projected out from the direct products

$$(n, n) \times (j, 0) \quad \text{and} \quad (n, n) \times (0, j) \quad (3.2)$$

Let us remark immediately that in general
 - independently of any physical interpretation -
 every IR of $SL(2, \mathbb{C})$ can be obtained - in
 several ways - by projecting it out from the
 direct product of a self-conjugate IR with a
 finite dimensional one. The quark model only
 makes this procedure unique by dictating the
 finite dimensional IR one has to choose.

Bearing these considerations in mind, we shall try and saturate the CR (2.11) with states which have the same quantum numbers as the $(q\bar{q})$ and (qqq) states (for mesons and baryons respectively) with the total energy-momentum of the system being set equal to zero. The last condition is motivated by the circumstance that we want to construct states which form representations of the TGA (2.5).

In this paper we shall concentrate upon baryonic representations, i.e., on the (qqq) system. Because of the special symmetry properties a system enjoys at vanishing energy momentum, we devote the next Section to their study.

4. - SYMMETRY PROPERTIES OF THE (qqq) SYSTEM AT VANISHING ENERGY MOMENTUM

We anticipate the result from the next Section that the orbital dependence of the matrix elements of the $F^{(\pm)}$ can be entirely factored out and hence the saturation problem of the CR (2.11) reduces to that of the algebra of spin generators in terms of the quark spin and unitary spin indices. This problem is familiar from the theory of static SU(6) and one knows that a solution is provided by a spin-unitary spin configuration, totally symmetric in the SU(6) indices. On terms of the SU(2) $\times$ SU(3) contents, this representation is characterized by $(4,10) \oplus (2,8)$ when we label the representations by their dimensionalities. Assuming that the quarks are fermions, the next task is to construct totally antisymmetric spatial wave functions. Let the four-momenta of the quarks be p_1, p_2, p_3 respectively ; introduce two other sets of momenta by the definitions :

$$\left. \begin{aligned} E &= p_1 + p_2 + p_3, \\ k &= 6^{-\frac{1}{2}} (p_1 + p_2 - p_3), \\ q &= 2^{-\frac{1}{2}} (p_1 - p_2), \end{aligned} \right\} \quad (4.1)$$

$$\left. \begin{aligned} &(\text{Set } E=0) \\ \tau_1 &= 2^{-\frac{1}{2}} (p_2 - p_3), \\ \tau_2 &= 2^{-\frac{1}{2}} (p_3 - p_1), \\ \tau_3 &= 2^{-\frac{1}{2}} (p_1 - p_2). \end{aligned} \right\} \quad (4.2)$$

($\tau_1 + \tau_2 + \tau_3 = 0$)

The set (4.1) is convenient because it exhibits the kinematic properties explicitly, while the set (4.2) reflects symmetry properties. Since $E = 0$, one should construct spatial wave functions which transform according to irreducible (in general infinite dimensional) representations of $O(3,1)$. Since $E = 0$, the vectors in each set can be expressed by

those in the other. We shall use the set (4.1) to check the consistency of the expressions obtained by the use of vectors from (4.2), which transform simply under the symmetric group, S_3 . We have explicitly :

$$\begin{aligned} \tau_1 &= \frac{1}{2}(\sqrt{3}k - q), \\ \tau_2 &= -\frac{1}{2}(\sqrt{3}k + q), \\ \tau_3 &= q. \end{aligned} \quad (4.3)$$

Since we are not interested in functions of mixed symmetry, our treatment simplifies considerably.

Construct two vectors, one totally symmetric, the other totally antisymmetric under S_3 . Choose :

$$\begin{aligned} S_\mu &= (p_2 \cdot p_3) p_{1\mu} + (p_1 \cdot p_3) p_{2\mu} + (p_1 \cdot p_2) p_{3\mu} \\ a_\mu &= (\tau_2 \cdot \tau_3) \tau_{1\mu} + (\tau_1 \cdot \tau_3) \tau_{2\mu} + (\tau_1 \cdot \tau_2) \tau_{3\mu} \end{aligned}$$

Expressing them through k_μ and q_μ we have :

$$\begin{aligned} S_\mu &= \frac{1}{\sqrt{6}} \left\{ 2(k \cdot q) q_\mu + (q^2 - k^2) k_\mu \right\}, \\ a_\mu &= \frac{3}{4} \left\{ (q^2 - k^2) q_\mu - 2(q \cdot k) k_\mu \right\}. \end{aligned} \quad (4.4)$$

The scalar product $A = (s \cdot a)$ is totally antisymmetrical and not identically zero, since using (4.4) we find

$$A = \frac{3}{4\sqrt{6}} (k \cdot q) \left\{ 3(q^2 - k^2)^2 - 4(k \cdot q)^2 \right\}. \quad (4.5)$$

5. - CONSTRUCTION OF THE BARYON REPRESENTATION

We construct the baryon representation in the following steps :

- a) We show that for zero mass the commutators can be saturated by taking irreducible representations of the Lorentz group.
- b) We factor out the orbital dependence of the matrix elements of $F_{\alpha}^{(\pm)}$ (where α stands for the pair of indices aA).
- c) We show that the orbital dependence can be factored out of the commutators; thereby reducing the problem to finding finite dimensional representations of the algebra $SU(6)$.
- d) Using these results, we give a full classification of the massless baryon states which saturate the supermultiplet algebra.

We now go into these points in some detail.

- a) We want to obtain a representation of the algebra in terms of single-"particle" states. Since we shall work with massless composite particles, we leave their spin structure open for the moment and insist only on the states having one definite mass, W (which we shall let go to zero afterwards). Also it will be sufficient to take diagonal matrix elements of commutators of the type :

$$[F_{\alpha}^{(\pm)}, F_{\beta}^{(\pm)}] = C_{\alpha\beta\gamma} F_{\gamma}^{(\pm)} \quad (2.11')$$

($C_{\alpha\beta\gamma}$ stand for the structure constants), since every other matrix element can then be obtained by the use of hermiticity properties of the operators. Let then $|E_{\mu}, s, \mu; X\rangle$ be a state with - timelike - total momentum E_{μ} , total spin and its projection being equal to s, μ , respectively; the rest of the quantum numbers is denoted by X . The mass of the state is $W = \sqrt{-E_{\mu} E_{\mu}}$. We have to study then equations which symbolically can be written as follows :

$$\sum_{\chi, \chi'} \left\{ \text{Tr} \left(\Omega_{\chi}(E) F_{\alpha}^{(\pm)} \Omega_{\chi'}(E) F_{\beta}^{(\pm)} \right) - \text{Tr} \left(\Omega_{\chi}(E) F_{\beta}^{(\pm)} \Omega_{\chi'}(E) F_{\alpha}^{(\pm)} \right) \right\} = \quad (5.1)$$

$$= C_{\alpha\beta\gamma} \sum_{\chi} \text{Tr} \left(\Omega_{\chi}(E) F_{\gamma}^{(\pm)} \right).$$

In the last equation we introduced the formal operator :

$$\Omega_{\chi}(E) = \sum_{s, \mu} |E_{\mu}, s, \mu; \chi\rangle \langle E_{\mu}, s, \mu; \chi|. \quad (5.2)$$

We again call the reader's attention to the fact that in Eq. (5.2) we summed over all spins: since we do not know beforehand that the algebra can be saturated by only one (or a few) spin values at $W=0$: as a matter of fact, previous work on "conspiracy theory" suggests rather that we shall need an infinite power of spins. In more physical terms, we use the set of vectors $|E_{\mu}, s, m; \chi\rangle$ since it furnishes a convenient complete set of states. Were we to leave W finite, we could expect just one such vector to represent a single particle state (since it spans an IR of the Poincaré group for timelike momenta). However, as W goes to zero, the structure of the little group (and hence the notion of spin) becomes different.

