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1. Motivation behind neutrino experiments

At the present time, the vast bulk of our knowledge of weak interactions, and certainly all the precise measurements, come from observations on natural $\beta$-decay processes. Such studies are inevitably restricted to the region of low centre-of-mass energy and momentum transfer (typically $q^2 < 0.01$ GeV$^2$). The reasons for this are fairly obvious. Weak interactions can be successfully observed only in the absence of the more prolific strong and electromagnetic transitions, and, for the hadrons, these are forbidden only by conservation of energy, baryon number or strangeness. So, the total list of phenomena is limited to muon decay and capture, nucleon and pion decay, and the strangeness-changing decays of kaons and hyperons.

By contrast, the weak interactions produced by neutrino beams in principle offer enormously greater flexibility and scope. Firstly, one can detect the inverse of $\beta$-decay transitions, for example

$$N^{*++}(\frac{3}{2},\frac{3}{2}) \rightarrow p + \mu^+ + \nu_\mu$$  \hspace{1cm} (1)

as the inverse process

$$\nu + p \rightarrow N^{*++} + \mu^-$$ \hspace{1cm} (2)

while one could never hope to observe (1) directly (branching
ratio \( \approx 10^{-14} \).

Secondly, and depending on the neutrino energy available, one can explore reactions like (2) over a wide range of momentum transfer \( q^2 \), investigate the associated weak form factors (as in high energy electron scattering one measures electromagnetic form-factors), make tests of CVC and PCAC - all difficult or impossible with decay processes.

Finally, and possibly most important of all, neutrino experiments offer the only practical possibility of exploring the weak interactions in the high-energy domain, where the classical Fermi theory\(^{(1,2)}\) of a 4-fermion point interaction must eventually fail. Such experiments are an essential pre-requisite to the formulation of a proper field theory of leptonic weak interactions (leptodynamics). It also goes without saying that a firm understanding of the purely leptonic processes is necessary before we can understand the semi-leptonic or non-leptonic weak interactions.

The experimental difficulties of neutrino experiments are well known. The interaction cross-section is of order

\[
\sigma \sim G^2 q^2
\]

where \( q^2 \) is the range of \((\text{momentum transfer})^2\), and in units \( \hbar = c = 1 \), the Fermi constant \( G = 10^{-5}/M_p^2 \). Thus, with \( q^2 = M_p^2 \), \( \sigma = 10^{-10} \times (\text{proton Compton wavelength})^2 = 10^{-10} \times 10^{-28} = 10^{-38} \text{ cm}^2 \). The corresponding neutrino mean free
path in a dense material like lead is $3 \times 10^3$ cm = 2 astronomical units! With the usual accelerator fluxes of $\sim 10^{12}$ protons/sec, and an efficient focussing system, the flux of neutrinos from decay of secondary pions or kaons produced in the target, is typically $10^6$ M$^{-2}$/sec at the detector; this results in an event rate in the region of $10^{-3}$/sec/ton of detector. Such low event rates means that high precision experiments with neutrinos will hardly be possible. The usual disadvantage of all collision experiments, that one is stuck with only electron or nucleon targets, also applies.

2. Historical background

As a starting point, let us try briefly to summarize a few of the salient features of what we know to date about weak interactions, insofar as they affect neutrino processes i.e. the so-called leptonic and semi-leptonic interactions. (1) The V-A theory of the current-current interaction

The current-current hypothesis states that the Lagrangian (interaction energy density) can be written as a product of 2 four-vector weak currents i.e.

$$\mathcal{L} = \frac{G}{\sqrt{2}} \mathcal{J}_a(x) \mathcal{J}_a^+(x)$$

$$a = 1 \ldots 4$$

where $\mathcal{J}_a^+$ is the Hermitian adjoint of $\mathcal{J}_a$. G is the weak interaction coupling constant (Fermi constant). This type of interaction is analogous to the coupling energy
\[ \int \frac{i_1 \cdot i_2}{r_{12}} \text{ d}^n \mathbf{r} \] of 2 electric currents. The important difference however is that the currents interact only at the same point \(x\) in space-time.

In general, \( \mathcal{J}_a \) is a sum of both lepton and hadron weak currents; they have in common that the electric charge changes by one unit at the interaction point. Experimentally, the following currents have been established:

(a) lepton currents \( j_a; (\mu, v_\mu), (e, v_e) \)

(b) hadron currents \( \bar{J}_a; (\Lambda^0, p), (\Sigma^-, n), (\pi^+, \pi^0) \) etc.

Thus \( \mu \)-decay results from the interaction of the two lepton currents, while nucleon \( \beta \)-decay corresponds to a term

\[ \frac{G}{\sqrt{2}} (j_a J_a^+ + \bar{J}_a j_a^+) \]  \hspace{1cm} (5)

in (4), as indicated in the sketch.

According to the 2-component theory, the lepton current has the form

\[ j_a = \bar{\nu} \gamma_a (1 + \gamma_5) \nu \]  \hspace{1cm} (6)

where \( \nu \) and \( \ell \) are neutrino and lepton spinors, \( \gamma_a \) is the Dirac current operator, and \( (1 + \gamma_5) \) projects out the unwanted helicity state. Thus (6) describes leptons
(ν, e⁻, μ⁻) of helicity -ν/c, and antileptons (ν⁺, e⁺, μ⁺) of helicity +ν/c. (Recall that the Dirac equation describing spin \( \frac{1}{2} \) particles has as solutions 4-component wave-functions; for particles of zero mass Weyl first showed that the equation splits into 2 simpler, decoupled equations; one describes L.H. particles and R.H. antiparticles, and the second equation R.H. particles and L.H. antiparticles. Which equation was correct (in our world) was settled experimentally in a classic and beautiful experiment by Goldhaber, Grodzins and Sunyar; they showed the lepton (neutrino) to be L.H.)

There is no simple form akin to (6) for the hadron current \( J_a \). However, from the form of the leptonic current, we know that the hadron current operators must have the transformation properties of vectors (V) and/or axial vectors (A). In β-decay, the \( ΔJ = 0 \) pure Fermi transitions result from the V coupling, and the pure Gamow-Teller \( (ΔJ = 1) \) from the A coupling. Generally one has a mixture of both. Thus

\[
J_a = A_a + V_a
\]

From a second classic experiment on polarized neutron β-decay, we know that the V and A couplings for the \( (n, p) \) current are comparable in magnitude and opposite in sign; in fact the amplitudes are related by \( C_A/C_V = -1.18 \). If \( C_A = -C_V \) exactly, the nucleon current would be of the same form as (6) - hence the so-called V-A theory. If we
put $C_A = -\lambda C_V$, the nucleon $\beta$-decay interaction becomes

$$\frac{G}{G_V} \cdot \frac{G_V}{\sqrt{2}} \left[ \frac{1}{p} \gamma_\alpha (1 + \lambda \gamma_5) n \right] \left[ \bar{e} \gamma_\alpha (1 + \gamma_5) v \right] + \text{h.c.}$$

(8)

$G_V$ is the coupling constant for Fermi transitions, and $G$ is, by definition, the constant for a purely leptonic transition ($\mu$-decay). Experimentally

$$\frac{G(\mu\text{-decay})}{G_V(0^+\text{decay})} = 1.02$$

(9)

Note that (8) applies to the $(n,p)$ current; for other hadron currents (e.g. $\Lambda,p$) one will in general expect different values of both $G_V$ and $\lambda$.

(ii) CVG and UFI

If in (9) above, we had strictly $G_V = G$, this would imply that, at least for the vector interaction, the Fermi "weak charges" ($g = \sqrt{G}$) not only of all the leptons (e-\mu universality) but also of the nucleons were identical - the famous Universal Fermi Interaction (UFI). This is either a sheer accident, or there is a conservation law at work, since a priori there is no reason to expect a lepton, which has no strong interaction, to have the same Fermi charge as a nucleon, which does. An additional (in this case strong) interaction, resulting in continuous emission and absorption of virtual mesons, might be expected to change or renormalise the weak charge, even if this were identical for all particles in the first place, before
additional interactions had been switched on. For example, the gyromagnetic ratios for electrons and muons are approximately equal to the Dirac value of 2, whereas strongly interacting particles, like p, n and λ, have large anomalous magnetic moments. In electromagnetic interactions, the bare electric charges are equal, and the renormalization does not destroy this. We say that the electric charge is conserved in the strong interactions, or that the electromagnetic current is divergenceless:

$$\frac{\partial J_{\alpha}^{\text{e.m.}}}{\partial x_\alpha} = 0 \quad (10)$$

The conserved vector current (CVC) hypothesis of Feynman and Gell-Mann\(^{(2)}\) and Gershtein and Zel'dovich\(^{(5)}\) states that the vector part of the weak current is also conserved:

$$\frac{\partial V_{\alpha}}{\partial x_\alpha} = 0 \quad (11)$$

Thus the vector weak charges of all particles, proton, neutron, pion, electron etc. should be equal. One of the best demonstrations\(^{(6)}\) of this equality is that the decay rates for the pure vector transitions:

\[
\begin{align*}
0^{+} &\rightarrow N^{+} + e^+ + \nu \\
\pi^+ &\rightarrow \pi^0 + e^+ + \nu
\end{align*}
\]

(12)
give (ft) values equal within experimental error (~10\%).

Formally, we can see that both (10) and (11) would
result if the appropriate charges were carried by a conserved isospin current; conservation of the z component of this current corresponds to the conservation of charge (10), and conservation of the (x + iy) component (converting n → p) corresponds to conservation of the vector weak charge (11). By extending the argument, one can surmise a strict proportionality between the isovector part of the electromagnetic current and the vector part of the weak current, so that the $q^2$-dependence of the form-factors introduced by strong-interaction effects is identical in the two cases.

(iii) PCAC

The hypothesis of a partially conserved axial vector current (PCAC) may be stated in a variety of ways. We need only discuss what happens from a qualitative viewpoint.

Let us start with a "bare" proton, that is one without strong interactions. Now turn on the S.I. The S.I. properties can be described (no-one knows exactly how) in terms of virtual emission and re-absorption of π, k, p mesons etc. The CVC hypothesis states that, for the vector part of the weak charge ie. $V^e$, the coupling is not renormalized by the strong interactions, ie. $g^e(β-decay) = g(μ-decay)$. In other words, the weak vector charge is distributed among the nucleon core and surrounding meson cloud in such a way that the "lepton emitting power" of the whole assembly is exactly the same as that of the bare nucleon, ie. the vector current is divergenceless.
On the other hand, this compensation does not work for the axial ($G_A$) coupling; $G_A = 1.18 G_V$ for the nucleon $\beta$-decay, whereas in $\mu$-decay they are equal. The renormalization of the axial charge when the S.I. is turned on, means that the change in "lepton-emitting power" of the core is not compensated exactly by that of the meson cloud. The PCAC hypothesis then states that this difference — the divergence $\partial A_\alpha / \partial x_\alpha$ — is dominated by single pion exchange, and is thus, as the diagram indicates, determined by the pion-nucleon scattering amplitude, the $\pi \rightarrow \mu$ decay rate and other constants. This leads essentially to the famous Goldberger-Treiman relation — see eqn. (39). One can also see qualitatively that the magnitude of the renormalization effect will depend on the pion-nucleon scattering cross-sections integrated over the initial pion momentum. This last result is essentially the Adler-Weisberger sum rule for $G_A$, derived from axial current algebra.$^\dagger$

$^\dagger$ Footnote: The sum rule is

$$ G_A / G_V = \left\{ 1 + \frac{2M_n}{\pi r \pi n} \int_0^\infty \frac{p_x dE_x}{[E_x - \frac{m_A}{2M_n}]^2} \times [\sigma(\pi^- p) - \sigma(\pi^+ p)] \right\}^{-1/2} $$
Finally, note that the pion decay rate → 0 as m_π → 0, so that on this model, ∂V_α/∂x_α → 0; in other words, the axial charge is conserved by all the possible strong-interaction effects, other than single-pion exchange.

(iv) Cabibbo theory

For β-decay of hyperons, it is well established that the coupling is an order of magnitude weaker than the nucleon vector coupling G_V. Furthermore, we have the 2% discrepancy (9) which according to CVC and the UFI should not exist. These anomalies have been resolved by Cabibbo(7) as follows. Suppose we think of the unit of weak charge as being attached to the baryon 1^+ octet (consisting of the nucleon, Λ, Σ and Ξ isospin multiplets). In the limit of exact SU_3 symmetry, the 8 states are degenerate, and have all the same unit of weak charge. The symmetry is broken along the I spin axis by electromagnetic interaction which split up the components of an I spin multiplet into states of slightly different mass. Similarly there is a mass-splitting along the hypercharge (or U-spin) axis. No principle exists to predict how the weak charge is to be divided between the different states; Cabibbo assigned a factor cosθ to the strangeness conserving (ΔS = 0) weak currents and sinθ to the ΔS = 1 currents. The value of θ is found experimentally from the 0^- meson octet decay rates, and is the same for both Λ and V transitions.
\[(\tan \theta)_A \text{ from } \frac{K^+ - \mu^+ + \nu}{\pi^- - \mu^+ + \nu} = (\tan \theta)_V \text{ from } \frac{K^+ - \pi^0 + \bar{c}^+ + \nu}{\pi^- - \pi^0 + \bar{c}^+ + \nu} = 0.26 \]

Thus, to compare with the constant \(G\) for the purely leptonic interaction (muon decay), we must take for the nucleon vector coupling \(G_V \sec \theta\); the ratio is then much closer to unity. Further, the weak decay rate of the hyperons is thus depressed by a factor \(\tan^2 \theta\) relative to the nucleon as required.

The formal \(SU_3\) theory of weak interactions assigns the vector weak current and the electromagnetic current to the same (octet) representation of \(SU_3\). The 8 generators \(F_i\) of the group (obeying the commutation rules \([F_i, F_j] = C_{ijk} F_k\)) can be identified with physical quantities:

- isospin: \(I_1 = F_1\)
- \(I_2 = F_2\)
- \(I_3 = F_3\)
- hypercharge: \(Y = \sqrt{2} F_8\)

To each \(F_i\) one can assign a 4-current:

\[F_i \rightarrow j^{(i)}_a, \quad i = 1, \ldots, 8\]

such that \(F_i = \int j^{(i)}_4 \, d^3 x\) \(j_4 = \) time component
Thus \[ I_3 = \int j_4^{(3)} \, d^3x, \quad Y = \sqrt{3} \int j_4^{(8)} \, d^3x \]

\[ \text{isospin current} \quad \text{hypercharge current} \]

From the relation

\[ Q = I_3 + \frac{i}{2} Y = F_3 + \frac{1}{\sqrt{3}} F_8 \]

we may write \[ j_a^{(e.m.)} = j_a^{(3)} + \frac{1}{\sqrt{3}} j_a^{(8)} \]

Since \( j^{(e.m.)} \) is conserved, it follows \( j^1 \ldots j^8 \) are all conserved by SU_3 symmetry. Thus, the \( \Delta S=0 \) and \( \Delta S=1 \) vector weak currents are

\[ V_a(\Delta S = 0) = j_a^{(1)} \pm ij_a^{(2)} \]

\[ V_a(\Delta S = \pm 1) = j_a^{(4)} \pm ij_a^{(5)} \]

The selection rules \( \Delta S = \Delta Y = 0, \Delta Q = \pm 1, \Delta I_3 = \pm 1 \), and \( \Delta S = \Delta Y = \Delta Q = \pm 1, \Delta I_3 = \pm \frac{1}{2} \), follow from the commutation rules for the generators \( F \).

3. Neutrino reactions involving hadron currents

3.1 General remarks

There are two types of process which can take place with nucleon targets:

(a) **Elastic reactions** not involving pion production.

These consist of those transitions inside the \( \frac{3}{2}^+ \) baryon octet allowed by the selection rules (i) \( \Delta Q = 1, \Delta I = 1 \) for a \( \Delta S = 0 \) transition, (ii) \( \Delta Q = 1, \Delta S = \Delta Q, \Delta I = \frac{1}{2} \) for a \( \Delta S = 1 \) transition; here \( Q, I, S \) refer to the baryon quantum
numbers. These selection rules follow from experiment, and from the definition \( Q = I_3 + \left( \frac{B+S}{2} \right) \), and appear naturally in the Cabibbo theory as indicated above.

Examples of elastic reactions are:

\[
\begin{align*}
\nu_\mu + n &\rightarrow \mu^- + p \quad \Delta S = 0, \quad \Delta I = 1 \\
\bar{\nu}_\mu + p &\rightarrow \mu^+ + n \quad \\
\bar{\nu}_\mu + p &\rightarrow \mu^+ + \Lambda \quad \Delta S = \Delta Q = -1, \quad \Delta I = \frac{1}{2} \\
\bar{\nu}_\mu + n &\rightarrow \mu^+ + \Sigma^- 
\end{align*}
\]

(13)

(b) Inelastic reactions These are reactions involving pion production, and may be thought of as transitions from the \( n, p \) states of the \( \frac{1}{2}^+ \) baryon octet, to states in other \( SU_3 \) multiplets. For example, \( \frac{1}{2}^+ \) (octet) \( \rightarrow \frac{3}{2}^+ \) (decuplet) transitions, resulting in single pion production, are of the form

\[
\begin{align*}
\nu_\mu + p &\rightarrow \mu^- + N^{*}(1238)^{++} \rightarrow \pi^+ + p \quad \Delta S = 0, \quad \Delta I = 1 \\
\nu_\mu + n &\rightarrow \mu^- + N^{*}(1238)^+ \rightarrow \pi^0 + p \\
\bar{\nu}_\mu + n &\rightarrow \mu^+ + Y_1^{*}(1385) \rightarrow \Lambda + \pi^- \quad \Delta S = \Delta Q = -1, \quad \Delta I = \frac{1}{2} \\
\end{align*}
\]

(14)
(c) General form of cross-section

The assumption made above (6) namely that the lepton variables enter the matrix element in the form of the local lepton current \( j_\alpha(x) \), puts strong restrictions on the form of the cross section \(^{(8,9)}\); namely it can be at most only a quadratic function of the neutrino energy. If we neglect the charged lepton mass (i.e. \( q^2 \gg m_\mu^2 = .01 \text{ GeV}^2 \), which is usually the case), and sum over lepton polarization, \( \sigma \) is a function of three real form factors, which take account of the structure of the baryon current \( J_\alpha \). A convenient form for the cross section for the general reaction:

\[ \nu + n \rightarrow \mu + \Gamma \]

is then

\[
\frac{d^2\sigma(\nu)}{dq^2dM_\Gamma^2} = \frac{A}{E_\nu^2} + \frac{B}{E_\nu} + C \tag{15}
\]

\( A, B \) and \( C \) are unknown functions of \( q^2 \) and \( M_\Gamma \). For \( \bar{\nu}, A, B \) and \( C \) are replaced by \( A', B' \) and \( C' \). For any one channel \( \Gamma \), the asymptotic cross-section is a constant \( \int dq^2 \).

3.2 Elastic reaction \( \Delta S = 0 \)

(a) Theory

By far the most comprehensively studied neutrino processes are

\[ \nu_\mu + n \rightarrow p + \mu^- \]

\[ \bar{\nu}_\mu + p \rightarrow n + \mu^+ \tag{13a} \]

In order to exploit known invariance principles, and
the conserved current hypothesis, (15) is not very informative. We start therefore by writing down the most general form of the matrix element for $J_\alpha$ allowed by the V-A theory and Lorentz invariance

$$\langle p|J_\alpha|n\rangle = \bar{u}_p \left[ g_\gamma \gamma_\alpha + \frac{i\mu G^I_V}{4\mu M} (\gamma_\alpha \gamma_\beta - \gamma_\beta \gamma_\alpha) q_\beta + \frac{iA}{m_\mu} q_\alpha \right. $$

$$+ G_A \gamma_5 \frac{iB}{4\mu} (\gamma_\alpha \gamma_\beta - \gamma_\beta \gamma_\alpha) \gamma_5 q_\beta + \frac{iB}{m_\mu} \gamma_5 q_\alpha \right] u_n$$

(16)

the first three terms being the vector contributions, the rest the axial vector. $q$ is the 4-momentum transfer $(p_v - p_\mu)$, $M$ the nucleon mass, and $m_\mu$ the muon mass. $\bar{u}_p$ and $u_n$ are the usual nucleon Dirac spinors. The coefficients $G_V$, $G^I_V$, $A$, $G_A$, $B$ and $b$ are functions of $q^2$.

(1) If we assume invariance under time reversal, all these coefficients are real.

(2) A further simplification is made by postulating G-invariance.

We are familiar with the fact that e.m. field $A_\mu$ changes sign under the operation of charge conjugation $C$ - because electric currents $J_{\text{em}}$ change sign (+ to -). There is an analogous behaviour for the weak currents $V_\alpha$ and $A_\alpha$. Under the combined operation of charge conjugation $+180^\circ$ rotation about the $y$-axis in isospin space

$$G = C e^{i\lambda I_y}$$
the currents transform according to
\[ V_a \rightarrow + V_a \]
\[ A_a \rightarrow - A_a \]  \hspace{1cm} (17)

Such currents are called "first-class currents"; if they transform in the opposite sense \( V_a \rightarrow - V_a \) and \( A_a \rightarrow + A_a \), they are called "second-class currents". All evidence suggests that we deal only with currents of the first kind. Invariance under \( G \) implies that the coefficients \( A \) and \( B \) in (16) should vanish.

(3) The final term in (16), the so-called induced pseudo-scalar term is dominated by one-pion exchange, is always small, and can safely be neglected at the momentum transfers \( q^2 > m^2 \) usually considered.

(4) Now consider the 2 remaining \( V_a \) terms. Recall that in electron-proton scattering, the baryonic factor is
\[ \langle p | J_1^{e.m.} | p' \rangle = \bar{u}_p e \gamma_\alpha \gamma^\mu u_p \text{ in the absence of S.I.} \]
\[ = \bar{u}_p \left( e F_1^D(q^2) \gamma_\alpha + \frac{i u_p}{4M} F_2^P(q^2)(\gamma_\alpha \gamma_\beta - \gamma_\beta \gamma_\alpha) q_\beta \right) u_p \]  \hspace{1cm} (18)

when the strong interactions are turned on. \( F_1^D \) is the Dirac form-factor, \( F_2^P \) the Pauli form-factor corresponding to the anomalous magnetic moment of the proton, \( \mu_p \).
\[ F_1^D(0) = F_2^P(0) = 1 \]; similarly for the neutron, where
\[ F_1^D(0) = 0 \] however. Note that the coefficient of the Dirac
or electric term $\epsilon \gamma_a$ is the same (at $q^2=0$), with or without strong interactions — because charge is conserved. Now, we get from the first line to the second by adding a current due to pions etc. Thus

$$J^e_m = \bar{\psi}_p \gamma_a \psi_p + \text{pion current, kaon current, etc.}$$

$$= \bar{\psi} \gamma_a \left( \frac{1+\tau_3}{2} \right) \psi + \text{(pion current)}$$

(19)

The isovector part of this is

$$(J^e_m \text{ isovector}) = \bar{\psi} \gamma_a I_3/2 \psi + \text{(pion current)}$$

and is conserved, since $J^e_m = J(\text{isoscalar}) + J(\text{isovector})$, and the first 2 are certainly conserved.

$$V_a = \bar{\psi}_p \gamma_a \psi_n + \text{pion current}$$

$$= \bar{\psi} \gamma_a \tau_+ \psi + \text{(pion current)}$$

(20)

where $\tau_+ = \frac{i}{2}(\tau_1+i\tau_2)$, and $\tau_1,2,3$ are the Pauli spin matrices.

CVC states that (19) and (20) are merely different components of one and the same isovector current

$$\bar{J}_a = \bar{\psi} \gamma_a \frac{q}{e} \tau_+ \psi + \text{(pion current)}$$

(21)

In other words, $V_a = \frac{q}{e} (J^e_m) \text{ isovector}$, and $V_a$ transform as the $I_3 = -1,0$ and +1 members of an $I=1$ triplet.

Conservation of $I$-spin ensures that, if (19) is a conserved current so also is (21), and hence also $V_a$. So
the vector coupling constant $G_V$, is not renormalized by the strong interactions. Further, from (18), (19) and the analogous relation for e-n scattering, we obtain for the isovector electromagnetic form factors

$$F^V_1(q^2) = F^P_1(q^2) - F^n_1(q^2)$$

for the electric interaction, and

$$(\mu_p - \mu_n)F^V_2(q^2) = \mu_p F^P_2(q^2) - \mu_n F^n_2(q^2)$$

where $\mu_p = 1.79 \approx -\mu_n = +1.91$. It turns out that $F^V_1$ and $F^V_2$ have a similar form, call them both $F^V$. Hence,

we can use for the $V_\alpha$ terms in (16)

$$G_V(q^2) = G_V(0) F^V(q^2)$$

$$\mu G^V_\alpha(q^2) = G_V(0) (\mu_p - \mu_n) F^V(q^2)$$

(22)

The second term is frequently called the "weak-magnetism" term, for obvious reasons.

(5) As a result of the above simplifications, the matrix element (16) now contains only one unknown parameter, the axial form factor

$$G_A(q^2) = F_A(q^2)/G_A(0)$$

with the normalization

$$G_A(0) = -\lambda G_V(0) \approx -1.2 G_V(0).$$

Multiplying through by the lepton bracket (6), we finally get an expression for the elastic cross-section. Again, neglecting lepton masses, this can be conveniently expressed in the form
\[
\frac{d\sigma_{\mu \nu}}{dq^2} = \frac{g^2}{32\pi E_{\nu}} \left[ X \pm Y(s-u)+Z(s-u)^2 \right]
\]

where

\[
X = q^2(4\mu^2 F^2_A - 4F^2_V) + q^4(F^2_V[1 + \frac{\mu^2}{M^2} + \frac{\mu^2}{M}] + \lambda^2 F^2_A)
\]

\[
Y = 4q^2\lambda F_A F_V(1 + \mu/M)
\]

\[
Z = F^2_V(1 + \frac{\mu^2}{M^2}) + \lambda^2 F^2_A
\]

\[
M = \text{nucleon mass}
\]

\[
s-u = 4ME_{\nu} - q^2 - m^2\mu \text{ in L.S.}
\]

\[
s \text{ and } u \text{ are 2 of the Mandelstam Variables}
\]

\[
s = -(k+K)^2 = -(k'+K')^2
\]

\[
t = -(k-k')^2 = -(K-K')^2
\]

\[
n = -(k-K')^2 = -(K-k')^2
\]

Now the reactions

\[
\nu + n \rightarrow p + \mu^-
\]

and

\[
\bar{\nu} + p \rightarrow n + \mu^+
\]

have the same matrix element:

provided we exchange the nucleon labels i.e. Set K=K' and K' = -K. Then (s-u) = -2k(K+K') changes sign. Thus, the
difference in $\nu$ and $\bar{\nu}$ cross sections

$$\frac{d\sigma_\nu}{dq^2} - \frac{d\sigma_\bar{\nu}}{dq^2} = \frac{q^2}{16\pi s^2} Y(s-u) \quad (24)$$

$$= \frac{q^2}{4\pi s^2} s^2 \left(1 + \frac{4}{M^2}\right) \lambda F_\nu F_\nu(s-u)$$

is directly proportional to $F_A$.

(b) **Experimental investigations of the $\Delta S=0$ elastic reaction**

Thus far, the processes (13a) have been investigated in experiments at BNL, CERN and ANL with thick-plate spark chambers or heavy-liquid bubble chambers, so that reactions have taken place on complex nuclei rather than free nucleons. There are severe uncertainties in interpretation of the data. I shall discuss only the bubble chamber events (15) since they are probably the least unreliable. Various backgrounds were first eliminated:

(i) **Pion background**

Pion background arises because $\nu$-interactions in the magnet yoke or other material near the bubble chamber can generate backward-moving $\pi^+$ interacting in the chamber and simulating genuine $\nu$-interactions, with an emergent $\mu^-$, in the liquid. These can practically all be removed by kinematic tests - especially if one considers only events with visible energy $E_{vis} > 1$ GeV.

(ii) **Inelastic interactions**

Since the events occur on complex nuclei (freon CF$_3$Br), a genuinely inelastic event can appear as non-pionic (i.e.
a lepton plus one or more protons) because of pion absorption. Monte-Carlo calculations demonstrated that, as a result of pion absorption, one would nearly always obtain multi-proton events, whereas the true elastic events would mostly (85%) result in a single proton. To study this in more detail, suppose the neutrino collides with a stationary free neutron, and the final-state hadron mass is $M^*$. Then, if $M$ is the nucleon mass, from simple kinematics we have:

$$M^{*2} = -q^2 + M^2 + 2M(E_v - E_{\mu})$$  \hspace{1cm} (25)

In this equation, $E_v$ is equated to the visible energy $E_{vis}$ in the event, $q^2 = (p_{v} - p_{\mu})^2 - (E_v - E_{\mu})^2$ and hence $M^*$ can be calculated. Of course, the target nucleon is in Fermi-motion so a distribution in $M^*$ is obtained on this account. The satisfactory feature is that the single-proton events have $M^{*2}$ clustering symmetrically around $M_p^2$, while the multiproton events show a distribution pushed to higher values.

Eliminating the above backgrounds, the number of "good" elastic candidates was reduced to 63 (from an initial total of about 330). In order to confront the theory with these survivors, it is necessary to allow for the effects of Fermi motion and the Pauli principle$^{14}$, which inhibits low momentum transfers to the nucleon. Fig. 1 shows the $q^2$-distributions observed for $v$ and $\bar{v}$ elastic events, and Fig. 2 shows the comparison between observed and calculated
elastic cross-sections, as a function of neutrino energy. The curves show the expected distributions on the following assumptions:

\[ F_V(q^2) = \left(1 + \frac{q^2}{M_V^2}\right)^{-2} \]  \hspace{1cm} (25)

with \( M_V = 0.84 \text{ GeV} \). This quadratic formula gives quite a good fit to the electron scattering data out to \( q^2 \sim 25 \text{ GeV}^2 \) - no-one knows why. Naively one would have expected a pole-type form-factor

\[ F(q^2) = \left(1 + \frac{q^2}{M^2}\right)^{-1} \]  \hspace{1cm} (25a)

but this does not happen.

\[ F_A(q^2) = \left(1 + \frac{q^2}{M_A^2}\right)^{-2} \]  \hspace{1cm} (26)

with \( M_A \) to be determined. The best fit then gives \( \frac{M_A}{M_V} \approx 1.0 \pm 0.3 \) \hspace{1cm} (27)

where a systematic error equal to the statistical error has been included. A point-like axial structure (\( M_A = \infty \)) is definitely ruled out.

(c) On the basis of results from current algebra \( ^{16} \), it has been proposed that

\[ F_A(q^2) = F_1(q^2) + \frac{q^2}{4M^2} F_2(q^2) \approx \left(1 + \frac{q^2}{4M^2}\right) F_V(q^2) \]  \hspace{1cm} (28)

This does not give quite such a good fit, if one sets \( M_V = 0.84 \text{ GeV} \), but one cannot exclude this form of \( F_A \) on the basis of existing data.
(c) Conclusions

The result (27) is of great interest from the point of view of nucleon structure. The pole-type form-factor (25a) - if it fitted the data - would suggest that the vector structure is dominated by exchange of a heavy vector meson of even G-parity - see eqn. (17); while exchange of an axial vector particle, of odd G-parity, is responsible for the axial vector structure:

\[ J^{PG} = 1^{-+} \]

\[ J^{PG} = 1^{+-} \]

\[ \rho \rightarrow 2\pi^2 \]

\[ A_1 \rightarrow 3\pi^2 \]

Vector \hspace{2cm} Axial Vector

If one tries to force a fit to a one-pole form factor, the ensuing value of \( M_V \) from the electron scattering data \(^{(17)}\) is \(~0.6\) GeV, substantially below the mass of the lightest known candidate (\( \rho \), 765 MeV). It is also hard to understand why the value of \( M_A/M_V \) is close to unity, since the lightest axial vector candidate, the \( A_1 \), of mass \(~1100\) MeV is substantially more massive than the \( \rho \). Of course, a picture of structure due to single-particle exchange is probably wrong anyhow; the form-factors may originate in something more fundamental or complex, such as the spatial distribution of quarks in the quark model.
3.3 Elastic reaction ΔS=1 (Hyperon production by antineutrinos)

The cross-sections for the reactions

\[ \bar{\nu} + p \rightarrow \Lambda + \mu^+ \]  \hspace{1cm} (29) \hspace{1cm} (a) \\
\[ \bar{\nu} + p \rightarrow \Sigma^0 + \mu^+ \]  \hspace{1cm} (b) \\
\[ \bar{\nu} + n \rightarrow \Sigma^- + \mu^+ \]  \hspace{1cm} (c)

have been calculated by Cabibbo and Chilton\(^{(15)}\), on the basis of the SU\(_3\) model of weak currents\(^{(7)}\). (Note that reactions (b) and (c) are related by the ΔI=\(\frac{1}{2}\) rule:

\[ d\sigma(\Sigma^0) = \frac{1}{2} d\sigma(\Sigma^-) \].

In general, if \(B^i\) and \(B^k\) are two members of the \(\frac{1}{2}^+\) baryon octet, the matrix of the weak current

\[ \langle B^i | J_\alpha | B^k \rangle \]

can be expressed as a linear combination of 2 reduced matrix elements, say \(O_\alpha\) and \(E_\alpha\), with coefficients which are the SU\(_3\) equivalent of Clebsch–Gordan coefficients. With the help of CVC, the vector parts of \(O_\alpha\) and \(E_\alpha\) can be found from the known matrix elements of the electromagnetic current \(\langle p | J_{\alpha}^{em} | p' \rangle\) and \(\langle n | J_{\alpha}^{em} | n' \rangle\). Thus

\[ \langle n | V_\alpha | \Sigma^- \rangle = \sin\theta (O_\alpha - E_\alpha)^V \]
\[ \langle p | V_\alpha | \Lambda \rangle = -\sqrt{3}/2 \sin\theta (O_\alpha + \frac{E_\alpha}{3})^V \]

where

\[ \langle p | J_{\alpha}^{em} | p \rangle = (O_\alpha + \frac{E_\alpha}{3})^V \]

and

\[ \langle n | J_{\alpha}^{em} | n \rangle = -\frac{2}{3} E_\alpha^V \]

For the axial coupling, \(A_\alpha\), \(O\) and \(E\) are not known individually. From the \(n \rightarrow p\) β-transition we know the sum:
\[ \langle p|A_\alpha|n \rangle = \cos \theta (\mathcal{O}_\alpha + E_\alpha)^A \]

Thus, to get the nE\(^-\) pA etc. axial couplings, one must put in the O(odd) to E(even) coupling ratios (f/d). Cabibbo and Chilton take \( x = f/d = 0.25 \) for all \( q^2 \).