We now demonstrate, by a formal calculation ²⁷⁾, that in the limit of vanishing total mass the operator Ω_{χ} becomes diagonal in the quantum numbers characterizing representations of the homogeneous Lorentz group [or rather its covering groups $SL(2, C)$]. First of all we expand the operator $\Omega(E)$ in terms of basis states of $SL(2, C)$. Omitting the labels χ for the time being, we have :

$$\Omega(E) = \sum_{j_0, j'_0} \iint_{-\infty}^{\infty} d\tau (\tau^2 + j_0^2) d\tau' (\tau'^2 + j_0'^2) \sum_{j, m, j', m'} |\tau, j_0, j, m\rangle \langle \tau, j_0, j'_0 | \langle \tau', j'_0, j', m' |,$$

where the kernel K is given by the following expression :

$$K_{jm,j'm'}^{(\tau_{j_0}, \tau'_{j'_0})}(E) = \sum_{s\mu} \langle \tau_{j_0} j m | E_\mu s \mu \rangle \langle E_\mu s \mu | \tau'_{j'_0} j' m' \rangle. \quad (5.3)$$

[Again, in the spirit of our cavalier treatment of analytic properties, we assumed that the integrations can be performed over the principal series only, and wrote $\sigma = -1 + i\tau$.]

Thus our objective is to show that K becomes diagonal in the limit $W \rightarrow 0$.

Now the bracket $\langle \tau_{j_0} j m | E_\mu s \mu \rangle$ is equal to a solid harmonic, suitably normalized :

$$\langle \tau_{j_0} j m | E_\mu s \mu \rangle = \frac{i W^{i\tau}}{\sqrt{8\pi(\tau^2 + j_0^2)^7}} \mathcal{D}_{jm, s\mu}^{(\tau_{j_0})}(L(E)), \quad (5.4)$$

where $L(E)$ stands for a boost which produces the vector E_μ from the corresponding vector in the rest frame, i.e., $(0, W)$; it will be convenient to choose it according to the helicity convention. $\mathcal{D}_{jm, j'm'}^{(\tau, j_0)}(\Lambda)$ is the matrix element of the Lorentz transformation Λ in the IR (τ, j_0) , as defined, e.g., by Sciarrino and Toller²⁸⁾ (Section 2, and Appendix A). On parametrizing the vector E_μ in the usual way, i.e.,

$$\begin{aligned} E_0 &= W \cosh \beta, \\ E_1 &= W \sinh \beta \sin \Theta \cos \Phi, \\ E_2 &= W \sinh \beta \sin \Theta \sin \Phi, \\ E_3 &= W \sinh \beta \cos \Theta, \end{aligned} \quad (5.5)$$

and using the helicity convention for $L(E)$, we can write :

$$\mathcal{D}_{jm, s\mu}^{(\tau, j_0)}(L(E)) = \mathcal{D}_{m\mu}^j(R(\Omega)) Z_{js; \mu}^{(\tau, j_0)}(\beta),$$

where $R(\Omega)$ is the rotation through Euler angles $(\phi, \theta, 0)$, $Z_{js; \mu}^{(\tau, j_0)}(\beta)$ stands for the matrix element of the boost along the 3rd axis.

Using the unitarity of the matrix elements (5.4), we can rewrite the kernel K as follows :

$$K_{jm, j'm'}^{(\tau, j_0, \tau', j_0')}(E) = \frac{W^{-2+L(\tau-\tau')}}{\sqrt{(\tau^2+j_0^2)(\tau'^2+j_0'^2)}} \times \quad (5.6)$$

$$\frac{1}{4\pi} \sum_{s\mu} \mathcal{D}_{m\mu}^j(R(\Omega)) Z_{js; \mu}^{(\tau, j_0)}(\beta) Z_{s'j'; \mu}^{(\tau', j_0')}(-\beta) \mathcal{D}_{\mu m'}^{j'}(R(\Omega)^{-1}).$$

Taking into account the explicit expression of the matrix elements of boosts given by CST (loc.cit), one easily finds :

$$(4\pi)^{-\frac{1}{2}} \mathcal{D}_{m\mu}^j(R(\Omega)) Z_{js; \mu}^{(\tau, j_0)}(\beta) = \left[\frac{(2j+1)(2s+1)}{4\pi} \right]^{\frac{1}{2}} \times \quad (5.7)$$

$$\times \int d\tau(\omega) \mathcal{D}_{m j_0}^j(R(\Omega) \tau(\omega)^{-1}) \exp[(1-i\tau)\beta] \left[\frac{\cos \frac{\theta}{2}}{\cos \frac{\theta'}{2}} \right]^{2(1-i\tau)} \mathcal{D}_{j_0 \mu}^s(\tau(\omega)^{-1})$$

and a similar expression for the second inverse boost.

The rotation $r(\omega)$, used as an integration variable, is characterized by the Euler angles $(\varphi, \vartheta, 0)$, whereas ω' stands for $(\varphi, \vartheta', 0)$, with

$$\operatorname{tg} \frac{\vartheta'}{2} = \exp \beta \operatorname{tg} \frac{\vartheta}{2}.$$

Inserting (5.7) into (5.6) and using the completeness relation :

$$\sum_{s\mu} (2s+1) \mathcal{D}_{j_0\mu}^s(+(\omega')) \mathcal{D}_{\mu j_0}^s(+(\omega)^{-1}) = \delta_{j_0 j_0} \delta(\omega' - \omega),$$

we can rewrite the kernel K as follows :

$$K_{jm, j'm'}^{(\tau, \tau')}(E) = \frac{W^{i(\tau - \tau')}}{\sqrt{(\tau^2 + j_0^2)(\tau'^2 + j_0'^2)}} \delta_{j_0 j_0'} F_{jm, j'm'}^{j_0}(E, \tau - \tau'), \quad (5.8)$$

where

$$F_{jm, j'm'}^{j_0}(E, \tau - \tau') = \frac{-1}{8\pi} \sqrt{(2j+1)(2j'+1)} e^{i\beta(\tau' - \tau)} \times \int d\omega \mathcal{D}_{mj_0}^j(R(\Omega) + (\omega)^{-1}) \left[\frac{\cos \frac{\vartheta}{2}}{\cos \frac{\vartheta'}{2}} \right]^{i(\tau - \tau')} \mathcal{D}_{j_0 m'}^{j'}(+(\omega) R(\Omega)^{-1}). \quad (5.9)$$