For the form factors, it is assumed that \( F_V(q^2) \) is of the same form as in the \( \Delta S=0 \) transition e.g.

\[ F_V(q^2) = (1 + \frac{q^2}{M^2})^{-1} \]

but, since one is dealing with \( \Delta S=1 \), take \( M = M_{K^*}(890) \) - the lightest \( S=1 \), \( J^{PG} = 1^{-+} \) vector meson - instead of \( M_\rho(765) \). For the axial vector mass, again \( M_{K^*}(890) \) was taken (although the lightest possible axial vector candidate (\( K^*\pi \)) has a mass of around 1300 MeV).

The experimental result in the CERN \( \bar{\nu} \) experiment\(^{(18)} \) was that no hyperon candidates were found, whereas, under the foregoing theoretical assumptions, 5 should have been observed. There are two factors affecting the interpretation of this result. From the experimental side, the collisions occurred in complex nuclei, so that hyperons can be absorbed (with formation of hyperfragments or cryptofragments) in their passage out of the nucleus. Rough estimates indicated that \( \approx 30\% \) of all hyperons would thus be lost. On the other hand, \( (29) \) refers to the elastic process only, and at least a comparable number of hyperons would be produced in inelastic processes, as for example in \( Y_1^* \) production - see eqn (14) above.
So there appears to be a discrepancy, which can probably be blamed on too "hard" form factors, as well as on the meagre experimental data. This is clearly a topic which will have to await future experiments with bigger chambers.

3.4 Inelastic reactions - single pion production

Next to the $\Delta S = 0$ elastic reaction, by far the best studied neutrino reaction channel is that of single pion production: $\nu + n, p \rightarrow \mu^- + \pi^- + n, p$.

3.4 (a) Theoretical considerations

In analogy with photopion and electropion production processes, the hadronic weak current can be approximated by summing the four graphs below:

Graph (a) represents the $N_{33}^*$ resonance production, (b) the pion exchange contribution, while (c) and (d) are nucleon exchange graphs. Many authors have discussed this reaction (20-25). At the neutrino energies presently available (i.e. a few GeV), the $N^*$ diagram (a) dominates the
Let \( p_1 \) and \( p_2 \) be the 4-momenta of the initial and final nucleon, \( p_\nu \), \( p_\mu \) and \( p_\pi \) those of neutrino, lepton and pion. The invariants relevant to the problem are then

\[
\begin{align*}
    t &= -(p_1 - p_2)^2 \\
    q^2 &= (p_\nu - p_\mu)^2 \\
    M^{*2} &= -(p_2 + p_\pi)^2
\end{align*}
\]

As in inelastic electron scattering, a Rosenbluth-type cross section is obtained, of the general form:

\[
\frac{d^2\sigma}{dM^{*2}dq^2} = \frac{1}{E_\nu E_\mu} \left[ q^2 R(q^2, M^{*2}) + (2E_\nu E_\mu - \frac{q^2}{2}) \right] \\
S(q^2, M^{*2}) + (E_\nu + E_\mu) T(q^2, M^{*2}, \frac{q_0}{|q|})
\]

(30)

of the same form as (15). Since the main contribution is from isobar formation, we may set

\[
M^{*} = M^{*}(3/2, 3/2)
\]

(31)

The structure functions \( R, S \) and \( T \) appear in the hadron current \( J_a = V_a + A_a \). The normalization for \( V_a \) at \( q^2 = 0 \) has been taken from photoproduction on the basis of the CVC theory. The axial coupling \( A_a \) has been estimated(21) using the Goldberger-Treiman relation (i.e. one pion exchange), or by use of dispersion relations(25).
A quite different approach was made\(^{(22)}\) on the basis of SU(6) symmetry. Since N (nucleon) and N\(^*(3,3)\) are members of the same (56) representation of SU(6), it is possible to relate N\(^*\) production to the elastic process $\nu n \rightarrow p \mu^-$. (By the same token it is possible to relate N\(^*\) and Y\(^*\) production.)

Although the different models employed yield slightly differing results on cross-sections, a meaningful comparison with experiment is difficult since the appropriate form-factors are essentially unknown. The corresponding electroproduction experiments have not yet reached the stage of accuracy where they can yield useful information on electromagnetic form-factors to serve as a guide. Thus, all the calculations made so far have assumed that the appropriate vector and axial-vector form-factors are the same as in the elastic nucleon case.

3.4 (b) Discussion of the experimental results

Since the one-pion production is assumed to be dominated by, or generated exclusively\(^{(21)}\) by, the N\(^*(3/2, 3/2)\) resonance, a first selection of the experimental data\(^{(8)}\) has been made by imposing $M^2 < 2.2$ GeV\(^2\) (as calculated from eqn. (25)). Both one-visible-pion and non-pionic multi-proton events (where the pion is absorbed in the nucleus, as discussed above) are included in the sample\(^{+}\). Figs. 3(a) and (b) show the observed and calculated\(^{(21)}\) $\nu$ and $\bar{\nu}$ cross-

\(^{+}\) Footnote: Contributions from 2$\pi$ events, with 1$\pi$ absorbed, are expected to be negligible.
sections as a function of neutrino energy. There is good agreement between theory and experiment for the choice

\[ F_A = F_V = (1 + \frac{q^2}{M^2})^{-2} \]

with \( M = M_A = M_V = 0.8 \text{ GeV} \) as in the elastic reaction.

The distribution of \( M^2 \) for the one-pion events has been calculated from eqn. (25), and is shown in Fig. 4. It can be seen that there is a significant peak near the \( N^*(3,3) \) mass, the shift to lower mass being accounted for by missing energy in the form of neutrons from nuclear scattering and absorption processes (note that \( E_V \) in (25) is taken equal to the "visible" energy release in the event.) A direct calculation of \( M^* \) from proton and pion 4-momenta in the process \( \nu + p \rightarrow (p + \pi^+) + \mu^- \) is not useful, because of scattering of the pion (or proton) in its passage through the nucleus.

Since the final hadron state has definite isospin, the possible neutrino reactions are related (the \( \Delta I = 1 \) rule is equivalent to replacing \( \nu \) and \( \mu^- \) by a \( \pi^+ \) on the L.H.S.)

\[
\begin{align*}
\text{Amplitude} & \\
(a) \quad \nu \mu + p \rightarrow \mu^- + p + \pi^+ & \quad A_3 \\
(b) \quad \nu \mu + n \rightarrow \mu^- + n + \pi^+ & \quad \frac{1}{3}A_3 + \frac{2}{3}A_1 \\
(c) \quad \nu \mu + n \rightarrow \mu^- + p + \pi^0 & \quad \frac{\sqrt{3}}{3}A_1 - \frac{\sqrt{2}}{3}A_3
\end{align*}
\]

Thus, if the \( I=3/2 \) amplitude \( A_3 \) dominates, the cross-sections for (a), (b) and (c) should be in the ratio

\[ p\pi^+: n\pi^+: p\pi^0 = 9:1:2 \quad \pi^+/\pi^0 = 5/1 \] (33a)
whereas if $I=\frac{1}{2}$ dominates, we would have
\[ n\pi^+ : p\pi^0 = 2:1 \quad \pi^+ / \pi^0 = 2/1 \quad (33b) \]

In single-pion events, the observed ratio (18) is, at first sight surprisingly:
\[ \frac{N\pi^+}{N\pi^0} = 2.3 \pm 0.9 \quad (34) \]

The fact that 5% of the events contain \(\pi^−\), where none should be observed and that in half the events (the multiproton), the pion is absorbed, is indicative of strong nuclear (charge-exchange) effects which distort the ratio. When these, and the neutron excess in the target nuclei are taken into account, the expected value of the ratio (33a) becomes \(\approx 2.2\), consistent with experiment.

It may also be remarked that, for \(M^*^2 > 2.2 \text{ GeV}^2\), one expects \(I=\frac{1}{2}\) resonances to dominate. Thus one expects a ratio of the order of (33b); the observed ratio is \(0.8 \pm 0.4\), again indicative of strong charge-exchange effects.

3.4 (c) Conclusions

To summarize therefore, the information on single pion production by neutrinos and antineutrinos indicates very strongly that production through the \(N^*(5,3)\) resonance dominates, and that the associated transition form-factors are closely equal to those pertaining to the elastic process.

3.5 Inelastic reactions - multiple pion production and other processes

On account of the poor statistics on the experimental
side, and almost complete absence of any theoretical models, only a qualitative discussion of the general features of the multi-pion and other highly inelastic events can be given, and will be discussed in a later section. We shall discuss some general conservation tests in inelastic reactions, since these are of great interest for weak interaction theory.

3.5 (a) **Adler tests of CVC and PCAC**

Adler (26) has proposed tests, applicable to inelastic neutrino interactions, of the conserved vector current hypothesis (CVC) (2,5) and that of the partially-conserved axial vector current (PCAC) (27). Consider the reaction

\[ \nu + N \rightarrow \mu + N^* \]  \hspace{1cm} (35)

where N is a nucleon, or nucleus, of mass M, and \( N^* \) is the final hadron state, mass \( M^* \). The matrix element is the product of the nucleon and baryon matrix elements, as in (5):

\[ \mathcal{M}_0 \propto (\bar{\nu} \gamma_\alpha (1+\gamma_5) \nu) \langle N^* | V_\alpha + A_\alpha | N \rangle + \text{h.c.} \]  \hspace{1cm} (36)

Let \( k = k, k_4 \) and \( k' = k', k_4' \) be the 4-momenta of the ingoing and outgoing leptons, and take the case where \( k \), and \( k' \) are parallel (i.e. the muon is emitted in the forward direction \( \theta = 0 \) - what Adler calls the "parallel configuration").

\[
\begin{array}{c}
\vec{k} \\
\hline
\hline
\vec{k} \\
\end{array} \quad \begin{array}{c}
\pi \\
\hline
\hline
\pi \\
\end{array} \quad \begin{array}{c}
\mu, \nu \\
\hline
\hline
\mu, \nu \\
\end{array} \quad \begin{array}{c}
\vec{k}' \\
\hline
\hline
\vec{k}' \\
\end{array}
\]

In the approximation that \( m_\mu = m_\nu = 0 \), k and \( k' \) are both null-vectors (i.e. \( k^2 = k'^2 = 0 \)) and proportional to
one another. (I have attempted to show this 4-dimensional situation on the 2-dimensional drawing above.) Thus, the 3-momentum \( q \) transferred to \( M \) is numerically equal to the energy transfer \( q_0 \), so that the 4-momentum transfer \( q = k-k' \) is also a null-vector \( (q^2=0) \). The energy transfer \( q_0 = \frac{M^2 - m^2}{2M} \) and is always finite, provided \( \stackrel{*}{M} \downarrow M \), i.e. the reaction is inelastic (otherwise the Adler theorem does not work). Under these conditions, the matrix element \( \langle j_a \rangle \) of the lepton current (summed over lepton spins) can contain only one 4-vector, namely \( q_a^* \). (It is impossible to construct any other 4-vector from the two, parallel, null 4-vectors \( k \) and \( k' \)). Further, since the matrix element of the divergence of the hadron current is

\[
\langle N^* \mid \partial (V_a + A_a) / \partial x_a \mid N \rangle
\]

\[
= -i q_a \langle N^* \mid V_a + A_a \mid N \rangle
\]

(37)

it follows that the square of the matrix element (36) i.e. the reaction rate, is given by

\[
\sigma \propto \left( \langle N^* \mid \partial (V_a + A_a) / \partial x_a \mid \rangle \right)^2
\]

(38)

which is the Adler theorem.

3.5 (b) CVC

Now, the CVC hypothesis states that \( \partial V_a / \partial x_a \) is zero, so that a first consequence of the theorem is that parity-violating effects (V-A interference terms) must be absent in the parallel configuration. This can be turned round and used as a test of CVC. If CVC is valid, then for
0, in the simplest reaction to test:
\[ \nu + N \rightarrow \mu + N' + \pi(1) + \pi(2) \]
there should be no terms of the type \( \bar{p}_\mu \cdot (p_{\pi 1} \cdot p_{\pi 2}) \). No such experimental tests have been made as yet, but will clearly be possible when neutrino reactions are studied in large hydrogen or deuterium bubble chambers.

3.5 (c) PCAC

Although the axial baryon current \( A_\alpha \) is not conserved like the vector current, it has been postulated that it might be conserved in the high \( q^2 \) limit (i.e. when \( q^2 \gg m_\pi^2 \)). Indeed, as explained previously, the picture is that the divergence of the axial current is dominated by the single pion pole, and the Goldberger-Treiman relation (one form of expressing PCAC) states that
\[
\frac{\partial A_\alpha}{\partial x_\alpha} = - \frac{1}{\sqrt{2}} \frac{m_N}{m_\pi} \frac{m_\pi}{g_{\pi N}} G_A \phi_\pi
\]
where \( (g_{\pi N}/4\pi) \approx 15 \) is the pion-nucleon coupling constant, \( G_A = 1.2 G_V \) is the axial vector coupling constant in \( \beta \)-decay, and \( \phi_\pi \) is the pion field. If \( m_\pi \rightarrow 0 \), \( A_\alpha \) would then be a conserved current.

It is not difficult to see (and Adler proves it rigorously) that, for the parallel configuration where we have only axial coupling dominated by the pion pole, that there is a proportionality between the weak and strong reactions, for a given \( N, N^* \):
\[ \nu + N \rightarrow \mu(\theta=0) + N^* \quad \text{and} \quad \pi^+ + N \rightarrow N^* \]

(a) \[ \langle N^* | \delta_{\alpha \beta} | N \rangle \]

(b) \[ \langle N^* | \phi_{\pi} | N \rangle \]

Provided that \( M^* - M \gtrsim 4m_{\pi} \) (a condition which does not apply well for the \((3,3)\) resonance, but for which modifications can be made), the difference in \( q^2 \) in the 2 reactions will be unimportant. Then it is possible to calculate the cross-section for (a) in terms of the cross-section (b) and known constants:

\[
|M^*(\nu + N \rightarrow \mu + N^*)|^2 = \frac{\gamma^2}{g_{\pi N}^2} \cdot G_{\mu}^2 \left[ 1 - \frac{m_{\mu}^2 (E_{\nu} - E_{\mu})}{2(m_{\pi}^2 + m_{\mu}^2 (E_{\nu} - E_{\mu}))} \right] \]

\[ \times |M^*(\pi^+ + N \rightarrow N^*)|^2 \quad (40) \]

where the second term in the square bracket is the correction for finite lepton mass.

A first test has been made \(^{28}\) of equation (40) by comparing the observed mass spectrum of \( N^* \) in inelastic reactions for which \( \theta_{\mu} < 13^\circ \) with that predicted by equation (40) – see Fig. 5. Because of pion absorption, the observed mass spectrum is distorted. Nevertheless, the absolute numbers of predicted and observed events, for \( M^{*2} < 5 \text{ GeV}^2 \) agree remarkably well. Another way of presenting the comparison is to form the ratio of different decay channels for given \( M^* \). In Fig. 6 is shown the ratio

\[ r = \frac{\text{No. of one-pion events}}{\text{Total no. of events (2\pi)}} \]
as a function of $M^*$. Again, the calculated and observed values are in fair agreement.

Unfortunately, there are doubts whether this agreement is meaningful, in the sense that (i) even for $\theta_\mu < 13^0$, the vector contribution (zero only for $\theta_\mu = 0^0$) may be substantial; (ii) even when the condition $\theta < 13^0$ is relaxed, the fit of Fig. 6 is unimpaired; (iii) the calculation of the neutrino cross-sections has been made under the assumption that, in a complex nucleus, all nucleons contribute equally i.e. $\sigma_{\text{nucleus}} \propto A$.

It has however been pointed out that $A^3$, in complex nuclei, the pion cross-section goes as $\sim A^{3/2}$; so also then should the neutrino cross-sections for the parallel configuration. The $A^{3/2}$ law for "forward muon" neutrino interactions has been the subject of a recent experiment with Pb, Fe and Al spark chambers at CERN; no results are yet available.

3.5 (d) **Sum Rules**

As an extension of the work on the renormalization of the axial vector coupling constant $G_A$ (the Adler-Weissberger sum-rule), Adler has obtained, on the basis of axial-vector current algebra, sum rules for high energy $\nu$, $\bar{\nu}$ cross-sections for the "parallel configuration". The relations, for $\Delta S = 0$ currents, are
\[
\left\{ \frac{d^2 \sigma}{d \Omega d \mathbf{E}_\mu} \right\}_{\theta_\mu = 0} = g_V^2 \cos^2 \theta \cdot f(M^*) N_p^+(M^*)
\]

and
\[
\left\{ \frac{d^2 \sigma}{d \Omega d \mathbf{E}_\mu} \right\}_{\theta_\mu = 0} = g_V^2 \cos^2 \theta \cdot f(M^*) N_p^-(M^*)
\]

where
\[
\theta = \text{Cabibbo angle}
\]

\[
f(M^*) = \frac{1}{2\pi^2} \left[ \frac{M^2 + 2 M E_V - M^*}{M^* - M^2} \right]^2
\]

and the quantities \(N_p^+\) and \(N_p^-\) obey the sum rule:

\[
(1 - \frac{g_A^2}{g_V^2}) = \int_{M+M_A}^{\infty} \frac{4 M^* \, dM^*}{(M^* - M^2)^2} \left[ N_p^-(M^*) - N_p^+(M^*) \right]
\]

A similar expression, with \(\sin^2 \theta\) instead of \(\cos^2 \theta\), and a different constant on the LHS of eqn (41), applies to \(\Delta S = -1\) (hyperon and \(Y^*\) production) processes.