Notice that

$$F_{jm, j'm'}^{j_0}(E, 0) = -\frac{1}{2} \delta_{jj'} \delta_{mm'}. \quad (5.10)$$

We now insert Eqs. (5.8) and (5.9) into (5.2) and find :

$$\Omega(E) = \sum_{j_0} \int_{-\infty}^{\infty} d\tau d\tau' \sqrt{(\tau^2 + j_0^2)(\tau'^2 + j_0^2)} e^{i(\tau - \tau')\zeta} \quad (5.11)$$

$$\sum_{j'm, j'm'} |\tau j_0 j m\rangle F_{j'm, j'm'}^{j_0}(E, \tau - \tau') \langle \tau' j_0 j' m' |$$

where $\zeta = -\log W$.

Noticing that the operator :

$$A(\tau, \tau') = \sum_{j_0} \sqrt{(\tau^2 + j_0^2)(\tau'^2 + j_0^2)} \times \sum_{j'm, j'm'} |\tau j_0 j m\rangle F_{j'm, j'm'}^{j_0}(E, \tau - \tau') \langle \tau' j_0 j' m' | \quad (5.12)$$

enjoys the property :

$$A(\tau', \tau) = A(\tau, \tau')^+$$

we can estimate the asymptotic behaviour of the double Fourier transform in (5.11) for $\zeta \rightarrow \infty$. Assuming sufficient convergence at infinity, after some simple calculations, one finds :

$$\Omega(E) = \int d\tau A(\tau, \tau) + o(\zeta^{-1}) \quad (\zeta \rightarrow \infty)$$

or, taking into account (5.11) and (5.12), we get the asymptotic expression :

$$\Omega(E) \sim \sum_{j_0 j m} \int d\tau (\tau^2 + j_0^2) |\tau j_0 j m\rangle \langle \tau j_0 j m| \quad (5.13)$$

(W → 0)

which is the result we wanted to demonstrate.

The preceding calculation teaches us that if we assume that the matrix elements of observables have sufficiently good analyticity properties, the limit $W=0$ is independent of the direction of E_μ in Minkowski space, [the polar angles β, θ, ϕ dropped out of the leading term in (5.13) !] and is equivalent to putting E_μ equal to a null vector, cf., (5.5). This is to be compared with the inverse result quoted at the beginning of Section 2.

Bearing these results in mind, we are now able to define more precisely what we should call a single composite particle state at $W=0$. (Composite, in order to have sufficiently smooth matrix elements !)

We shall assume that the states $|A\rangle$ represent a single, massless composite particle if :

- i) the matrix elements of observables - in particular, those of the $F_\alpha^{(\pm)}$ - taken between the states $|A\rangle$ are smooth functions of W so that the asymptotic estimate given above holds, and
- ii) at $W=0$ the set of states $|A\rangle$ spans an IR of the algebra $SL(2, \mathbb{C})$.

The IR of $SL(2, \mathbb{C})$ spanned by these states is, in general, a non-unitary, infinite dimensional one, as one learns it from scattering experiments where massless, composite states are exchanged. Hence, re-writing Eq. (5.1) with the help of (5.13)

$$\begin{aligned}
 0 = & \sum_{j_0 j'_0} \sum_{j m j' m'} \sum_{\chi \chi'} \int d\tau (\tau^2 + j_0^2) d\tau' (\tau'^2 + j_0'^2) \\
 & \times \left\{ \langle \tau_{j_0} j m \chi | F_\alpha^{(\pm)} | \tau'_{j'_0} j' m' \chi' \rangle \langle \tau'_{j'_0} j' m' \chi' | F_\beta^{(\pm)} | \tau_{j_0} j m \chi \rangle - \right. \\
 & \left. - (\alpha \neq \beta) - \delta_{\alpha\alpha'} \frac{\delta(\tau - \tau')}{\tau^2 + j_0^2} \delta_{j_0 j'_0} \delta_{j j'} \delta_{m m'} C_{\alpha\beta\gamma} \langle \tau_{j_0} j m \chi | F_\gamma^{(\pm)} | \tau_{j_0} j m \chi \rangle \right\}, \\
 & (W=0)
 \end{aligned} \tag{5.14}$$

and allowing for a deformation of the integration path into non-unitary representations, we shall say that we find a single-particle representation of (5.14), if the right-hand side can be made to vanish for a finite set of values of τ and j_0 .

b) We proceed by trying to find a single-particle representation of the algebra (2.11) in the framework of the orbital excitation model discussed in the previous Section. To this end, we observe that the orbital dependence can be factored out from the matrix elements - taken between eigenstates of the total angular momentum - of such operators which act on the spin variables only. Indeed, one can decompose a state $|jm\rangle$ as follows :

$$|jm\rangle = \sum_{st, nr} (nr, st | jm) |nr, st\rangle, \quad (5.15)$$

where nr and st are the orbital angular momentum, its projection and the spin and its projection, respectively. Suppose A_M^S is an irreducible tensor of $(2S+1)$, components acting on the spin variables only; thus, using the Wigner-Eckart theorem, we can write

$$\langle j'm' | A_M^S | jm \rangle = (jm, SM | j'm') \langle j' || A^S || j \rangle. \quad (5.16)$$

Further, using (5.15), the orthogonality of the orbital states and again the Wigner-Eckart theorem, one finds :

$$\begin{aligned} \langle j'm' | A_M^S | jm \rangle &= \\ &= \sum (nr, st | jm) (nr, st' | j'm') (st, SM | st') \langle s' || A^S || s \rangle. \end{aligned} \quad (5.17)$$

Comparing (5.16) and (5.17), we find, after a little algebra :

$$\langle j' \| A^S \| j \rangle = (-1)^{-(n+s+S+j')} \sqrt{(2j+1)(2s'+1)} \begin{Bmatrix} n & s & j \\ S & j' & s' \end{Bmatrix} \langle s' \| A^S \| s \rangle \quad (5.18)$$

thus factoring out the dependence on n into the 6- j symbol.

c) Next we factor out the orbital dependence from the product of two operators. Once we achieve that, the factorization of the orbital dependence from the commutator is trivial. Consider the matrix element of the product of the irreducible tensor operators, $A_M^S, A_{M'}^{S'}$, acting on the spin variables only. Using the same technique as in the previous paragraph, we find after a straightforward calculation :

$$\begin{aligned} \sum_{j''} \langle j' m' | A_{M'}^{S'} | j'' m'' \rangle \langle j'' m'' | A_M^S | j m \rangle = \\ = \sum_{j'' \mp} (-1)^{-(j+S+S'+j'')} \sqrt{(2j'+1)(2\mp+1)} \begin{Bmatrix} j & S & j' \\ S' & j'' & \mp \end{Bmatrix} \quad (5.19) \\ \times \langle j' \| A^{\mp} \| j \rangle (S M, S' M' | \mp, M+M') (j m, \mp, M+M' | j' m'). \end{aligned}$$

In writing down Eq. (5.19), we wrote the product of the operators A_M^S and $A_{M'}^{S'}$ in reduced form as

$$A_{M'}^{S'} A_M^S = \sum_{\mp} (S M, S' M' | \mp, M+M') A_{M+M'}^{\mp} \quad (5.20)$$

We now insert (5.18) into (5.19), use Elliott's formula ²⁹⁾ :

$$\begin{aligned}
& \sum_{j''} (-1)^{j''} (2j''+1) \begin{Bmatrix} j j'' S \\ S' \mp j' \end{Bmatrix} \begin{Bmatrix} n j'' s'' \\ S s j \end{Bmatrix} \begin{Bmatrix} j' j'' S' \\ s'' s' n \end{Bmatrix} \\
& = (-1)^{-A} \begin{Bmatrix} n s j \\ \mp j' s' \end{Bmatrix} \begin{Bmatrix} S' \mp s \\ S s' S' \end{Bmatrix}
\end{aligned} \tag{5.21}$$

where $A = s+s''+s'+S+J+S'+j+n+j'$, to give us finally :

$$\begin{aligned}
& \sum_{j''} \langle j' m' | A_{M'}^{S'} | j'' m'' \rangle \langle j'' m'' | A_M^S | j m \rangle = \\
& = \sum_{s, t} \sum_{\mp s''} (-1)^{-(s+s''+S+S')} \sqrt{(2s'+1)(2\mp+1)} \begin{Bmatrix} s S s' \\ S' s'' \mp \end{Bmatrix} \\
& \times (SM, S'M' | \mp M+M') (s, t; \mp M+M' | s', t') \langle s'' || A^{S'} || s' \rangle \langle s'' || A^S || s \rangle.
\end{aligned} \tag{5.22}$$

On the right-hand side of Eq. (5.22) the first summation is extended over all spin variables, subject to conservation laws. The right-hand side is expressed entirely in terms of spin variables. The first Clebsch-Gordan coefficient (after proper symmetrization or antisymmetrization) is just equal to the $SU(2)$ factor in the corresponding structure coefficient of the algebra (2.11) written out in spherical components, while the second Clebsch-Gordan coefficient is the one appearing on the right-hand side of a commutator when sandwiched between two states.

Thus this formalism allows us to write out the commutators entirely in terms of reduced matrix elements taken between spin states only.

The generalization from $SU(2)$ to $SL(2, \mathbb{C})$ is immediate. First of all we observe that the angular momentum algebra as it was used here can be generalized to "fake" representations of $SU(2)$; i.e., to those where the angular momenta j, n , etc., are not half integers ³⁰⁾. Taking this for granted, one can decompose the states spanning an IR of $SL(2, \mathbb{C})$ in the framework of the orbital excitation model as follows :

$$\begin{aligned}
 |(j_1 j_2) j m\rangle &= \sum_{m_1 m_2} (j_1 m_1, j_2 m_2 | j m) | j_1 m_1, j_2 m_2 \rangle \\
 &= \sum_{\substack{m_1 m_2 \\ n_1 n_2 t_1 t_2 s_1 t_1}} (j_1 m_1, j_2 m_2 | j m) (n_1 t_1, s_1 t_1 | j_1 m_1) \delta_{n_2 j_2} \delta_{t_2 m_2} \quad (5.23) \\
 &\quad \times |(n_1 t_1, s_1 t_1), (n_2 t_2, 00) \rangle.
 \end{aligned}$$

We wrote down the decomposition into orbital and spin states, assuming that the spin states transform according to the finite dimensional representation $(s_1, 0)$ of $SL(2, \mathbb{C})$. The representation on which act the operators $F^{(+)}_{-}$ The orbital momentum is assumed to transform according to the - generally infinite dimensional - IR (n_1, n_2) . (Not necessarily self-conjugate, cf., Section 4 !). When an operator - say an $F^{(+)}$ - is sandwiched between two states of the type (5.23), it acts on the variables s, t only, whereas the "dotted" variables, $n_2 r_2$, drop out by orthogonality. The previously developed technique can then be used to factor out the orbital dependence from the "undotted" variables $(n_1 r_1, s_1 t_1)$. Evidently, analogous relations hold for the operators $F^{(-)}$ acting on a state with a dotted spin variable, whereas, say, an $F^{(-)}$ acts on a state with an undotted spin variable as the zero operator in the spin variables (and vice versa). It follows that the orbital dependence can be factored out from the algebra (2.11) and the problem of finding its representations reduces to finding finite dimensional representations of the chiral algebra $SU(6) \times SU(6)$.

The problem is reduced further if one observes that the algebras spanned by the $F^{(+)}$ and $F^{(-)}$ are conjugate [in the sense of $SL(2, \mathbb{C})$] to each other, further that in order to find a set of states spanning an IR of the (covering group of the) extended Lorentz group, one must find two conjugate representations of $SL(2, \mathbb{C})$. Hence, it is sufficient to find finite dimensional representations of, say, $SU(6)_{\text{LEFT}}$ and add to it its conjugate representation.

d) We are now able to enumerate the massless baryon states which saturate the CR (2.11). Since the operators $F^{(\pm)}$ act as the unit operator on orbital states, the orbital momentum (n_1, n_2) and the orbital signature, $(-1)^{(n_1-n_2)} = (-1)^{\ell}$ are conserved. [It is not excluded that in the real world there is configuration mixing even at zero total energy momentum; however, since it is not dictated by the algebra (2.11), we try out a model without spin orbit coupling first.]

The solution of the $SU(6)$ problem is well known ³¹⁾. Using the symmetric quark model, we assign the baryons to the IR (56), with $SU(2) \times SU(3)$ contents $(\underline{2}; \underline{8}) + (\underline{4}; \underline{10})$.

The representations of $SU(2)$ contained in 56 span finite dimensional IR's of $SL(2, \mathbb{C})$, $(\frac{1}{2}, 0)$ and $(\frac{3}{2}, 0)$, respectively. We now combine these spin representations with the orbital momentum, (n_1, n_2) ; the resulting $SL(2, \mathbb{C}) \times SU(3)$ contents of the state will be denoted by $(j_1, j_2; N)$, where N is the dimensionality of the IR of $SU(3)$. There are infinitely many branches of the spectrum, corresponding to $\ell_0 = 0, 1, 2, \dots$ and with alternating orbital signature, each of them generating a distinct representation of the algebra. [Since the $F_{\alpha}^{(\pm)}$ conserve orbital momentum, we can consider one branch at a time, there being no transition between orbital branches.]

We tentatively identify the known baryon states with the representation generated by the lowest branch, $\ell_0 = 0$.