4. **Leptodynamics**

Our discussion so far has dealt primarily with the properties of the hadronic weak currents, their selection and conservation rules, and the associated form factors. These properties are determined to a large extent by the properties of the strong interactions themselves. One can imagine that there are "strong" currents which are carriers of the electromagnetic and weak "changes". Confronted with an electromagnetic field \(A_\mu\) (not to be confused with \(A_\alpha\)),
the interaction energy density is
\[ j_{\mu}^{\text{strong}} \cdot e A_{\mu} \quad (42) \]

For the corresponding weak interaction, the corresponding Lagrangian density is
\[ j_{\mu} \cdot G j_{\mu} \quad (43) \]

where \( j_{\mu} \) is the lepton current. In each case, the appropriate components of \( j_{\mu}^{\text{strong}} \) must be taken (as regards space and isospin transformation properties).

In this section we consider in more detail the properties of the lepton currents themselves, that is the structure of the weak interactions proper.

4.1 Conservation rules

(a) Lepton conservation

All leptons are given a lepton number \( \pm 1 \), as indicated in Table 1. The law of lepton conservation states that the total lepton number must be additively conserved, so that in any reaction, the number of leptons minus the number of antileptons must be the same on both sides. The best demonstration of this law is the absence of neutrinoless double \( \beta \)-decay

<table>
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<th>Lepton number</th>
<th>( e^- )</th>
<th>( \mu^- )</th>
<th>( \nu_e )</th>
<th>( \nu_{\mu} )</th>
</tr>
</thead>
<tbody>
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<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>-1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

for all other particles
Muon number

\[
\begin{align*}
+1 & \quad \mu^- \quad \nu_\mu \\
-1 & \quad \mu^+ \quad \bar{\nu}_\mu \\
0 & \quad \text{for all other particles}
\end{align*}
\]

In the high energy neutrino experiments\(^{(34)}\), violation of lepton number has been looked for, in terms of a charged lepton of the wrong sign. Neutrinos should only generate negative leptons. The limit to this form of lepton violation is <1%.

(b) Muon number conservation

The existence of two neutrinos, \(\nu_e\) and \(\nu_\mu\), coupled with electrons and muons respectively, was first established many years ago (1962). The muon number, defined in the above table, is additively conserved. High energy neutrino beams are generated by decay in flight of pions and kaons, and are therefore predominantly of the type \(\nu_\mu\). There is a background of \(\nu_e\) (\(\sim 0.5-1\%\)) from the decay \(K^+ \rightarrow \pi^0 e^+ \nu_e\). The small proportion of events in which the emerging lepton is an electron rather than a muon, can all be attributed to this background.

Both \(\nu_\mu\) and \(\nu_e\), according to the 2-component theory, have zero rest-mass. The experimental limits on the mass are <0.2 keV for \(\nu_e\) (from \(K^0\) \(\beta\)-decay), and <1 MeV for \(\nu_\mu\) (from \(\pi^-\mu\) decay). A substantial prize should be awarded to anyone bright enough to think of a way to reduce the \(\nu_\mu\) limit by an appreciable factor. It is difficult to
believe that both masses are identically zero, otherwise $v_e$ and $v_\mu$ apparently would have identical properties, and not "know" whether to produce $e$ or $\mu$ in their subsequent interactions. In the distant future, it might be possible to demonstrate the finiteness (or otherwise) of the mass of the neutrino, from its electromagnetic interaction (see below) and the existence (or not) of a magnetic dipole moment.

4.2 The intermediate vector boson

(a) Historical

The idea that weak interactions might be mediated by a charged, spin 1, vector boson $W$, just as electromagnetic interactions are mediated by the photon, is a very old one, and indeed is even implicit in the original formulation of the Fermi theory of $\beta$-decay. (It will be remembered that Fermi wrote down the $\beta$-decay interaction in analogy with the electromagnetic coupling). On the $W$-theory, the two types of couplings are as follows:

The coupling of a single charged particle (or lepton) to the electromagnetic (or weak) field is then:
The 4-lepton point interaction yields a cross-section

$$d\sigma(\text{point}) = (\frac{g}{\sqrt{2}})^2 dq^2$$

whilst the boson theory gives

$$d\sigma(\text{boson}) = (g^2)^2 \frac{dq^2}{(M_w^2 + q^2)^2}$$

For $q^2 \ll M_w^2$, the two cross-sections are identical provided

$$\frac{G}{\sqrt{2}} = \frac{g^2}{M_w^2}$$

Since $g = (\frac{G}{\sqrt{2}})^{\frac{1}{2}} M_w = 10^{-2.5} \frac{M_w}{M_p}$, the coupling of the boson to the lepton current is intermediate in strength between strong and weak interactions - hence the terminology "intermediate boson". The detailed properties of the $W$-boson have been discussed at length by Lee(36) and others; its magnetic moment should be close to the normal value
g=1, and its quadrupole moment should be close to zero. The W decays into leptons

$$W \rightarrow \ell + \nu$$  \hspace{1cm} (47)

with a rate

$$R_\ell \sim \frac{g^2 M_W}{6\pi} = 5.6 \times 10^{17} \left(\frac{M_W}{M_p}\right)^3 \sec^{-1} \hspace{1cm} (48)$$

Since the W can also decay into pions e.g. $W^+ \rightarrow \pi^+ \pi^0$, the overall rate $R_{\text{tot}} = R_\ell + R_\pi + \ldots$ will exceed (48).

Until 1962, the only information one had on the W mass was that $M_W > M_K$, since the process $K \rightarrow W + \gamma$ was not observed. Since then, the limit has been increased to $M_W > 2 \text{ GeV}$, as discussed below, but there is as yet no real evidence, direct or indirect, for its existence.

(b) Production of the $W$-boson

Serious attempts have been made to detect the $W$-particle in both strong interactions and in neutrino experiments.

(1) The Columbia group have looked for evidence of the $W$ in the reaction

$$p + p \rightarrow W + 2n + \ldots \hspace{1cm} (49)$$

by firing a 30 GeV proton beam into a massive shield, and observing energetic muons emerging from the shield at wide angles. If the $W$ is very massive, it will decay with branching ratio B in the mode $W^+ \rightarrow \mu^+ + \nu$ with a large Q-value; the muons can therefore possess transverse momenta (relative to the proton beam) up to $(p_T)_{\text{max}} \sim \frac{M_W}{2}$. 
Of course, muons are produced prolifically from decay in flight of pions generated in the same target, but with an exponential $p_T$ distribution, of coefficient $\sim 0.2$ GeV. Thus the experiment seeks to detect an anomaly in the $p_T$ distribution of muons emerging from the shield. None was found, and it was concluded that, if $M_w < 6$ GeV, the product
\[ \sigma_w B < 4 \times 10^{-34} \text{ cm}^2/\text{nucleon} \quad (50) \]

The limit of 6 GeV was imposed by the proton energy available, even with the help of a bound target proton (e.g. in a copper nucleus), so that advantage could be taken of Fermi motion to boost the cms energy. No reliable calculations of $\sigma_w$ in (49) can be made, to compare with the result (50). It seems likely however, that even if $M_w$ was somewhat below the 6 GeV limit, the expected cross-sections would be well below the limit observed.

(2) Both BNL and CERN groups have searched for the electromagnetic production of $W$'s by neutrinos. The relevant diagrams for this process are as follows:

Diagram (b) is the more important. $Z$ stands for a nucleon or nucleus. When the minimum momentum transfer to the nucleus
\[ q_{\text{min}} \approx \frac{M_w^2}{2E_v} \quad (51) \]

is such that \( q_{\text{min}} < 1/R \) where \( R \) is the nuclear radius, the whole nucleus acts coherently. At very high energies, the cross-section then becomes

\[ \sigma_{\text{coherent}} = \frac{GZ^2}{6\pi \lambda^2} a^2 \left( \ln \frac{2E_v M_w}{W} \right)^3 \quad (52) \]

\[ \sim GZ^2 a^2 \sim 10^{-37} \text{ cm}^2 \]

Note that, because the \( W \) is coupled semi-weakly, the cross-section \( \sim G\lambda^2 \) is considerably higher than for the first-order weak interaction, \( \sim G^2 \).

Near threshold, on the other hand, the coherence condition is far from satisfied, and the cross-section is proportional to the number of nucleons, \( Z \). Many detailed numerical calculations have been made of \( \sigma_{\text{coherent}} \) and \( \sigma_{\text{incoherent}} \) for neutrino energies up to 20 GeV, and \( M_w \) up to 2.5 GeV. The most recent is that of Wu et al.\(^{(38)}\).

Subject to assumptions about the boson gyromagnetic ratio - which do not change the cross-section very much - most of the calculations agree to within 30% or so.

The experiments\(^{(37,39)}\) carried out in CERN and BNL in 1963/4 searched, in large magnetized spark chambers, for energetic pairs of muons of opposite charge - see the above diagrams; and in a heavy liquid bubble chamber, for evidence of the decay \( W \rightarrow \) pions. I shall not describe these in detail. The result was that, if leptonic decay
modes $W \rightarrow \ell + \nu$ dominate, $M_w > 2$ GeV; and if pionic decay dominates, $M_w > 1.7$ GeV.

(c) Future possibilities

Reference to (51) shows that, for a given cross-section for $W$-production, the typical neutrino energy $E_\nu$ required increases as $M_w^2$. Since the typical neutrino energy, under optimum conditions of pion decay path and shield length, rises roughly as the bombarding proton energy, the boson mass limit which could be set at other accelerators varies crudely as $(\text{proton energy})^{3/2}$. Integration and extrapolation of the Wu et al cross-sections over the neutrino energy spectra confirms this variation. Because of the intense importance of the $W$-boson for our understanding of weak interactions, it is worth displaying the results obtained or to be (perhaps) expected in the future:

<table>
<thead>
<tr>
<th>Year</th>
<th>Available proton energy (GeV)</th>
<th>$M_w$ limit</th>
</tr>
</thead>
<tbody>
<tr>
<td>1960</td>
<td>-</td>
<td>$&gt; 0.5$ GeV (from $K \rightarrow W + \gamma$)</td>
</tr>
<tr>
<td>1964</td>
<td>25/30</td>
<td>$&gt; 2$ GeV</td>
</tr>
<tr>
<td>1972?</td>
<td>76 (Serpukhov)</td>
<td>$&gt; 4$ GeV</td>
</tr>
<tr>
<td>1977?</td>
<td>450 (Europe or USA)</td>
<td>$&gt; 9$ GeV</td>
</tr>
</tbody>
</table>

These figures refer to a fixed number of accelerated protons; for a given beam, the event rate falls off exponentially with increasing $M_w$, so that the proton energy is more important than running for extended periods, or very
massive detectors.

Theoretical attempts have frequently been made to guess at the probable boson mass (if it exists) by carrying out higher-order calculations in weak interactions, and inserting a cut-off (e.g. a boson mass) to eliminate divergences. To quote a recent example, the Rochester group (10) obtain a rather low cut-off, \( \Lambda \sim 4 \text{ GeV} \), in equating the calculated \( K_L - K_S \) mass difference with the observed value.

4.3 Locality of the lepton current

Although the 4-fermion point interaction of the Fermi theory is consistent with experimental results to date, it was pointed out by Heisenberg over thirty years ago that it led to nonsensical results at high energies. Thus, for the neutrino-electron scattering process,

\[
\nu + e \rightarrow e + \nu
\]

where no form factors are involved, the cross-section is proportional to phase-space i.e.

\[
\sigma \sim \frac{q^2}{\pi} \frac{E^2}{E_c^2}
\]

where \( E_c \) is the cms energy. On the other hand, from ordinary wave theory we know that, for s-wave scattering,

\[
\sigma_{\text{max}} = \frac{i\pi}{2} \kappa^2
\]

where the \( \frac{1}{2} \) comes from the average over initial lepton spin. \( \kappa = 1/p_{\text{cms}} \rightarrow 1/E_c \). Thus \( \sigma_{\text{Fermi}} = \sigma_{\text{max}} \) when \( E_c^2 \sim \pi/G \sim 10^5 M_p^2 \); so when \( E_c > 300 \text{ GeV} \), the cross-section exceeds the unitarity limit. This means that the Lagrangians written
down earlier are only the ones effective at very low energies. For a proper field theory of weak interactions, it is necessary to include higher-order terms, so that $\nu, e$ scattering becomes (in terms of a boson theory):

\[
\begin{align*}
\nu & \rightarrow \nu & \nu \rightarrow \nu \\
\nu & \rightarrow e^+ & e^- \rightarrow e^+ \\
\nu & \rightarrow \nu & \nu \rightarrow \nu
\end{align*}
\]

A similar situation exists in quantum electrodynamics. Higher-order terms give rise to divergences, but it is found that all these can be dumped into the mass and charge of the electron, which we know are finite, no matter what the mathematics says. This procedure is known as renormalization. The remaining, finite terms give answers agreeing with experiment to very high accuracy. Unfortunately no-one has yet been able to find a weak interaction field theory (with or without $W$'s) which can be renormalized\(^{44,45}\).

In this situation, it is certain that experiments directed towards much higher energies, or to processes which can only occur to higher order in the weak interaction will be vitally important. We discuss these experimental possibilities in the following sections.

4.4 Neutrino Electron scattering

The allowed (first-order) reactions are:

\[
\begin{align*}
\nu_e + e^- & \rightarrow \ell^- + \nu_e \\
\bar{\nu}_e + e^- & \rightarrow \ell^- + \bar{\nu}_e
\end{align*}
\]
the appropriate diagrams being

\[ \nu_e \quad \text{and} \quad \bar{\nu}_e \]

The threshold neutrino energy, for a stationary electron target, is

\[ (E_\nu)_{\text{min}} = \left( \frac{m_\ell^2 - m_e^2}{2m_e} \right) = 10 \text{ GeV for } \ell = \mu \]

\[ = 0 \text{ for } \ell = e \] (57)

If one looks at (55) in the CMS, one can see that it proceeds via \( J=0 \)

\[ p_1 \rightarrow \nu_e \rightarrow p_2 \rightarrow p_3 \rightarrow e^- \]

so that the CMS angular distribution must be isotropic. Indeed the squared matrix element for (55) has the form

\[ |\mathcal{M}|^2 \propto \frac{(p_1 p_2)(p_3 p_4)}{E_1 E_2 E_3 E_4} \propto \text{constant} \] (58)

where \( p_1 \ldots p_4 \) are the 4-momenta indicated. Reaction (56) is obtained from (55) by the interchange \( p_2 \leftrightarrow p_4 \), yielding

\[ |\mathcal{M}|^2 \propto \frac{(p_1 p_4)(p_2 p_3)}{E_1 E_2 E_3 E_4} \propto (1 - \cos \theta)^2 \] (59)

Thus, for the antineutrino reaction, the charged lepton is peaked backwards in the CMS. In the laboratory system, the leptons in either case are emitted under very small angles,
of order $\sqrt{\frac{m_e}{E_y}}$. The total cross-sections are:

$$\sigma_y \approx \frac{G^2}{\pi} E_c^2 (1 - \frac{m_e^2}{E_c^2})^2; \sigma^\gamma_y = \sigma_y / 3$$

(60)

where $E_c^2 \approx 2m_e E_y(lab)$, is the (CMS energy)$^2$. The low target mass, and consequently small momentum transfer, keeps the cross-section down in the region of 1% of the $\nu$-nucleon elastic cross-section, for any possible $\nu$-beam to operate before 2000 A.D. Typical members are:

**Table 3**

| $\nu_\mu + e^- \rightarrow \mu^- + e_\nu$ |

<table>
<thead>
<tr>
<th>$E_y$ GeV(lab)</th>
<th>$q^2_{max}$ (GeV$^2$)</th>
<th>$\sigma_{tot}$ cm$^2$ per electron</th>
<th>$\frac{\sigma_{ve}}{\sigma_{np}}$</th>
<th>$\frac{\langle \theta \rangle_{\mu}^l_{ab}}{\sqrt{2m_e E_y}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>-</td>
</tr>
<tr>
<td>20</td>
<td>0.01</td>
<td>$5 \times 10^{-41}$</td>
<td>$10^{-2}$</td>
<td>$7$ mr</td>
</tr>
<tr>
<td>60</td>
<td>0.05</td>
<td>$4 \times 10^{-40}$</td>
<td>$6 \times 10^{-2}$</td>
<td>$4$ mr</td>
</tr>
<tr>
<td>300</td>
<td>0.29</td>
<td>$3 \times 10^{-39}$</td>
<td>0.5</td>
<td>$1.5$ mr</td>
</tr>
<tr>
<td>$10^8$</td>
<td>$10^5$</td>
<td>$(\sqrt{2})X^2 = 2.5 \times 10^{-32}$</td>
<td>$10^{-7}$</td>
<td>$10^{-2}$ mr</td>
</tr>
</tbody>
</table>

It will be seen that, apart from the low cross-section, the range of $q^2$ is not much greater than that in $\mu$-decay, so that deviations from the Fermi theory are not expected. Such reactions are therefore unlikely to be of great H.E. physics interest. (On the contrary, feeble cross-sections
are no hindrance on the scale of our universe, and the \( \nu_e - e \) reaction is of considerable cosmological significance.)

4.5 **Neutral currents and electromagnetic scattering**

All evidence at present suggests the absence of first-order neutral lepton currents, which would result in non-charge-exchange neutrino scattering:

\[
\begin{align*}
\nu & \quad \rightarrow \quad \nu \\
\mid & \quad \rightarrow \quad W^0 \\
\mid & \quad \rightarrow \quad p \\
\phi & \quad \rightarrow \quad \nu + p
\end{align*}
\]

Thus, processes such as \( K^0 \rightarrow \mu^+ + \mu^- \)
do not occur, although no other reason exists for their suppression.