Writing then for the orbital momentum $\sim(n,n)$, the physical points correspond to $n = 0, 1, 2, 3, \dots$.

The possible states contained in this branch are listed in Table II. Notice that the parameter n is not fixed by the algebra itself, so, in principle, we have a continuous family of representations parametrized by the intercept of one of the leading Regge trajectories (i.e., by the intercept of one of the Lorentz trajectories). The fact that the elements of the algebra carry at most spin one, restricts further the number of states available for saturation. Sandwiching, for example, the algebra spanned by the $F^{(+)}$ between the states $(n+\frac{1}{2}, n; 8)$, the possible intermediate states are those marked with an asterisk in Table II.

To summarize the result, we found that the algebra (2.11) can be saturated by the tribe, consisting of five Lorentz states, two octets and three decuplets. In order to identify these (or at least some of them) with known baryon Lorentz families, we invoke the assumption about the existence of a smooth CST limit. Looking at Table II, we find that the most reasonable assumption is to identify the highest lying octet and decuplet states with the physical nucleon and delta families, respectively. Indeed, if we make this identification, we find that the physical states on the leading trajectory of the octet family have spins $1/2, 5/2, 9/2, \dots$, whereas the leading trajectory of the decuplet family gives rise to particles of spin $3/2, 7/2, 11/2, \dots$. There exist particles of both natural parities, because of the presence of two chirality states at $W=0$. In the limit when symmetry breaking is turned off completely, the intercept of the delta trajectory at $W=0$ lies one unit higher than the nucleon. The existence of the low-lying octet and decuplet trajectories (states Nos 2 and 5 in Table II) is not very disturbing at the moment, since in the CST limit they would produce physical particles of rather high masses only (and so far off the point $W=0$ it is hard to trust the no-mixing and no-symmetry breaking approximation anyway). What is disturbing at first sight, is the existence of the state No. 4, which in the CST limit would give rise to a physical decuplet of spin one-half, at the mass of the nucleon. We shall add further comments on this state in a moment.

6. - SYMMETRY BREAKING MECHANISM

The TGA we are describing here can be broken at least by three different mechanisms.

a) At $W=0$ one can break the structure $O(3,1) \times SU(6)_L \times SU(6)_R$ by introducing spin orbit coupling terms. Such a term would be of the form $\sim \sigma_{\mu\nu} \ell_{\mu\nu}$ at the quark level. It couples the orbital group $O(3,1)$ with the generators $F_a^{(\pm)}$ of the spin group, but maintains the over-all classification according to $SL(2,C)$ of the states at $W=0$. We shall not study terms of this type in the present paper, but continue to assume that there exists an - approximate - spin independence of the dynamics.

b) The internal symmetry $SU(3)$ itself is not exact, but broken by medium strong interactions even at $W=0$. In the framework of the large algebra we are considering, various assumptions can be made about the nature of the $SU(3)$ breaking term. We shall make the assumption that $SU(3)$ is broken by an operator which is independent of the orbital state and transforms as the $S=I=Y=0$ octet member of the $(\underline{35}+\underline{405}+\underline{2695}) + (\underline{35}+\underline{405}+\underline{2695})$ of $SU(6)_L \times SU(6)_R$ reducible representation.

[In the framework of $SU(6)_L \times SU(6)_R$ it is necessary to consider at least the $\underline{405}$ contained in $\underline{56} \times \underline{56}$, since otherwise D terms would be absent.] The mixing angles fix essentially the F/D ratio.

c) Finally, as one goes away from $W=0$, the $SL(2,C)$ classification is broken by the energy momentum vector E_μ .

At $W \neq 0$ the contribution of the symmetry breaking terms b) and c) to the Regge trajectories is not additive, as one might perhaps expect, but there are mixed terms which cause differences between the slopes of Regge trajectories even within the same $SU(3)$ multiplet.

The general theory of treating symmetry breaking terms in Regge trajectories was described in Ref. 8), and we refer the reader to that paper for details. In treating the present problem, one should modify in principle that theory, since we have now degenerate levels (states Nos 2 and 5 in Table II).

However, it is easy to see that symmetry breaking terms of the type a) or b) alone cannot mix those states. Indeed, so far as there is no spin orbit coupling, s is a good quantum number, so there are no transitions caused between Nos 2 and 5 by $SU(3)$ breaking alone. A term of type a) in turn does not cause mixing because $SU(3)$ forbids it. Hence mixing would take place only if both terms (or a term of the form $\sim \lambda_8 \sigma_{\mu\nu} \ell_{\mu\nu}$) would be present. As we stated already, this problem goes beyond the scope of the present paper. Hence we can treat a non-degenerate problem here.

Our aim is to derive a "mass formula" for the trajectories which were identified with the known octet and decuplet family. As in Ref. 8), we assume that the Regge trajectories, α , are solutions of a non-linear eigenvalue problem

$$H(\alpha(W), W) |\varphi\rangle = 0 \quad (6.1)$$

We write H as the sum of $SU(3)$ singlet and octet terms : $H = H_1 + \lambda H_8$, and, correspondingly, $|\varphi\rangle = |\varphi_1\rangle + \lambda |\varphi_8\rangle$, $\alpha = \alpha_1 + \lambda \alpha_8$. We shall assume that all but the singlet and octet terms are suppressed by the dynamics, hence Eq. (6.1) is split into singlet and octet parts as follows

$$H_1(\alpha_1, W) |\varphi_1\rangle = 0, \quad (6.2a)$$

$$H_1(\alpha_1, W) |\varphi_8\rangle + \left(\alpha_8 \frac{\partial H_1(\alpha_1, W)}{\partial \alpha} + H_8 \right) |\varphi_1\rangle = 0. \quad (6.2b)$$

Equation (6.2a) is of the same structure as those treated in Ref. 8), hence we can use the results of that calculation. In order to calculate the dependence of the reduced matrix elements on the intercept, $\bar{\sigma}$, and j_0 , one has to remember that the total energy momentum vector, E_μ can couple both to the orbital and spin parts, the coupling terms being proportional to $E_\mu p_{\mu L}$ and $\sum_i E_{\mu i} p_{\mu i}$, respectively,