In the CERN neutrino experiment, the bubble chamber

data puts a limit on the cross-section for \( \nu + p \rightarrow \nu + p \),
of < 10% of that for the charge-exchange process
\( \nu + n \rightarrow \mu + p \), for \( q^2 \geq 0.5 \text{ GeV}^2 \). The \( q^2 \) limit was
imposed to reduce effects of neutron background. With
improvements in technique (and particularly the use of
hydrogen chambers) it will be possible to considerably
improve the limit on the neutral current coupling.

Even in the absence of direct \( W^0 \) coupling, the
neutrino can undergo "contact" electromagnetic interactions
by virtue of its charge form factor \((41, 42)\). In general, the
electromagnetic current of the neutrino contains up to 4
form factors. For
a particle of zero rest-mass, magnetic and electric dipole, and electric quadrupole, moments vanish, and one can have only a spherically symmetric charge distribution, (a). The form factor is then of the form

\[ F(q^2) = -\left( e/6 \right) q^2 \langle r^2 \rangle \; ; \; F(0) = 0 \]  \hspace{1cm} (61)

where \( \langle r^2 \rangle \) is a mean square charge radius. The factor 'e' in (61) tells us that the cross-section for the process shown in (b) is \( \sigma_{\nu} (\text{e.m.}) \sim a^2 \sigma_{\nu} (\text{elastic}) \) i.e. \( \sim 10^{-42} \text{cm}^2 \) at \( E_{\nu} \sim 1 \text{ GeV} \).

For a boson of g=1, Lee and Sirlin have calculated \( \langle r^2 \rangle \) exactly and find

\[ \langle r^2 \rangle = \frac{3G}{8\sqrt{2}\pi} \left[ \ln \frac{5}{3} 137 - 2 - \frac{8}{3} \ln \frac{M_w}{m_e} \right] \]  \hspace{1cm} (62)

which has the nice feature that it increases with \( M_w \). For \( M_w = 5 \text{ GeV} \),

\[ \langle r^2 \rangle_\nu \approx 0.5 \times 10^{-33} \text{ cm}^2, \quad \langle r^2 \rangle_\mu = 2.2 \times 10^{-33} \text{ cm}^2 \]

\( \left( \frac{d\sigma}{dq^2} \sim a^2 \langle r^2 \rangle^2 F^N (q^2), \right. \) where \( F^N (q^2) \) is the nucleon e.m. form factor.

The experimental limit on \( \langle r^2 \rangle_\nu \) is \( \leq 10^{-29} \text{ cm}^2 \). This
comes from the Reines-Cowan experiment, and from astrophysics. In the latter case, it is conjectural that, in stellar interiors, the photons in the plasma have an effective mass \( \sim KT \), and thus, by means of the reactions
\[
\gamma \rightarrow e^+ + e^-; \quad e^+ + e^- \rightarrow \nu + \bar{\nu}
\]
can transform into neutrino pairs, which escape and thus drain the thermonuclear energy source directly; from the observed evolutionary rate one can thus put limits on the neutral current cross-section.

The present limit on \( \langle r^2 \rangle_{\nu_\mu} \) is \( <10^{-32} \text{ cm}^2 \), which corresponds to a limit on cross-section \( \sim 500 \) times greater than (62) predicts. In the future, it will be of great interest to push the experimental limit down as far as possible, although it is doubtful if one will be able to approach the predicted value, because of the low rate and the monumental problem of experimental (neutron) background.

Electromagnetic scattering of neutrinos can take place on electrons as well as on nucleons. The cross-section is down by a factor \( \alpha^2 \) on that of equation (60). However, Lee and Sirlin show that there is an interference term between weak \( \nu-e \) and electromagnetic \( \nu-e \) scattering, which term is only down by a factor \( \alpha \), and thus offers a faint possibility of experimental investigation.

4.6 Total inelastic cross-sections on nucleons

The discussions in sections 4.4 and 4.5 show that the possibilities of investigating the purely leptonic currents
in the high $q^2$ region, or of second-order effects which might reveal evidence of non-locality, are fairly bleak. The only remaining possibility of testing locality was already discussed in section 3.1, eqn. (15). It is perhaps worth discussing this somewhat further.

We expect the breakdown of the Fermi theory to occur at high $q^2$, and thus, since we are limited to low neutrino energy ($E_\nu < 100$ GeV or so in the foreseeable future), we are compelled to use nucleon targets. If we neglect the lepton mass, the maximum $q^2$ which can be explored with a neutrino energy $E_\nu$ and target mass $M$, in an elastic collision, is

$$ (q_{\text{max}}^2)_{\text{el.}} = \frac{4M^2E_\nu^2}{(M^2+2ME_\nu)} $$

(63)

Thus, the maximum possible variation of $q_{\text{max}}^2$ - and hence $\sigma$ - is as $(\text{neutrino energy})^2$. For $E_\nu > M$, the variation is more nearly linear with $E_\nu$ and $M$. For an inelastic collision, giving a final state mass $M^*$, we have

$$ (q_{\text{max}}^2)_{\text{inel.}} = (q_{\text{max}}^2)_{\text{el.}} \left[ 1 - \left( \frac{M^*-M^2}{2ME_\nu} \right)^2 \right] $$

(64)

For neutrino energies in the GeV region, and nucleon targets, and $M^* < E_\nu$, one therefore expects $q_{\text{max}}^2 \sim 2E_\nu$ GeV$^2$. This limit is generally large compared with that imposed by the strong-interaction form factors, for the region $E_\nu > 5$ GeV.
The total neutrino cross section, as observed in the CERN HLBC data - see Fig. 7 - increases rapidly in the region up to about 7 GeV, at least as fast as $E_{\nu}$. Above 7 GeV the statistics are too poor to draw reliable conclusions. The increase in $\sigma$ is certainly partly due to increase in the range of $q^2$ - (63) and (64) - in a particular channel, but presumably mostly due to the increasing number of channels which open as the energy increases. Thus, $\sigma_{\text{total}}$, as the sum of individual channels, rises as indicated in the sketch.

As we have seen from equation (5), the total cross-section, for a particular $M^*$ channel, must reach an asymptotic value $\int C(q^2) dq^2$ as $E_{\nu} \to \infty$, if the hypothesis of a local leptonic current is valid. Without a reliable model of high-energy strong interactions, we do not know the form factors A, B and C ($= f(q^2, M^*)$), but it is clear that, on this hypothesis, the cross-section will grow indefinitely with energy. Thus, a failure of the locality assumption might well be manifest by a damping out of the high $q^2$ reactions, or a dependence, for a given $M^*$ and $q^2$, of the cross-section on $E_{\nu}$. This situation is familiar in strong interactions, where the total cross-section flattens off and in any one
channel (e.g. the elastic channel) the cross-section at even moderate energies (GeV) starts to decrease. The main points are that high energy neutrino collisions on nucleon targets offer the possibility of reaching values of $q^2$ in excess of the present limit on $M_w^2$; that the value of $q_{\text{max}}^2$ rises much faster with increasing proton energy than does $M_w^2$; that we do not have any idea how the eventual breakdown of the point interaction will be manifest. A serious investigation of the total neutrino cross-sections, particularly at the new generation of accelerators, will therefore be highly interesting and possibly even lead to unexpected breakthroughs in our understanding of the weak interactions.
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a) ELASTIC NEUTRINO EVENTS WITH $E_{\text{vis}} > 1$ GeV

- $M_A = 0.81$ GeV
- $M_A = 0.50$ GeV

b) ELASTIC ANTI-NEUTRINO EVENTS WITH $E_{\text{vis}} > 0.5$ GeV

Fig. 1
Fig. 2. Elastic neutrino and antineutrino cross-sections for "best fit" value of $M_A$. 
(a) Neutrino cross-section for single pion production, with different values of $M_A$.

\begin{align*}
\sigma_{\pi^0} \times 10^{-36} \text{cm}^2/\text{nucleon}
\end{align*}

\begin{align*}
\sigma_{\pi^0} \times 10^{-36} \text{cm}^2/\text{nucleon}
\end{align*}

(b) Neutrino events $\nu$ and antineutrino events $\bar{\nu}$.

Fig. 3
Fig. 4. Invariant mass distribution of "single-pion events"
Muon angular distribution in all inelastic events.

319 EVENTS

Distribution of $M^2$ for events of $\cos \theta > 0.975$

Comparison of data with the Adler prediction.

Fig. 5. Tests of Adler Theorem.
$q^2$ distribution of events of $\cos \theta_1 > 0.975$

Ratio (1π events)/(all inelastic events) for parallel configuration, and comparison with the Adler prediction.
THREE LECTURES ON BARYON RESONANCES

M. Ferro-Luzzi

CERN
Geneva
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THREE LECTURES ON BARYON RESONANCES

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FIRST LECTURE

I. Introduction

Avowed purpose of this school is to give a theoretical background to young experimentalists. So, I don't know where the baryon resonances come in. The art here is mainly at an experimental stage and the best thing that most people could learn is how to deal with the experimental data. Still, in order to comply with the above purpose, I will try to skip all or almost all of the experimental details and concentrate on what remains: the results and their meaning. Theoretical interpretations of baryon resonances as such, particularly when it comes to quantitative predictions, are rather skimpy. Detailed calculations do not exist and the predictions of the various symmetry schemes are somewhat flexible. The reason for all this is probably that much more information has to be put together on the experimental side before realistic calculations can be performed. What remains, then, and what I will try to present here, is a phenomenological description of these resonances and some general recipe on how they can be studied. The emphasis will be on methods rather than on systematics. Special cases will be discussed only when illustrative of a certain procedure. Although the bibliography will contain as much as possible of the relevant literature, these lectures have no pretense of offering an updated review of baryon resonances. Examples of the latter do come out regularly once a year and those interested are referred to them. Finally, I should mention that most of the examples and approaches discussed are related to the work of the CES group\(^1\) during these last few years and should not be taken as the most objective and unbiased view of the subject.
Since there are only three lectures, the best approach is to start as quickly as possible. The plan of the lectures can be understood from the table of contents. The first subject to be treated will be the hyperon resonances; along with them, the basic definitions and necessary formalism will be recalled.

II. Production and formation experiments

I suppose that everybody here is already familiar with the quantum numbers of the elementary and not-so-elementary particles. For those who aren't, the Rosenfeld tables 2) will provide an exhaustive collection of such and other numbers. Hyperon resonances have in common a baryon number $B = 1$ and a strangeness $S$ different from zero. Convenient historical symbols 3) are $\Lambda$ for $S = -1$ and isotopic spin $I = 0$, $\Sigma$ for $S = -1$ and $I = 1$, $\Xi$ for $S = -2$ and $I = 1/2$, $\Omega$ for $S = -3$ and $I = 0$. A generic symbol $Y^\pi$ is also frequently used to denote any hyperon resonance. Additional conventions do exist, identifying the angular momentum $J$ and parity $P$ of the particles 3), but their usefulness is strongly limited by the very lack of knowledge existing in many cases about those numbers.

There are many possible ways of presenting the known hyperon resonances, from a simple table of masses to a sophisticated (and often unwarranted) collection of $SU(3)$ multiplets perhaps even arranged along Regge-trajectories. In fig. 1 the representation chosen 4) has the advantage of exhibiting the most relevant experimental features while avoiding any specific theoretical point of view. All reasonably well established $S = -1$ resonances are here plotted on a "mass" versus "$J^P$" diagram. When the latter quantity is not known, a vertical line replaces the point. Open circles indicate uncertainty in $J^P$ attribution or resonance existence. The horizontal bars indicate the full width $\Gamma$ of the resonances. Only $\Lambda$ and $\Sigma$ type of resonances are shown, no other types being known at present.
The following features are worth digressing on:

a) Points and lines are crowded towards low masses; it looks like resonances fade away with increasing mass. This is a simple reflection of the fact that the low mass region has been studied first and more thoroughly. A few years ago the whole map would have been just as unpopulated as its present high mass region.

b) The values of $J$ seem to increase with mass. Perhaps some physical reality hides behind this. Still, it is a fact that resonances with high $J$ are more easy to detect, particularly at high energy.

c) The width also becomes larger with increasing mass. Here again one should keep in mind that a good energy resolution, such as is necessary in order to detect narrow resonances, is increasingly lacking when going up in energy.

Let us now quickly review the methods and techniques for finding and studying these resonances. Figs. 2 and 3 show two examples of what is known as a "production" type of experiment: the oldest\(^5\) and one of the most recent\(^6\), respectively. The general procedure consists of investigating
(Year 1965)

$\bar{p}p$ annihilations at 5.7 GeV/c.

(a) KN and KN$\pi$ combinations.

(b) $\Sigma\pi$ combinations

\begin{align*}
1520 & & 1660 & & 1820 & & 2095 & & 2300 \\
1405 & & 1660 & & 1820 & &
\end{align*}

\begin{align*}
\bar{p}p & \rightarrow \Lambda K \\
\Lambda K & \rightarrow \Sigma K \\
\Sigma & \rightarrow \Sigma\pi
\end{align*}

Fig. 2

Fig. 3
the spectrum of effective mass of two (fig. 2) or more particles (fig. 3) emitted in certain reactions. All enhancement not explainable in terms of kinematics (for example "phase-space") or statistical fluctuations is assumed to come from the decay of a resonance, its height being related to the production rate and its mass spread to its width. Ambiguities in interpretation arise when several resonances are simultaneously produced, as in the overlap region of the plot of fig. 2 and in the more complicated combinations possible for most of the reactions of fig. 3. The sketches of fig. 4 show how a 3-particle final state (the simplest in this type of experiment) can originate from three distinct resonant configurations.

![Fig. 4](image_url)

It is clear from just these examples how anything more than an attribution of mass, width and isospin must heavily depend on specific assumptions on the final state interactions of the particles present, on the production mechanism and other possible effects which are usually far from well understood. This is reflected on the rarity of meaningful $J^P$ determinations achieved through these experiments. Reference 7 discusses in detail the methods devised for measuring spin and parity of resonances "produced" in the above manner. It should be noticed, finally, that, in addition to the above mentioned problems, these types of experiments as performed in practice are usually very much limited by lack of statistics.

The situation of the $S = -2$ resonances, shown in fig. 5, exemplifies dramatically the difficulties of the "production" experiments; only the latter are indeed available in this case. The resonances are much fewer,
uncertain and little is known about their quantum numbers. The statistical limitations are here much more severe than for the $S = -1$ case. This can be seen, for example, in the collection of data from a large experiment\(^6\) reproduced in fig. 6. One can imagine, of course, that there is a natural lack of these resonances; still, it may be more likely that the experimental difficulties are such as to hinder both their identification and the determination of their quantum numbers.
Turning now to the other possible approach - the one referred to as "formation" - Fig. 7 shows a schematic parallel between this and the previous approach. The advantage here is that no third partner is around to confuse things; the limitation, that only those resonances which are sufficiently well coupled to the experimental systems used in the "formation" can be studied. Apart from the problems introduced by the "background" (reactions not proceeding through the resonant state), a study of the angular distributions of the final state provides all the information necessary to determine the quantum numbers of the decaying state. Figs. 8 and 9 give two examples, one very early, one very early, the other more recent, of how the formation of resonances may appear in practice. The process in Fig. 8 shows one aspect of the formation of Λ(1520) via K⁻p collisions. The resonance is here seen to decay into the Σπ⁻ mode: Σ⁺π⁻, Σ⁻π⁺ and Σ⁰π⁰. The formation is superposed to a sizeable background of non-resonant processes. The resonance being in a 3/2⁻ state and the background in a predominantly 1/2⁻ state, the change in differential cross section when traversing the resonant energy is quite drastic. A detailed analysis of this change and the related variations in the other
channels ($K^+ p$, $K^0 n$, $\Lambda n^+\pi^-$) allows a complete identification of the resonance. Although by today's standards the experiment would not be considered as statistically very rich, the information provided was sufficient to pinpoint all the relevant quantum numbers and branching ratios of the resonance.

Fig. 9 shows, for a similar situation, how even a simple but accurate measurement of the total $\bar{K}N$ cross section (separated into its two isospin components) can already yield an impressive amount of useful information on resonances whose $\bar{K}N$ branching fraction may be as small as a few percent. Mass, width, isospin and the product $(J + 1/2)x$ where $x$ is the elastic branching fraction (the "elasticity" of the resonance), is the information obtainable through this type of measurements. Any further information, like $J^P$ and branching fraction into different channels, requires additional measurements of the type mentioned in connection with fig. 8.

Before ending this hurried survey of the methods to identify resonances, one more possibility is worth mentioning. It may happen that the decay products of resonances occurring in formation experiments are themselves other resonances which then decay according to their own mode. Fig. 10 shows examples of this mixed "formation - production" behaviour: case (a) has actually been observed, (b) is a possibility which for the moment needs more
statistics to be confirmed. Apart from the folkloristic side, these phenomena are quite useful. Angular correlations directly connected with the $J^P$ of the various resonances will be present throughout the decay chain; if the statistics are sufficient and the resonances are well-behaved (widths small, backgrounds low, etc.) then these correlations can be easily exploited to determine the unknown $J^P$ values. This was the case for example (a) of fig. 10, where the known information was the $J^P$ of $\Lambda(1520)$ and the unknown was the $J^P$ of $\Sigma(1760)$. Fig.11 is an excerpt from ref.11 showing the evidence for $\Sigma(1760)$ formation and subsequent decay into $\Lambda(1520)$. A discussion of the angular correlations and the details of the analysis can be found in the above reference.