In the first term the spin dependence, in the second one the orbital dependence can be factored out, using the technique developed in the previous Section. The value of the orbital momentum is the same for the whole tribe; expressed in terms of the singlet part of the nucleon intercept, $\bar{\sigma}$, we have $n = 2^{-1}(\bar{\sigma} - \frac{1}{2})$. Combining all these results and using the expressions of the Clebsch-Gordan coefficients as calculated in Ref. 8), one gets the singlet parts of the known octet and decuplet trajectories :

$$\alpha_{1\kappa}^{(8)}(W) = \bar{\sigma} - \kappa + \frac{1}{8} A_{11} (\bar{\sigma} - \kappa + \frac{1}{2}) W + \frac{1}{8} [A_{01} + A_{11}^2 (\bar{\sigma} - \kappa + \frac{1}{2}) + A_{21} F_1(\bar{\sigma}) G_{\kappa}(\bar{\sigma}, \frac{1}{2})] W^2 \quad (6.3)$$

$$\alpha_{1\kappa}^{(10)}(W) = \bar{\sigma} + 1 - \kappa + \frac{1}{10} [A_{01} + A_{21} F_3(\bar{\sigma}) G_{\kappa}(\bar{\sigma} + 1, \frac{3}{2})] W^2 \quad (6.4)$$

where the functions F_{2s} , $G_{\kappa}(\bar{\sigma}, j_0)$ are defined as follows (s is the spin of the state, see Table II):

$$F_1(\bar{\sigma}) = \left[\frac{(\bar{\sigma} - \frac{1}{2})(\bar{\sigma} - \frac{3}{2})}{(\bar{\sigma} + \frac{1}{2})(\bar{\sigma} + \frac{3}{2})} \right]^{\frac{1}{2}},$$

$$F_3(\bar{\sigma}) = \frac{2 - (\bar{\sigma} + \frac{3}{2})(\bar{\sigma} - \frac{3}{2})}{\bar{\sigma} + \frac{3}{2}} \left[\frac{(\bar{\sigma} + \frac{7}{2})}{(\bar{\sigma} - \frac{1}{2})(\bar{\sigma} + \frac{1}{2})(\bar{\sigma} + \frac{5}{2})} \right]^{\frac{1}{2}}$$

$$G_{\kappa}(\bar{\sigma}, j_0) = \frac{4(\bar{\sigma} - \kappa + \frac{1}{2})^2 - (\bar{\sigma} + j_0 + 1)^2 - (\bar{\sigma} - j_0 + 1)^2 + 1}{[(\bar{\sigma}^2 - j_0^2)((\bar{\sigma} + 1)^2 - j_0^2)]^{1/2}}.$$

In Eqs. (6.3) and (6.4) the subscript "1" on the left-hand side means "SU(3) singlet part of the trajectory", the superscripts "8" and "10" refer to the IR of SU(3) which the unperturbed states belong to. The subscripts of the reduced matrix elements on the right-hand side refer to their transformation property under $SL(2, \mathbb{C}) \times SU(3)$: A_{nN} is the reduced matrix element of an operator transforming as the $j=m=I=Y=0$ member of the IR $(n/2, n/2; N)$ of $SL(2, \mathbb{C}) \times SU(3)$. [The factors 8^{-1} and 10^{-1} , etc. are just Clebsch-Gordan coefficients of $SU(3) \times SU(3)$.]

We now calculate α_8 from (6.2b). Multiplication with $\langle \varphi_1 |$ gives, in view of (6.2a) :

$$\alpha_8 = - \frac{\langle \varphi_1 | H_8 | \varphi_1 \rangle}{\langle \varphi_1 | \frac{\partial H}{\partial \alpha}(\alpha_1 W) | \varphi_1 \rangle}.$$

The denominator is identically equal to unity because of the normalization condition imposed on Eq. (6.2a). [Cf., Eq. (2.10) of Ref. 8).] Hence

$$\alpha_8 = - \langle \varphi_1 | H_8 | \varphi_1 \rangle \quad (6.5)$$

Taking into account that H_8 is independent of the orbital state, we have

$$\alpha_8(0) = - \langle\langle H_8 \rangle\rangle \langle \varphi_1 | \varphi_1 \rangle,$$

where the double bracket refers to an expectation value with respect to $SU(6)_L \times SU(6)_R$. At the point $W=0$ the bracket $\langle \varphi_1 | \varphi_1 \rangle$ is proportional to $(2s+1)^{-1}$. This spin factor cannot be extracted explicitly from the derivatives with respect to W , since E_μ is coupled both to the spin and orbital variables.

Thus the energy dependence of α_8 , the octet part of the trajectory, is entirely given by the scalar product $\langle \varphi_1 | \varphi_1 \rangle$.

As shown in Ref. 8), when the trajectory is calculated in the parabolic approximation, the state vector has to be calculated up to terms $O(W)$. The first total derivative was shown to be expressible through the derivatives of the trajectory :

$$\left[\frac{d}{dW} \langle \varphi_1 | \varphi_1 \rangle \right]_{W=0} = \frac{1}{4} \left[\frac{\partial \alpha''}{\partial \alpha'} \right]_{W=0}, \quad (6.6)$$

[see Eqs. (2.13) and (2.17) of Ref. 8]. Using Eqs. (6.3) and (6.4), we get :

$$\left[\frac{d}{dW} \langle \varphi_1 | \varphi_1 \rangle \right]_{W=0} = \begin{cases} \frac{1}{16} A_{11} & (\text{octet}), \\ 0 & (\text{decuplet}). \end{cases} \quad (6.7)$$

Similarly, the second order term is given by :

$$\begin{aligned} \left[\frac{d^2}{dW^2} \langle \varphi_1 | \varphi_1 \rangle \right]_{W=0} = \\ = \frac{1}{16} \left[\left(\frac{\partial \alpha''}{\partial \alpha'} \right)^2 \langle \varphi_1 | \varphi_1 \rangle \right]_{W=0} + \left[\langle \varphi_1 | \frac{dH}{dW} \frac{1}{P} \left(\frac{dH}{dW} \right)^2 \frac{dH}{dW} | \varphi_1 \rangle \right]_{W=0}. \end{aligned} \quad (6.8)$$

In the last equation, the projection operator, P , annihilates the state $|\varphi_1\rangle$, hence the second term in the right-hand side of (6.8) contains "inelastic" contributions only (i.e., from bound states different from $|\varphi_1\rangle$ and the continuum). In all the preceding equations the total derivative means differentiation along the characteristic surface of H :

$$\frac{d}{dW} = \frac{\partial}{\partial W} + \frac{d\alpha}{dW} \frac{\partial}{\partial \alpha}.$$

Finally, we calculate the expectation value $\langle\langle H_8 \rangle\rangle$. As indicated already, we assume that H_8 consists of three terms, transforming according to $\underline{35}+\underline{35}$, $\underline{405}+\underline{405}$ and $\underline{2695}+\underline{2695}$ under $SU(6)_L \times SU(6)_R$, respectively. We introduce the mixing angles γ, δ by the definition :

$$H_8 = (\cos\gamma H_8^{(35)} + \sin\gamma H_8^{(405)}) \sin\delta + H_8^{(2695)} \cos\delta.$$

Extracting a common scale factor, the expectation value of H_8 between the octet and decuplet parts is proportional to the expressions :

$$\begin{aligned} \langle\langle H_8 \rangle\rangle_8 &\div \frac{Y}{2} \left[\left(-\frac{1}{\sqrt{6}} \cos\gamma + \frac{1}{\sqrt{2}} \sin\gamma \right) \sin\delta + \frac{1}{\sqrt{3}} \cos\delta \right] + \\ &+ \frac{1}{5} \left[I(I+1) - \frac{Y^2}{4} - 1 \right] \left[-\sqrt{2} \sin\gamma \sin\delta + \sqrt{3} \cos\delta \right], \end{aligned} \quad (6.9)$$

$$\langle\langle H_8 \rangle\rangle_{10} \div -\frac{Y}{\sqrt{8}} \left[\left(\frac{\sqrt{5}}{6} \cos\gamma + \frac{1}{\sqrt{10}} \sin\gamma \right) \sin\delta + \frac{1}{\sqrt{15}} \cos\delta \right]. \quad (6.10)$$

The Clebsch-Gordan coefficients entering the last equations were taken from Cook and Murtaza ³²⁾ and Kuriyan et al. ³³⁾, respectively. The extraction of the reduced matrix elements from Eq. (6.8) proceeds along the same lines as described for the singlet part.