III. Partial wave decomposition

From now on we shall concentrate on the "formation" type of experiments and, more in general, on what is called a "partial wave analysis". A brief review of the most relevant formulae connected with this analysis is in order first, so as to lay the ground for the next lecture. In an accelerated
course such as this there is clearly no time to go into derivations or
comprehensive discussions of the formalism outlined below. I will simply
list the main results and will convey a visual impression of how and from
where they come out. Full treatments can be found practically in any of the
available textbooks or lecture notes (see, for example, refs. 4 and 13).

Referring first to the simpler case of a spinless particle incident on
a spinless target, Figs. 12 to 14 recall how the incident beam can be thought
of as a plane wave, how the latter can be expanded into an infinite sum of
spherical waves corresponding to all possible values of the orbital angular
momentum ℓ, how the effect of the target can be represented by a "phase shift"
and an "absorption" of the outgoing spherical waves, how finally the scattered
wave can be factorized into a radial term and a term depending only on the
scattering angle θ. The latter is called "scattering amplitude" and its
components are products of Legendre polynomials P_ℓ(cos θ) times complex
numbers called "partial wave amplitudes". It is the knowledge of these
partial waves which is sought in the analyses discussed in the next lecture,
BEAM OF PARTICLES INCIDENT ON A SPINLESS TARGET.

\[ e^{i k r} \sum_{l=0}^{\infty} \frac{1}{2k} \left\{ (-1)^l e^{-i k r} + e^{i k r} \right\} P_l (\cos \theta) \]

Fig. 12

The effect of the target is only on the outgoing spherical wave:

\[
\begin{align*}
\text{Target off:} & \quad e^{i k r} \\
\text{Target on:} & \quad e^{i k r} e^{2i\phi} = e^{i k r} \eta e^{i 2\phi}
\end{align*}
\]

\[ \eta : \text{absorption parameter} \]
\[ \phi : \text{phase shift} \]

All this depends on \( l \), of course

If \( \eta = 1 \), there is no absorption = all particles emerging do also come out.
If \( \phi = 0 \), there is no scattering = beam is undisturbed.

Fig. 13
Thus, according to its energy dependence, the \( l \)-th partial wave will be labelled as "resonant" or as "background" or as a mixture of the two. It is then important to examine in some more detail how the partial waves are related to the observable quantities. What can be measured, in practice, is the flux of scattered particles in a certain direction, normalized to the flux of incident particles; this is what is called "scattering" cross section. Fig. 15(a) shows the connection between the partial waves and the "scattering" or "elastic" cross section in the differential and integrated form. When the final state particles are not the same as in the initial state, then the partial waves describing the phenomenon are those referring to the "absorption" or "reaction" processes. In a manner analogous to that of Fig. 15(a) one can show that the flux of the absorbed part of the incident wave, again normalized to the incident flux, is related to the "absorption" or "reaction" cross section through the expression of fig. 15(b). Due to the unitarity requirements, the two cross sections, elastic and reaction, must be related to one another. The connection, as can be seen in fig. 15, is through the absorption parameter \( \eta \). Unitarity requires that \( \eta^2 \) should not exceed 1. Notice that the
reaction cross section should be understood as the sum over all possible reaction channels; one can, of course, subdivide $\eta$ into as many absorption parameters, $\eta_i$, as there are reaction channels, with the unitarity condition becoming then $\sum \eta_i^2 \leq 1$. The relation between $\sigma_{el}$ and $\sigma_r$ is given graphically in fig. 16 for different values of the phase shift $\delta$. From the above results and the scattering amplitude in fig. 14 one can easily put together the so-called "optical theorem" (fig. 17). The latter is quite useful because it connects our unknown partial wave amplitudes to an easily measurable quantity, the total cross section.

The next three figures, from 18 to 20, provide the extension to the case of a spin 0 particle incident on a spin 1/2 target. This is our practical case of interest, applying to $\pi N$ and $KN$ scattering. A straightforward extension, valid for the integrated cross sections is shown in fig. 18 and consists in replacing the statistical factor $2l + 1$ with $J + 1/2$ and introducing for each value of $l$ a partial wave amplitude $T_l^J$ corresponding to $J = l + 1/2$ and another $T_l^{-}$ corresponding to $J = l - 1/2$. The extension of the differential cross section to this case is not straightforward and
OPTICAL THEOREM

\[
\sigma_{d} = 4\pi k^{2} \sum_{l} (2l+1) \left( \frac{\xi_{l}}{\xi_{0}} \right)^{2}
\]

\[
\sigma_{r} = \pi k^{2} \sum_{l} (2l+1) \left( 1 - \eta_{l}^{2} \right)
\]

\[
\sigma_{d} + \sigma_{r} = 2\pi \sum_{l} (2l+1) \left( 1 - \eta_{l} \cos 2\theta_{l} \right)
\]

\[
\sigma_{\text{tot}} = \frac{4\pi}{k} \Im \phi(\theta)
\]

is "optical theorem".

scattering amplitude \( [d\theta/d\Omega + \Im(\phi)] \)

\[
\phi(\theta) = \frac{1}{k} \sum_{l} (2l+1) \left( \frac{\xi_{l}}{\xi_{0}} \right)^{2} P_{l}(\cos \theta)
\]

\[
\phi(\theta,\phi) = \frac{1}{k} \sum_{l} (2l+1) \left( \frac{\xi_{l}}{\xi_{0}} \right)^{2} P_{l}(\cos \theta) \sin \phi
\]

\[
\Im \phi(\theta) = \frac{1}{k} \sum_{l} (2l+1) \left( 1 - \eta_{l} \cos 2\theta_{l} \right) P_{l}(\cos \theta)
\]

\[
\Im \phi(\theta,\phi) = \frac{1}{k} \sum_{l} (2l+1) \left( 1 - \eta_{l} \cos 2\theta_{l} \right) P_{l}(\cos \theta) \sin \phi
\]

polarized wave amplitude

\[
\sigma_{\text{tot}} = 4\pi k^{2} \sum_{l} (2l+1) \Im \frac{1}{\xi_{l}}
\]

involves \( \sqrt{1+2\eta_{l}} \) and \( \frac{1}{\xi_{0}} \) when target has spin \( \frac{1}{2} \).
TARGET WITH SPIN $\frac{1}{2}$:

a) 2L+1 → $\frac{2L+1}{2}$ → $S = \frac{1}{2}$ → $S_{1} = 0$

incident spin $(S_{2} = 0)$
target spin $(S_{1} = \frac{1}{2})$

b) $\gamma_{L}$ → $\gamma_{L}^{\pm} e^{i \frac{2\pi}{2S}}$ (spin up) → $S = \frac{1}{2}$

$\gamma_{L}^{-} e^{i \frac{2\pi}{2S}}$ (spin down) → $S = \frac{1}{2}$

partial wave amplitudes for orbital angular mom. $L$ and spin up ($\uparrow$) or down ($\downarrow$)

\[
\begin{align*}
\sigma_{d} & \rightarrow 4\pi^{2} \sum (S_{z} = \frac{1}{2}) |T|^{2} \\
\sigma_{t} & \rightarrow 3 \sum (S_{z} = \frac{1}{2}) (1 - \eta_{S}^{2})
\end{align*}
\]

EXTENSION TO $\frac{d\sigma}{d\Omega}$ IS NOT SO SIMPLE (SEE NEXT FIGURE)

**Fig. 18**

SCATTERING OF SPIN 0 ON SPIN $\frac{1}{2}$

(for example, $\pi N$ and $KN$ scattering)

\[
\begin{align*}
\frac{d\sigma}{d\Omega} = (\psi_{f}^+, \psi_{f}) = \frac{1}{2} \text{Tr} (M^+ M^z) = |f|^2 + |g|^2 \\
\psi_{f} = M \chi_{f} \\
\psi_{i} = M \chi_{i}
\end{align*}
\]

initial spin state → operates on the spin state as it is

transition matrix $\left\{ \begin{array}{c}
\psi \rightarrow g^z \text{ spin operator}
\end{array} \right\}$

polariization of the final state spin $\frac{1}{2}$ particles operates on the flipped spin state

\[
\begin{align*}
\frac{d\sigma}{d\Omega} = (\psi_{f}^+, \sigma_{f} \psi_{f}) = \frac{1}{2} \text{Tr} (M^+ \sigma^z M^z) = 2 \text{Re} (f^z g^z)
\end{align*}
\]

**Fig. 19**
EXPANSION OF THE NON-SPIN-FLIP (f) AND THE
SPIN-FLIP (g) AMPLITUDE IN TERMS OF PARTIAL WAVES (T^±_l)

\[ \begin{align*}
\{ f_l(q) &= \frac{1}{M} \sum_k [ (2l+1) T^+_l + l T^-_l ] P^l_k(q) \\
g_l(q) &= \frac{1}{M} \sum_k [ T^+_l - T^-_l ] P^l_k(q)
\end{align*} \]

first associated Legendre polynomial

\[ \frac{d\sigma}{d\Omega} = \frac{1}{2} \sum_l \left[ T^+_l P^+_l + (2l+1) T^-_l P^-_l \right] + \frac{1}{2} \left( T^+_l T^-_l \right) P^l_0 \]

Example: only l = 0 and l = 1 partial waves:

\[ \sigma = \frac{d\sigma}{d\Omega} \]

\[ \frac{d\sigma}{d\Omega} = \sum_l \left[ \frac{1}{k^2} \left( |T^+_l|^2 + |T^-_l|^2 + \frac{2}{3} |T^+_l T^-_l|^2 \right) \right] \]

\[ \text{notation: } S_{\ell_k}, P_{\ell_k}, P_{\ell_k} \]

\( \sigma \approx 2 \lambda \sum \left| T^+_l \right|^2 \]

**Fig. 20**

FUNDAMENTAL AMBIGUITIES IN THE DETERMINATION OF THE
PARTIAL WAVES (T^±_l).

A. MINAMI AMBIG. : \( \sigma \mapsto c \) is scalar operator, odd under parity

A'. \( \sigma \mapsto c \) is odd under parity

B. COMPLEX CONJUGATION AMBIG. : \( T \mapsto T^* \)

**Fig. 21**
requires the steps indicated very briefly in fig. 19. A new quantity directly measurable is also introduced in this case; the polarization $\hat{P}$. Absent in the spinless case (see fig. 20: $T^+_L = T^+_L$ gives $g(\theta) = 0$, thus $\hat{P} = 0$), the polarization becomes an additional piece of information when dealing with $\pi N$ and $K N$ scattering.

We have now in our hands all the practically useful relations connecting partial waves ($T^+_L$) and measurable quantities ($d\sigma/d\Omega$, $\hat{P}$). Even before writing down explicitly the expression, one can already see that, no matter how skillful one is when solving the problem, the above formulae present certain properties which impede a complete solution. Referring to fig. 21, one can see that, under what is called a "Minami transformation", the differential cross section remains unchanged and the polarization changes sign. But this is also what happens under a "complex-conjugation transformation, so that the combined application of (a) and (b) leaves all our measured quantities as they are, while turning the amplitudes into the complex conjugate of their Minami transform. This is more embarrassing than it may look at first sight, because it is equivalent to saying that the parity of all our partial waves is undefined, thus depriving us of what seemed like a very good tool for finding out the spin and parity of the resonances. Fortunately there is something else that comes to help here. Let us suppose that the energy available in the centre of mass is just above threshold for the reaction to occur (it may be elastic scattering or an absorption process); then the above embarrassment of choice between, say, $l = 0$ and $l = 1$ is much less severe because all possible dynamical considerations point to the lower value of $l$. Thus we may feel reasonably sure that $S$-waves predominate near threshold (and this, for example, has been verified in all the detailed studies of low energy $K N$ and $\pi N$ interactions). It goes by itself that, once one amplitude is identified, then it suffices to follow that amplitude throughout its energy variation in order to have always a fixed reference point for the other amplitudes. In this connection, fig. 22 shows a possible behaviour expected for the partial waves near threshold; this is the "effective-range expansion", based on dynamical assumptions and valid only
at low energies. An amplitude, for example, which follows the prescriptions of fig. 22(b) will have a different energy dependence according to its \( l \)-value and, although at a certain moment the effective range approximation

\[
\text{EFFECTIVE-RANGE EXPANSION}
\]

(a) \[ k^{2l+1} \csc \theta \Delta = \frac{1}{A} + \frac{1}{2} \sqrt{V + \cdots} \]

(b) \[ T_k = \left( \frac{1}{z} \right)^2 \frac{1}{\csc \Delta} = \frac{1}{1 - \sqrt{\text{effective range}}} \]

\[ | \frac{\sqrt{k}}{1 - \sqrt{k}^{2l+1}} | = \frac{1}{4} \left( \frac{1}{1 - |e^{-}\Delta|^2} \right) = \frac{1}{4} \left( 1 + k^{2l+1} \right) \]

\[ \sigma_{(l=0, \text{res.}, K \rightarrow \omega)} \]

\[ \sigma_{(l=0, \text{res.}, K \rightarrow \omega)} \]

\[ \sigma_{(l=0, \text{res.}, K \rightarrow \omega)} \]

Fig. 22

will no more be valid, the knowledge of this amplitude in its early development provides a natural basis for its continuation at higher energies. One should not neglect the possibility, on the other hand, that different threshold behaviours are also possible. The example in fig. 23 shows that, at least for the \( K^p \rightarrow \Lambda \eta \) reaction near threshold, a resonant S-wave behaviour fits the data better than the S-wave predictions of a zero-effective-range expansion. A P-wave behaviour is not excluded either; however, the complete absence of S-waves (as required by the differential cross sections which are isotropic in the whole energy region, thus excluding S-P interference) is not a very likely situation. Thus the preference to a resonant S-wave when explaining the data\(^{14} \).
\( K^- \text{ c.m. energy (MeV)} \)

\[
\begin{array}{cccc}
1665 & 1675 & 1690 & 1710 \\
\end{array}
\]

\( K^- p \rightarrow \Delta \eta \)

- \( s_1 \) scatt. length.
- \( p_1 \) " "
- resonance.

Cross section (m.b.)

\[
\begin{array}{cccc}
0 & 100 & 200 \\
\end{array}
\]

\( \Delta \text{ c.m. momentum (MeV/c)} \)

**Fig. 23**

**BREIT-WIGNER FORMULA**

(limited near resonance)

(a) \( \cot \theta \sigma = \frac{1}{\sqrt{A}} \)

shows that \( \cot \theta \sigma \) is a function of \( E \).

\[
\cot \theta \sigma = 0 \Rightarrow (E - E) \left[ \frac{\Delta \theta \sigma}{\Delta E} \right] \left[ \frac{d\theta}{dE} \right] \left[ \frac{dE}{dE} \right] = (E - E) \frac{2}{\Gamma} \]

expansion around \( E \), where \( \frac{2}{\Gamma} \) is half width at rest.

(b) \( \frac{1}{T} = \frac{1}{\sqrt{A}} \frac{1}{E - z} \)

\[
\begin{array}{c}
T = \int_{E_z}^{\infty} \frac{1}{(E - E)^{3/2}} \frac{1}{E - z} \\
\end{array}
\]

\( z = \frac{k}{\hbar} \) from the uncertainty principle.

**Fig. 24**
IV. Breit-Wigner resonance formula

This brings up the last item of this lecture: the behaviour of resonant partial waves. The term, as applied until now, was used rather loosely and it is now time to see what we mean. In fig. 24 two examples are outlined of how the Breit-Wigner resonance formula can be simply derived \(^4\). The formula arrived at, although not relativistic, is good enough for most present-day applications. In any case, it should not be extended too far from resonance (no more than a couple of widths, say). The energy dependence of the width itself introduces another source of controversy; we shall come back to this point later. For the moment let us neglect the energy dependence and assume \( \Gamma \) constant. Under the above assumptions, it is easy to visualize the energy behaviour of a Breit-Wigner resonant amplitude. Fig. 25 shows how this amplitude describes a circle in the complex plane and how the size of the circle is related to the "elastic" or "reaction" branching fraction of the resonance. The relation is more apparent in

![Diagram of Breit-Wigner resonances]

**Fig. 25**
fig. 26, where three typical cases are given for the elastic channel: 
\( x = 1 \) (maximum elasticity), \( x = 1/2 \) (maximum reaction), \( x = 1/4 \) (small elasticity, small reaction). The corresponding circles shrink progressively (from the maximum, called "unitary circle", with unit diameter, to one having

![Diagram of three circles with different diameters and angles](image)

Fig. 26

a diameter of 1/4). It is interesting to notice that, while the resonant amplitude will always go along a circle, the behaviour of the phase shift \( \delta \) and absorption parameter \( \eta \) as a function of energy (fig. 27) is instead very different according to the different cases. Although the information contained by the two representations is clearly the same, the above example speaks much in favour of the complex plane representation when studying the behaviour of an unknown amplitude.

![Graphs showing phase shift and absorption parameter](image)

Fig. 27
As was already pointed out at the beginning (figs. 7 to 9), the most obvious manifestation of a resonant amplitude can usually be found in the shape of its cross section. Fig. 28 shows what to expect from the latter on the basis of the previous formula. Elastic, reaction and total cross sections all vary with energy in the same manner, the size of their respective enhancements being in the ratio of $x^2 : x(1-x) : x$. Since $x$ is always smaller than 1, a notable consequence is that looking for resonant enhancements in the elastic cross section is in principle more difficult than in the total cross section. However, there are other considerations - like the size and shape of the background - which must be taken into account before agreeing with the above simple conclusion.

Finally, one word about the energy dependence of the width. Fig. 29 shows the two main approaches to this problem. It should be stressed that, in most practical cases, the difference between the two is well beyond the experimental accuracy; furthermore neither case is appropriate enough when too far away from the resonance.
ENERGY DEPENDENCE OF PARTIAL WIDTHS

\[ \Gamma \sim \left( \frac{K}{E} \right) \times B_{\ell}(k) \]

- phase space
- centrifugal barrier

\( K \): final state wave number
\( E \): total energy

\( (a) \) relativistic parametrization:
\[ B_{\ell} = \left( \frac{k^2}{k^2 + \chi^2} \right) \ell, \quad \chi \approx 350 \text{ MeV} \]

\( (b) \) non-relativistic expression (square-well potential):
\( B_0 = 1 \)
\( r \approx 1 \text{ fermi} \)
\[ B_1 = \frac{(kr)^2}{10(kr)^2} \]
\[ B_2 = \frac{9 + 3(kr)^2}{4(kr)^4} \]
[small \( \ell \)]

\textbf{Fig. 29}
V. Partial wave analysis of the $\bar{K}N$ system

We can now go back to our subject, i.e. hyperon resonances and how to study them. The most accessible experiment in order to "form" hyperon resonances is to scatter $K^-$ mesons on protons. This has been done with both counters and bubble chambers, up to energies of $\sim 3$ GeV/c, i.e. masses of the $\bar{K}N$ system of 2.6 GeV. The two techniques are complementary: the total and elastic cross section (with or without a polarized target) being measured better and more easily with counters, the other channels being fully exploited only with bubble chambers.

Fig. 30 shows an early collection (circa 1962) of total cross section data resulting from various experiments. Quite outstanding is a pronounced enhancement near 1300 MeV, ascribed then to the formation of a resonance, $\Lambda(1320)$. We have already seen in fig. 9 how much better known this momentum region is nowadays and how many more resonances have come up when the same measurements were repeated with better accuracy and statistics. We
shall see now that even the situation represented in fig. 9 is still inadequately for a full understanding of the phenomena taking place in this region. In order to do so, we shall examine in detail the procedure and results of a partial wave analysis of the \( \bar{K}N \) system in the region of the above enhancement 1) Fig. 31 shows what reactions one has to deal with in this energy region. Notice that the analysis in the form described below has only been performed on the 2-body processes, the remaining reactions being either too scarce for a meaningful interpretation or too complicated for this simplified approach. To have an idea of the behaviour of the data in our and other momentum regions, let us examine figs. 32 and 33. They give the momentum dependence of the ratio between cross section \( \sigma \) and \( 4\pi k^2 \) (the "geometrical" factor in the formulae of fig. 18) for the two elastic channels. The arrows indicate the position of the better established resonances (as identified by this and other experiments 16), the full circles refer to the data of this experiment. It is interesting to notice how fundamentally different is the "background" behaviour of these two
channels. In the $K^0\pi$ case the resonances are superposed to an almost constant and small background, they are clearly distinguishable and a more accurate study of this cross section may well uncover some yet unknown structure. The background for the $K^-p$ case is, instead, of a quite different nature: it is large and appears to be growing steadily with increasing momentum. This is clearly not the best channel to look into when hunting for small resonant structures.

![Fig. 32](image1)

![Fig. 33](image2)
Turning now to the partial waves, the starting point is the expression of \( ds/d\Omega \) in fig. 20. Writing it explicitly in terms of partial waves gives the formula of Fig. 34. It is possible and convenient to expand this formula in a series of Legendre polynomials; the coefficients of the

\[
\frac{ds}{d\Omega} = \sum \frac{1}{k} \left| \left( \sum \frac{(l_r \cdot l_f) T_r^* T_f^*}{k} \right) P_j \right|^2 \frac{1}{k} \left| \sum \frac{(l_r \cdot l_f) T_r^* T_f^*}{k} \right|^2
\]

Recall scattering amplitudes for spin 0 on spin \( \frac{1}{2} \) -

Expand in terms of Legendre polynomials -

Coefficients can be calculated -

Example:

\[
A_n = \sum a_{n,j} \text{Re} \left( T_i \cdot T_j \right)
\]

Example: \( A_1 = 2 \text{Re}(S,P_1) + 4 \text{Re}(S,P_3 + P_1 P_3) + \frac{4}{5} \text{Re}(P_3 D_3) \)

Fig. 34

expansion can be calculated and tabulations exist\(^4\) connecting them with the partial wave amplitudes \( T_i \). The same operation can be performed on \( \hat{P}(ds/d\Omega) \) (fig. 35), except that here the most convenient expansion is one

\[
\hat{P} \frac{ds}{d\Omega} = \frac{2}{k} \sum_j \sum \text{Im} \left( \frac{(l_r \cdot l_f) T_r^* T_f^*}{k} \right) P_{\frac{3}{2}} \left( \lambda \right)
\]

Recall scattering amplitudes for spin 0 on spin \( \frac{1}{2} \) -

Expand in terms of associated Legendre polynomials -

Coefficients can be calculated -

Example:

\[
B_0 = 0; \quad B_1 = 2 \text{Im}(S, P_1) - 2 \text{Im}(S, P_3 + P_1 P_3) + \frac{8}{5} \text{Im}(P_3 D_3)
\]

Fig. 35
in terms of associated Legendre polynomial of the first order. Here also
the coefficients of the expansion can be calculated or read out from existing
tabulations. Exactly the same expansions can be performed on the
experimental differential cross sections (fig. 36) and we finally end up with
experimental coefficients $A_n$ and $B_n$ to be compared with the expressions

FROM ANGULAR DISTRIBUTIONS TO LEGENDRE COEFFICIENTS

![Diagram of angular distributions and Legendre polynomials]

expressing the desired partial waves (see some examples of these in figs. 34
and 35). Fig. 37 shows how some of the angular distributions look and
fig. 38 gives the experimental coefficients of their Legendre polynomial
expansion as a function of the incident $K^-$ momentum. These coefficients
(and those of the other reactions) are the data and from them we shall try
to extricate the partial waves. It should be mentioned that the procedure
of going through the Legendre coefficients is by no means universal; the
connection with the partial waves can be made directly through the
differential cross sections, comparing the experimental values of $d\sigma/d\Omega$
subdivided in convenient intervals of $\cos \theta$, with the corresponding
expressions of fig. 34 and 35 with $P_n$ and $P_l$ evaluated at the central value
of $\cos \theta$ for the interval in question. The advantage of using the Legendre
Fig. 37

Fig. 38
coefficients, although criticizable as statistically less sound, lies mainly in the intuitive appreciation they allow for the behaviour of the partial waves, appreciation which is totally lost in the other approach. Thus the behaviour of a certain coefficient as a function of energy can be easily related to the nature of the partial waves contributing to it.

As an example, let us consider the data of fig. 38. The conspicuous enhancement of $A_0$ near 1 GeV/c tells us that something must be resonant in this neighbourhood. On the other hand, any of the partial waves could be responsible for the enhancement ($A_0 = S^2 + P_1^2 + 2(P_2^2 + D_2^2) + 3(D_5^2 + F_5^2) + \ldots$). But $A_5$, together with the higher coefficients, is consistent with zero, thus implying that waves with $J > 5/2$ are absent or at least negligible in this region. $A_5$ is instead different from zero; with the maximum value of $J$ being limited to 5/2, the expression of $A_5$ as a function of the partial waves is particularly simple: $A_5 = \frac{100}{7} \text{Re} \left( D_5^2 \cdot F_5^2 \right)$. Thus the fact that $A_5$ shows a sudden and large negative excursion conveys that either $D_5$ or $F_5$ must be both present and at least one of them quickly varying. Remembering now the properties of a resonant amplitude in the elastic channel as illustrated in fig. 25, one can explain the enhancements in $A_0$ and $A_5$ as for position, magnitude and shape by simply assuming that both $D_5$ and $F_5$ are resonant at about the same energy. The above qualitative argument is further strengthened by the behaviour of the other elastic channel, $K^-p \rightarrow K^-p$ (a different combination of the same isotopic spin amplitudes; see fig. 40 below). In this way one can also show that the resonant $D_5$ and $F_5$ amplitudes must be in different states of isotopic spin (the excursion of $A_5$ in $K^-p \rightarrow K^-p$ having opposite sign than that in $K^-p \rightarrow \pi^0 n$). There are considerations of background which have been left out of this simplified discussion and we also have neglected the information coming from the other coefficients. Still the above conclusions are valid and borne out by the quantitative analysis discussed below. It goes by itself that the same type of considerations are much more difficult, if not impossible, when staring at the differential cross sections themselves (fig. 37). There, rather, one may well be tempted into invoking phenomena such as "baryon exchange" (look at all those backward peaks ...) or some other unnecessarily
involved mechanism which may not offer as simple and as consistent a picture as that in terms of resonant partial waves in the direct channel.

Proceeding with the partial wave analysis, fig. 39 shows what one knows or suspects (from previous experiments, qualitative considerations of the type indicated above, personal prejudices etc...) about the amplitudes present near 1 GeV/c. This list needs further to be enlarged in order to make space for the two possible isotopic spin states of the K^+p system. The isotopic spin composition of the channels considered in the analysis is given in fig. 40 together with a quick reminder of how it was obtained. We are now ready to investigate the energy dependence of each amplitude; from now on, the assumptions, simplifications and procedures are all specific to the problem in question and refer only to the particular approach adopted in ref. 1.

As it will be more clear later on, there isn't enough experimental
information to solve the problem at each momentum. One is instead obliged to

\[ \begin{aligned}
K \rightarrow K^- & \rightarrow \frac{1}{2} + \frac{1}{2} \\
& \rightarrow \frac{1}{2} + \frac{1}{2} \\
I_3 = 0
\end{aligned} \]

**Example of Isospin Decomposition**

\[
\begin{aligned}
|K^0 \rangle &= \langle \frac{1}{2} | A_0 - \frac{1}{2} A_1 | T \rangle |A_0 + \frac{1}{2} A_1 \rangle = \frac{1}{2} T_0^+ + \frac{1}{2} T_1^-
\\
|K^- \rangle &= \langle \frac{1}{2} | A_0 - \frac{1}{2} A_1 | T \rangle |A_0 + \frac{1}{2} A_1 \rangle = \frac{1}{2} T_0^- + \frac{1}{2} T_1^+
\\
|\Lambda^+ \rangle &= \langle \frac{1}{2} | A_0 + \frac{1}{2} A_1 | T \rangle |A_0 - \frac{1}{2} A_1 \rangle = \frac{1}{2} T_0^+ - \frac{1}{2} T_1^-
\\
|\Sigma^+ \rangle &= \langle \frac{1}{2} | A_0 + \frac{1}{2} A_1 | T \rangle |A_0 - \frac{1}{2} A_1 \rangle = \frac{1}{2} T_0^- - \frac{1}{2} T_1^+
\\
|\Sigma^- \rangle &= \langle \frac{1}{2} | A_0 + \frac{1}{2} A_1 | T \rangle |A_0 - \frac{1}{2} A_1 \rangle = \frac{1}{2} T_0^+ + \frac{1}{2} T_1^-
\\
\end{aligned}
\]

Fig. 40.

assume a certain momentum dependence of the amplitudes and then try to solve
over many momenta at the same time. Three explicit parametrizations are
shown in fig. 41, together with the number of unknowns required by each one
of them. The parametrization indicated as (c) has not really been used;
it is mentioned here only as a warning against such a simple minded approach.

\[
|\Sigma^+ \rangle = \frac{1}{2} (c_0 - c_1) |T\rangle |A_0 + \frac{1}{2} A_1\rangle = \frac{1}{2} T_0^+ - \frac{1}{2} T_1^-
\\
\]

\[
|\Sigma^- \rangle = \frac{1}{2} (c_0 + c_1) |T\rangle |A_0 + \frac{1}{2} A_1\rangle = \frac{1}{2} T_0^+ + \frac{1}{2} T_1^-
\\
\]

Fig. 41.
The correct way of adding up a background and a resonance with the same quantum numbers is more complicated, as can be seen for example in fig. 42 which shows the elastic case only. The parametrizations of fig. 41 are now imposed on the waves of fig. 39 according to the known or suspected character of these waves. Different combinations of resonant and background are then tried until a satisfactory solution is eventually reached and one is reasonably sure that alternative possibilities do not exist. How this is done in practice can be followed through figs. 43 and 44. First, our 16 waves are parametrized as (for example) in the manner listed at the left of fig. 43. In the same figure one can also see how many data are available for each channel and at each momentum. It immediately appears that only in the case of the Am reaction is there an equal number of data and unknowns (an overall undetermined phase brings down the 16 to unknowns to 15). Thus our system of equations (each data point is an equation) is largely underconstrained; a solution momentum by momentum is clearly impossible. The situation instead looks different when we count the free variables of our parametrization: already 3 different momenta are sufficient in order to have more equations than unknowns. The momenta available in the experiment are about 20, spread out over some 400 MeV/c (corresponding to ~200 MeV in centre of mass energy). The system is thus well overconstrained and the existence of a solution depends only on the validity of the hypotheses and the reliability of the data. It is also clear that such a large amount of computation cannot be carried out by hand; the flow diagram in fig. 44 shows the main steps followed in a computer search for the best solution. The method used is that of finding the parameters corresponding to a minimum of \( \chi^2 \); one could, alternatively, use a maximum-likelihood
Fig. 43

\[ \text{data points (regression coefficients)} \]
\[ \text{at each momentum (Nช momentum available)} \]

\[ \begin{align*}
\text{data} & \quad \begin{cases}
\text{K}^+ \quad 8 \\
\text{K}^- \quad 8 \\
\sigma^+ \quad 2 \\
\text{reaction channels} \\
\text{(2-body)} \\
\text{resonant} \quad 15 \\
\text{free} \quad 15 \\
\text{reaction channels} \\
\text{(2-body)} \\
\text{resonant} \quad 15 \\
\text{free} \quad 24 \\
\end{cases}
\end{align*} \]

\[ \begin{align*}
\text{total no.} & \quad 15 \text{ N}_m \\
\text{data} & \quad 16 \text{ N}_m \quad 45 \text{ par.} \\
\text{data} & \quad 15 \text{ N}_m \quad 21 \text{ par.} \\
\text{data} & \quad 24 \text{ N}_m \quad 45 \text{ par.} \\
\end{align*} \]

Fig. 44

\[ \text{example} \quad \begin{cases}
\text{data (K}_\text{n}, \text{n}, \Sigma \text{n}) \quad \text{over ~20 momenta} \quad \rightarrow \quad 1000 \\
\text{parameters} \quad \text{(apportioned between back. res.)} \quad \rightarrow \quad 100 \\
\text{unknowns} \\
\end{cases} \]

\[ \begin{align*}
\text{how to solve:} \\
\text{enter data} \\
\text{and set of initial} \\
\text{parameters (tentative)} \\
\text{calculate } A^+_n \text{ and } B^+_n \text{ from parameters} \\
\text{calculate } \chi^2 = \sum \frac{(A_{\text{calc}} - A_{\text{meas}})^2}{\text{d}A_{\text{meas}}} \\
\text{minimize } \chi^2 \text{ by changing} \\
\text{parameters. If } \chi^2 \text{ no min. step,} \\
\text{parameters at min. iterate.} \\
\text{result: parameters corresponding to smallest } \chi^2 \\
\text{i.e. satisfying in the best possible way the} \\
\text{system of 1000 equations.} \\
\end{align*} \]
method or other approaches. Without going into the details of the computer search and of the various combinations of resonant and background waves attempted, it suffices to say that the system of equations can indeed be satisfactorily solved and the minimum $\chi^2$ found to be in good agreement with that expected. This shows that the parametrization employed, with all its approximations and arbitrariness, is an acceptable representation of what is going on in reality.

Let us now look at the results. Fig. 45 shows how the two elastic channels are fitted under the above conditions. The total cross section (not shown) is also fitted simultaneously. This example, referring only to the elastic amplitudes, gives a minimum $\chi^2$ of 3.50 for 346 data points and 39 free parameters, i.e., a confidence level of better than 1 part in 20. The partial waves corresponding to the above solution are shown in fig. 46 together with results of other experiments.

Fig. 45

Fig. 46
Figs. 47 and 48 show the amplitudes obtained from the analysis of the other channels. An overall fit to all the channels at once (K\ensuremath{\bar{N}}, \Delta n, \Sigma n) has also been performed with results in good agreement with those of the separate fits.\(^\text{12}\)

Before leaving the subject, it should be pointed out that the successful outcome of the above approach may be attributed, in great part, to the relative simplicity of the region examined. The partial waves are present in limited numbers and their behaviour as a function of energy is apparently not very peculiar. That the same approach should also be valid at other (higher) momenta remains to be seen. As an example,
Fig. 49 shows the total $\bar{K}N$ cross section between 2 and 3 GeV/c\textsuperscript{19}). The resonant enhancements are here minuscule when compared to those near 1 GeV/c.

Furthermore, one knows that partial waves up to $J = 11/2$ and higher are definitely important at these momenta \textsuperscript{20,21}). Last, but not least, the considerable decrease in cross section for all the important two-body channels has the practical effect of reducing the statistics available in reasonably economic experiments.