In this way, after a suitable redefinition of the constants, one arrives at the following expression of the octet part of the trajectory :

$$\begin{aligned} \alpha_{8\chi}^8(W) &= \sqrt{\frac{1}{28}} \lambda \langle\langle H_8 \rangle\rangle_8 \times \\ &\times \left\{ \frac{1}{2} + A_{18}W + [A_{08} + A_{28}F_1(6)G_\chi(6, \frac{1}{2})]W^2 \right\}, \end{aligned} \quad (6.11)$$

$$\alpha_{8\kappa}^{(10)}(W) = \sqrt{\frac{1}{14}} \lambda \ll H_8 \gg_{10} \times$$

$$\times \left\{ \frac{1}{4} + \left[A_{08} + A_{28} F_3(\sigma) G_{\kappa}(\sigma+1, \frac{3}{2}) \right] W^2 \right\}. \quad (6.12)$$

The complete expression for both trajectories is given by

$$\alpha_{\kappa}^{(N)}(W) = \alpha_{1\kappa}^{(N)}(W) + \alpha_{8\kappa}^{(N)}(W). \quad (6.13)$$

One can proceed in an analogous way in calculating trajectory formulae for other members of the tribe. The trajectory formulae, as written down in the previous equations, exhibit the maximal information which can be extracted from the knowledge of the algebraic structure of the theory. The details of the dynamics are "hidden" in the parameters σ , $A_{01}, \dots, A_{28}$. Confrontation of these trajectory formulae with experimental data is not easy, since the formulae are rather complicated. In the next Section we develop a crude approximation - valid for low-lying states only - which, however, can be easily checked.

7. - THE FLEA. MASS FORMULA

Let us observe that in the $SL(2,C)$ fits to baryons the numerical coefficients before $SL(2,C)$ breaking terms (A, B_2) are small compared to the "invariant" ones (B_1) both for the octet and decuplet trajectories (cf., Table I). This suggests to neglect $SL(2,C)$ breaking in a zeroth approximation altogether (CST limit). In the notation of this paper, the CST limit means $A_{iN} \approx 0$ for $i \neq 0$ in the trajectory formulae. Clearly, the CST limit may be a good approximation only if the coefficients of the A_{iN} ($i \neq 0$) are small, i.e., both W and χ are small. On the basis of the figures quoted in Table I, we estimate that the error we commit in the masses is $\lesssim 20\%$ for $\chi = 0$ and $|W| \lesssim 2 \text{ GeV}$.

One can go a step further and notice that in the CST limit A_{08} is determined by inelastic contributions only [the first term on the right-hand side of Eq. (6.8) vanishes on account of $A_{11} = 0$], so, hopefully, it is small and can be set equal to zero. Combining both approximations, one finds an extremely simple trajectory formula in the low excitation approximation (FLEA), viz.,

$$\alpha_0^8(W) = 6 + \frac{1}{\sqrt{28}} \frac{\lambda}{2} \ll H_8 \gg_8 + \frac{A_{01} W^2}{8}, \quad (7.1a)$$

$$\alpha_0^{10}(W) = 6 + 1 + \frac{1}{\sqrt{14}} \frac{\lambda}{4} \ll H_8 \gg_{10} + \frac{A_{01} W^2}{10} \quad (7.1b)$$

Equations (7.1) give for the masses of the first physical states ($j = \frac{1}{2}$ and $\frac{3}{2}$ for the octet and decuplet, respectively) :

$$M_8^2 = 8M_0^2 - \frac{4}{\sqrt{28}} \ll H_8 \gg_8 \overline{M}_1^2, \quad (7.2a)$$

$$M_{10}^2 = 10M_0^2 - \frac{5}{2\sqrt{14}} \ll H_8 \gg_{10} \overline{M}_1^2, \quad (7.2b)$$

with $M_0^2 = A_{01}^{-1}(\frac{1}{2} - \epsilon)$, $\bar{M}_1^2 = A_{01}^{-1} \lambda$. We now invoke as an empirical fact that the coefficients of Y are - to a very good approximation - equal in the octet and decuplet. This gives an equation between χ and δ :

$$\sqrt{2} \cos \delta = \frac{8 - 5\sqrt{5}}{8 + \sqrt{5}} \cos \chi - \sqrt{3} \sin \chi. \quad (7.3)$$

Inserting (6.9), (6.10) and (7.3) into (7.2), we arrive at the formulae :

$$M_8^2 = 8M_0^2 + M_1^2 \frac{\sqrt{30}}{8 + \sqrt{5}} \cos \chi \cdot Y + \\ + M_1^2 \sqrt{2} \left(\sin \chi + \frac{\sqrt{3}}{5} \frac{5\sqrt{5} - 8}{8 + \sqrt{5}} \cos \chi \right) \left[I(I+1) - \frac{Y^2}{4} - 1 \right], \quad (7.4a)$$

$$M_{10}^2 = 10M_0^2 + M_1^2 \frac{\sqrt{30}}{8 + \sqrt{5}} \cos \chi Y, \quad (7.4b)$$

where

$$M_1^2 = \bar{M}_1^2 \frac{\sin \delta}{\sqrt{7}}.$$

Equations (7.4) give an octet mass formula which fits reasonably well for baryons. Moreover, they allow to calculate the absolute masses of the members of the decuplet in terms of those of the octet. One indeed finds :

$$M_{10}^2 = \frac{5}{8} (\Lambda^2 + \Sigma^2) - \frac{1}{2} (\Xi^2 - N^2) Y \quad (7.5)$$

(particle symbols stand for the corresponding masses). The calculated values of the decuplet masses are found in Table III.

All the calculated masses are too low, but the error does not exceed 10%, as one indeed can expect from the estimate made at the beginning of this Section. It is worth repeating that this, extremely simplified form of the trajectory formulae does not exhaust the predictions of the

theory, but the rough comparison we made, already indicates its power. In particular, Eq. (7.5), unlike static SU(6) formulae, predicts absolute values of the masses, not just mass ratios. This is a consequence of the fact that if we rewrote Eq. (7.4) in the standard static SU(6) form, say,

$$M = M_0 + M_S S(S+1) + FY + D \left[I(I+1) - \frac{Y^2}{4} \right],$$

we would find that the ratio M_0/M_S is determined by the larger algebra.

8. - DISCUSSION

To summarize, we constructed an algebra that unifies the "conspiracy algebra", $SL(2, \mathbb{C})$ with internal symmetries. Due to the somewhat unexpected result that the spin-part of the conspiracy algebra can be identified with space integrals of the appropriate components of the vector and axial vector currents, the theory outlined here provides a natural link between the so-called "conspiracy theory" of Regge poles in scattering amplitudes and the algebra of observables. While the S matrix formalism used in "conspiracy theory" is a natural one in studying the effect of irreducible cross-channel states on scattering amplitudes, the present theory provides - in our opinion - a more appropriate framework to study the spectroscopic implications of various trajectory generating algebras.

Taking a different, - but not unrelated - point of view, the study of charge algebras is facilitated by the identification just mentioned. It turns out that in order to find a basis of single-particle states which saturate charge commutation relations, one has to start from massless and hence "unphysical" hadron states. This recognition, by the way, immediately removes some of the paradoxes encountered in that theory, in particular those arising from the reality requirements imposed upon "coupling constants". As emphasized particularly strongly by Fubini ³⁴⁾, when one saturates charge commutation relations by physical states, the positivity requirement can be satisfied only by using unitary representations of the Lorentz group - which, in turn leads to some difficulties in the physical interpretation. We believe that the answer is provided by our analysis of the meaning of a "particle" in a massless world. What one has to call a "particle" there, is linked to physical particle states by analytic continuation only and there is no reason why positivity requirements (coming from the conservation of probability in the physical world) should be imposed. We hope to be able to return to these questions in future publications.

Finally, let us indicate briefly some possible subjects of future investigations. Two generalizations seem to be immediately possible. First, one can ask whether the formalism can be extended to the investigation of current (rather than charge) algebras. We believe that the answer is in the affirmative in the sense that the massless world described here may serve as a useful starting point in finding physically meaningful representations of those algebras as well. The second question concerns "symmetry" breaking. At the moment it is not clear how important the interactions we neglected (spin-orbit coupling, etc.) really are. In particular, it may happen that a very strong spin-orbit coupling washes out entirely the (n,s) structure used in this paper ; in that case the physical states may look quite different from those described here. Also, the theory of representation mixing at $W \neq 0$ needs further investigation. Our point of view adopted in this paper was that none of the symmetry breaking interactions can cause very violent changes in the spectrum, and we used this assumption as a principle to search for solutions which are likely to be stable against symmetry breaking. It would be interesting to see whether this is indeed the situation realized in nature.

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T A B L E I

Experimental values of the parameters
in the $SL(2,C)$ mass formula

	σ	$A_0 [GeV^{-2}]$	$A_1 [GeV^{-1}]$	$A_2 [GeV^{-2}]$
Nucleon	-0.37	1.04	0.054	0.08
Delta	0.15	0.89	0.0	0.07

T A B L E II

States contained in the first branch
of orbital excitations. Those marked by
an asterisk (*) are connected by the operators $F_{\alpha}^{(\pm)}$

#	S	$(j_1 j_2; N)$	Intercept of trajectory $\sigma = j_1 + j_2$	Minimal physical spin $j_0 = j_1 - j_2 $	Chirality $\chi = \frac{j_1 - j_2}{ j_1 - j_2 }$
1	1/2	$(n+1/2, n; 8) (*)$	$2n + 1/2$	1/2	+
2	1/2	$(n-1/2, n; 8) (*)$	$2n - 1/2$	1/2	-
3	3/2	$(n+3/2, n; 10) (*)$	$2n + 3/2$	3/2	+
4	3/2	$(n+1/2, n; 10) (*)$	$2n + 1/2$	1/2	+
5	3/2	$(n-1/2, n; 10) (*)$	$2n - 1/2$	1/2	-
6	3/2	$(n-3/2, n; 10)$	$2n - 3/2$	3/2	-

T A B L E III

Decuplet masses calculated from Eq. (7.5).

Masses are calculated in MeV. The
quantity δ is the relative error :

$$\delta = 100 M_{\text{exp}}^{-1} [M_{\text{exp}} - M_{\text{calc}}]$$

	Δ	Σ^*	Ξ^*	Ω
M_{exp}	1236	1382	1528	1672
M_{calc}	1120	1290	1460	1590
δ	1.3	6.7	4.5	4.9

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- 20) We then call $W = \sqrt{-p_\mu p_\mu}$ the mass of the pseudostate. Our metric is : $-g_{00} = g_{ii} = 1$, $g_{ik} = 0$ for $i \neq k$. The summation convention over repeated indices is used throughout.
- 21) The definition of such scalar products involves very difficult mathematical problems ; a partial answer to the problems arising in this connection was given in the article by G. Mattioli, *Nuovo Cimento* 56A, 144 (1968). In the present paper we continue to use this terminology, since its intuitive meaning is clear.
- 22) S. Fubini, private communication.
- 23) We purposely use the term "identical" instead of "isomorphic" to indicate that the algebra in question is actually spanned by the operators $M_{\mu\nu}$. One can easily construct counter examples. Consider, e.g., in a free quark theory the operator $X_\mu = \int \bar{\Psi} \gamma^0 x_\mu \Psi d^3x$ and the total momentum operator, $P_\mu = - \int \bar{\Psi} \gamma^0 \partial_\mu \Psi d^3x$. The set of six operators : $K_{\mu\nu} = i(P_\mu X_\nu - P_\nu X_\mu)$ spans an algebra isomorphic to that of $M_{\mu\nu}$, but it is not identical to it, evidently, the spin part of the $M_{\mu\nu}$ is missing.
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- 25) See, e.g., I.M. Gelfand, M.I. Graev and H. Ia. Vilenkin, *Generalized Functions*, Vol. 5, Ch. 3 (Fizmatgiz, Moscow 1962 - in Russian). These authors characterize the irreducible representations of $SL(2, C)$ by the numbers (n_1, n_2) which are related to our notation by $n_1 = 2j_1 + 1$, $n_2 = 2j_2 + 1$. We shall also use the pair of numbers $\{\sigma, j_0\}$ defined by $\sigma = j_1 + j_2$, $j_0 = |j_1 - j_2|$ to characterize a representation. Toller et al., loc. cit. use $\lambda = 1 + \sigma$ and $M = j_0$ instead.
- 26) Notice that in the analogous three dimensional case the orientation of the momentum triangle is characterized by a (pseudo-) vector, because of the existence of a vector product.
- 27) We call the following calculation formal, since, strictly speaking, it should be performed on matrix elements of the operator Ω , accompanied by an investigation of the analyticity properties of the matrix elements

in W and the eigenvalue of the first Casimir operator of $SL(2, \mathbb{C})$. Such a calculation could proceed along lines similar to those leading to "conspiracy conditions" for scattering amplitudes. Here we take all the requisite analyticity properties for granted.

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