VI. Partial wave analysis of the pion-nucleon system

As compared to the $\bar{K}N$, a study of the $\pi N$ system at equivalent energies is in principle much easier. First of all, pions have been around much longer than kaons and their low energy interaction has had the time to be studied extensively and conclusively. Then, they are considerably cheaper than kaons, thus affording better and more detailed measurements. Finally, they are much less prolific than kaons as for possible final states. The above reasons (and others) have substantially contributed to the advanced state of the partial wave analysis of this system and, consequently, of
the knowledge of baryon resonances with $S = 0$.

Fig. 50 summarizes the state of these resonances; all candidates have been plotted even if in some cases (open circles) there is uncertainty about their existence. Figs. 51 and 52 illustrate the experimental situation showing, by way of example, a compilation of the total cross sections $^{22}$. Differential cross sections of the elastic channels have also been measured and with comparable accuracy. Finally, polarization measurements have been performed over quite a large range of momenta, as shown in fig. 53 $^{23}$. It should be pointed out that the great majority of the data collected has been obtained via counter experiments.

Let us see now what has been done for the partial wave analysis of this system. Only the elastic channel has been thoroughly analysed, the study of inelastic processes like AK, ZK, etc... being still in a very early
stage. The main difference with respect to the \( \bar{K}N \) system is that here we have enough data at each momentum setting to afford an energy-independent partial wave analysis. Without going into a detailed description of the unknown quantities and the available equations, it is enough to say that the knowledge of polarization, total cross section and differential cross sections for the elastic channels available in \( \pi^+p \) and \( \pi^-p \) scattering provides us with an overconstrained system of equations. At each momentum this system can be solved using methods of the same type as those mentioned for the \( \bar{K}N \) case. One usually ends up with several solutions (i.e., several sets of partial waves which fit equally well our system of equations) for each one of the momenta considered. Up to this point the method is more or less general; the only differences which exist in practice between the various analyses are on the way in which the data are chosen or the particular procedure adopted to obtain the fitted amplitudes. From here on, instead, there is more freedom as to what to do. The problem is that, having a set of possible solutions, we must decide which is the good one.

Fig. 54 outlines the simple procedure used in the first analysis of this type which successfully yielded a wealth of unexpected new

\[ \text{ENERGY-INDEPENDENT PARTIAL WAVE ANALYSIS (SACNAS GROUP)} \]

**FIRST STEP**: at energy \( E \)

\[
\frac{d\sigma}{d\Omega} = \sum b_i(T_i, \omega \theta)
\]

This give \( \chi^2 \) (all with "acceptable" \( \chi^2 \)s)

**SECOND STEP**: over a range of energies \( E \)

[many solutions]

\[ \text{continuity imposed by hand} \]

[only one solution chosen]
resonances). The good solutions are here chosen by requiring that they satisfy the maximum possible "continuity" as a function of energy. How to define this "continuity" is of course the problem. This was solved by hand (or rather by eye) in the analysis of ref. 24; referring to fig. 54, one should keep in mind that the operation of smoothing out the energy dependence of the amplitudes must be done simultaneously over all amplitudes (each one of them represented by two quantities: phase shift and absorption parameter or, alternatively, real and imaginary part). Thus one should imagine that the plot of $T_i^i$ vs E in fig. 54 represents in reality a whole series of graphs where each point corresponds to another point in a different graph. A certain degree of arbitrariness is undoubtedly implicit in this procedure; on the other hand, the absence of any particular model for the energy dependence of the amplitudes (what one could call a "theoretical prejudice") may well compensate for the above subjectivity. Fig. 55 shows some of the results of this analysis. The points (if one can see them) represent the amplitudes chosen at each energy; the curves are an additional smoothing out of the results, suggesting the most likely trajectory described by the amplitudes as a function of energy.

The arbitrariness mentioned above can be somewhat reduced if a more impartial computer-controlled method is introduced at the "second step" of fig. 54 when imposing continuity. This has been done in another analysis 23) which otherwise is not very different from the preceding one. The continuity criterion takes the form of a "shortest possible path" over a certain energy interval, of conditions on the derivatives between adjacent energy settings, etc. Fig. 56 shows one of the resulting amplitudes. As compared to the
same in fig. 55 the points are less scattered but the curve is almost identical.

Let us now examine a different and more sophisticated approach to the problem of selecting our good solutions. Instead of simply requiring a smooth energy dependence of the amplitudes without invoking any specific pattern, one can demand that the energy dependence should be such as to satisfy in the best possible way certain conditions between the real and imaginary part of the amplitudes called "dispersion relations". A discussion of dispersion relations is well outside the scope of the present lectures; a good place to find a description of this technique, particularly in connection with partial wave analysis, is the series of lectures of ref. 25. Fig. 57 gives a very sketchy summary of the

\[
\text{STARTING FROM THE GENERAL RELATION:}
\]

\[
\text{Re}\left(\frac{T_k^e}{k}\right) = \frac{P}{\pi} \int_{S}^{\infty} \frac{\text{Im}\left(\frac{T_k^e}{k}\right)}{s-s_k} ds'
\]

\[
\int_{S}^{\infty} \frac{\Delta \left(\frac{T_k^e}{k}\right)}{s-s_k} ds'
\]

\[
\text{ONE OBTAINS 2 SEPARATE RELATIONS BETWEEN Re AND Im:}
\]

\[
\begin{align*}
\text{Re}\left(\frac{T_k^e}{k}\right) &= \sum_{n=1}^{N} a_n g_n(e) \\
\text{Im}\left(\frac{T_k^e}{k}\right) &= \sum_{n=1}^{N} a_n h_n(e)
\end{align*}
\]

\[
M = \sum_{n=1}^{M} \frac{r_m}{s-b_m}
\]

N and M can be changed; \(T_k^e\) are the "experimental" amplitudes; \(a_n\) depend on \(e\) and can be fitted for each part. wave.

Fig. 57
formulae used in the analysis of ref. 26. The main point is that from the
general expression (which does not have a form that can easily be exploited
for the purpose mentioned above) one obtains, through the magic of something
called "conformal mapping", two expressions valid separately for the real
and imaginary part of the amplitudes 25. The $g_n(E)$ and $h_n(E)$ in these
expressions are known functions of the energy and the summation over $n$,
which in principle should go to $\infty$, can be truncated at some value $N$. The
summation over "poles" is also finite, with $M$ usually small. The early
arbitrariness in energy dependence is now reduced (not eliminated) and, in
addition, a connection is established between the previously uncorrelated
real and imaginary part of each amplitude.

All this wealth of fresh information does not come completely free,
of course. Without raising doubts as to the basic validity of dispersion
relations, their practical use does necessarily imply a series of
approximations, assumptions and other mundane involvements which can indeed
be questioned. Thus, for example, one may have preferred more (or less)
poles, a different order of approximation in the sum over $n$, a stronger (or
weaker) effect of the long-range forces, etc... All this should not be
taken as a detracton of the method but rather as a warning against possible
biases which (just because they are much more subtle than those of the
rudimentary procedures described previously) may indeed still linger on
even in these aseptic surroundings.

Fig. 58 shows schematically what may have been the approach used in
this method 26 (no official detailed description is available as yet). What
is referred to as "free parameters" are the coefficients $a_n$ of fig. 57;
the "order of complexity" refers to the number of coefficients ($N$) and poles
($M$). The "iteration" procedure involves the re-calculation of the
amplitudes directly from the data starting each time from the previously
obtained "corrected" amplitude $T_{corr}$ and its error.

The results, shown as solid curves in fig. 59, are spectacular. All
the inaesthetic wiggles of the previous solutions have disappeared and, if nature is indeed smooth, there is no doubt that this is the smoothest representation. Still, it may be that in some case the solutions are too smooth. As in the case of $S_{11}$, which can be compared with that in figs. 55 and 56: the small secondary loop is gone. Perhaps it shouldn't.
On the other hand, the procedure seems to be very valuable when trying to make sense of small amplitudes. Thus, for example, look at $D_{33}$ in fig. 60. Here the open circles are from the best solution of ref. 24; there is little doubt that the full circles and the curves (from ref. 26) make more sense than the former. Similarly, one can see in fig. 61 a collection of small amplitudes$^{24}$ where the first approach (solid curves) would demand such an erratic behaviour that one is instinctively tempted in trusting the more dignified path described by the second approach (dotted curves). All this, if correct, would speak very much in favour of the second method in spite of its added complications and possible biases.

Fig. 60

Fig. 61
THIRD LECTURE

VII. Applications to $SU(3)$

Now that we know how and where to look for baryon resonances, we may perhaps wonder why we are doing it and whether it is really worth the effort. So many of them already exist on the market that the mere pleasure of a search and analysis may not be a reward by itself any more. We may then go one step further and try to see if the theoretical models that have been proposed make sense or not. This, of course, could have been done much earlier (and indeed it was\(^2\)) but it is only at present that we dispose of a sizeable body of information and that meaningful tests can be done.

The simplest level of model predictions is offered by $SU(3)$. I will assume here that everybody is already familiar with at least the more fundamental aspects of this theory. In what follows we shall concentrate on the existence and requirements of the baryon "multiplets": how many can be made up, what is their composition and how well do the branching fractions of their members agree with the predictions. This lecture follows very closely the articles of ref. 23.

A multiplet is usually proposed when a sufficient number of particles is found, all with the same $J^P$ and with masses such as to obey, within broad limits, some so-called "mass law". Corrections on the mass values deriving from "mixing" with other multiplets are sometimes taken into account. Finally, one can check that the decay modes of the members of the multiplet are in agreement with the assumption that there is only one "coupling constant" characteristic of the multiplet in question. It is particularly the latter point which is worth insisting upon; indeed, the fact that a mass happens to be correct may well be a coincidence due to the large variety of resonances existing throughout the mass spectrum.

An additional self-consistency check, which is quite useful in this
identification procedure, is offered by the different decay modes that the same resonance can have. Let us see, for example, what this means. We are already familiar with the isotopic spin requirements for the various charge combinations of the same decay mode of a resonance. Thus, for example, an I = 0 resonance decaying into ΔK will distribute itself equally into Σ⁺π⁻, Σ⁰π⁰, Σ⁻K⁺; an I = 1 resonance will not go into Σ⁰π⁰, etc... The relative rates are given by Clebsch-Gordan coefficients, as everybody knows. What is less evident (and indeed it demands the validity of SU(3) and the attribution to a definite multiplet) is that one can also do the same calculation to see how many times for example, the resonance decays into ΔK and how many times into Δπ. The relative importance of the various decay modes are given by Clebsch-Gordan coefficients of SU(3), which are available in several tabulations (for example in ref. 29 and 30). These coefficients, on the other hand, do not tell us the whole story. We must remember that the particles into which the resonances decay may have masses quite different from each other; so we must introduce a "phase space" factor to take this into account. Furthermore, when the orbital angular momentum of the system is different from zero, there will be centrifugal barrier effects which must also be considered.

All this is explicitly written out in fig. 62, which gives the formula

\[ \Gamma = C^2 \frac{2}{d} \frac{L}{B(p)} \left( \frac{M_N}{M_R} \right) P \]

SU(3) Clebsch-Gordan coefficient.
Barrier penetration factor for angular momentum L and c.m. momentum p
C.m. momentum of the decay products
Man of the resonances

Fig. 62
used in ref. 23 for the case of singlets and decuplets. The partial width $\Gamma$ measures the rate of a certain decay mode, $p$ is the phase space factor, $B$ is the centrifugal barrier factor of fig. 29, $(M_N / M_K)$ is put in to normalize things. The factor $g^2$ we can call "coupling constant" and it is this which should remain constant for all the members of the same multiplet.

Fig. 63 shows the Clebsch-Gordan coefficients for the most familiar decay modes of hyperon resonances. Here the symbol $Y$ stands for hypercharge $(B + S)$, $I$ and $I_3$ for magnitude and third component of the isotopic spin.

<table>
<thead>
<tr>
<th>SOME CLEBSCH-GORDAN COEFFICIENTS OF ${8} \otimes {8}$ going into:</th>
<th>${10}$</th>
<th>${1}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>example $\downarrow$ $\downarrow$ $\downarrow$ $\downarrow$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$Y$ $I$ $I_3$</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>$0$ $0$</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

| $PK^- \quad 1/2^- \quad 1/2^- \quad 1/12 \quad 1/8$ |
| $K^0 \quad 1/2^- \quad 1/2^- \quad 1/12 \quad -1/8$ |
| $\Sigma^+ \pi^- \quad 011 \quad 011 \quad 1/12 \quad 1/8$ |
| $\Sigma^0 \pi^- \quad 010 \quad 010 \quad 0 \quad -1/8$ |
| $\Sigma^- \pi^- \quad 010 \quad 000 \quad 1/4 \quad 0$ |
| $\Sigma^- \pi^0 \quad 011 \quad 011 \quad 1/12 \quad 1/8$ |
| $\Lambda \pi^0 \quad 000 \quad 010 \quad 1/4 \quad 0$ |

Fig. 63

Let us now make a tour of the most reliable multiplets and see if and how the relation in fig. 62 is satisfied in practice. In fig. 64 we see the well-known $3/2^+$ decuplet and a rather hypothetical $7/2^+$ decuplet which, through some stretch of imagination, can also be thought of as the Regge-recurrence of the first. The numbers given as branching fractions (in this and the following figures) come out only partly from ref. 2; full details can
be found in ref. 28. The most likely candidates for singlets are those in fig. 65 (a possible choice for Regge-recurrences) plus \( \Lambda(1405) \) with its main decay mode into \( \Sigma \kappa. \)

Now, instead of contemplating the above arrangements and congratulating ourselves for having found so many resonances with the correct spin-parity...
and mass value, we can do something more and see if the 16 branching fractions associated with these resonances make sense or not. This is shown in fig. 66 where the value of $g^2$ (and its error) as obtained through the formula in fig 62 is plotted for each one of the above decay modes. Right away we can see that, even taking into account the errors, there are serious discrepancies between the values of $g^2$ inside each multiplet. Even

Fig. 66

the $3/2^+$ decuplet, that most respectable and famous multiplet, gives values of $g^2$ differing as much as a factor 2. It is clear that if we start questioning also this multiplet, then we may as well give up the whole idea and do something else. Instead, let us blame ourselves for the inadequate connection between $\Gamma$ and $g^2$ and let us consider discrepancies of this order of magnitude as unavoidable and quite acceptable. Having thus broadened our tolerance, we can then inspect fig. 66 with a different eye. The coupling constants are in satisfactory agreement within multiplets and actually even between different multiplets. This agreement is certainly not trivial and we are then encouraged to proceed with our simple formula towards more questionable multiplets.

In order to do so, and having exhausted all the reasonably well-established singlets and decuplets, we must modify slightly the previous
formula so as to adapt it to the octet case. Fig. 67 recalls that there are two possible octets to be considered and that what one sees in reality is a mixture of the two. Thus the expression relating the coupling constant to

\[ \{8\} \otimes \{8\} = \{27\} \otimes \{10\} \otimes \{10\} \otimes \{8\} \otimes \{8\} \otimes \{1\} \]

In general, the observed baryon resonance octet will be a linear combination of \( \{8\} \) and \( \{8'\} \):

\[ \psi = \psi \cos \Theta + \psi \sin \Theta \]

The mixing parameter defined as \( \alpha = 1 - \frac{\sqrt{5}}{3} \tan \Theta \)

Fig. 67

the observed partial width must now contain a linear combination of the coupling constants \( g_d \) and \( g_f \) for the two octets. Fig. 68 shows the new

\[ \Gamma = \left( C \cos \Theta + C' \sin \Theta \right) g_d^2 B(p) \left( \frac{M_N}{M_R} \right) \]

Knowing \( \Gamma \) is not sufficient to obtain \( g_d^2 \). Only linear combination of \( g_d \) and \( g_f \) can be obtained. See graph

\[ (C g_d + C' g_f)^2 = \text{constant} \]

Fig. 68
expression and how, from it, one can only obtain a set of straight lines in the $g_d$, $g_f$ plane. The knowledge of only one partial width is clearly insufficient for determining $g_d$ and $g_f$ separately. On the other hand, if more partial widths are available, the problem can be solved (or better "fitted", because at a certain point there will be more constraints than unknowns). Alternatively, when this occurs, we have a way of knowing if the members of the octet in question were indeed well matched. The probability that this could happen by chance for a random set of resonances is admittedly very remote.

Let us now see if all this works in practice. Fig. 69 gives some examples of Clebsch-Gordan coefficients valid for the octets, again from

<table>
<thead>
<tr>
<th>$P \ K^-$</th>
<th>1/2</th>
<th>-1/2</th>
<th>$\sqrt{3}/2c$</th>
<th>$\sqrt{1}/2$</th>
<th>$\sqrt{1}/2$</th>
<th>$\sqrt{1}/4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n \ K^0$</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{1}{2}$</td>
<td>$\sqrt{3}/2c$</td>
<td>$\sqrt{1}/2$</td>
<td>$-\sqrt{1}/20$</td>
<td>$\sqrt{1}/4$</td>
</tr>
<tr>
<td>$\Sigma^+ \pi^-$</td>
<td>011</td>
<td>011</td>
<td>0</td>
<td>$\sqrt{1}/3$</td>
<td>$\sqrt{1}/5$</td>
<td>0</td>
</tr>
<tr>
<td>$\Sigma^0 \pi^0$</td>
<td>010</td>
<td>010</td>
<td>0</td>
<td>0</td>
<td>$\sqrt{1}/5$</td>
<td>0</td>
</tr>
<tr>
<td>$\Sigma^- \eta$</td>
<td>010</td>
<td>000</td>
<td>$\sqrt{1}/5$</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\Sigma^- \pi^+$</td>
<td>011</td>
<td>011</td>
<td>0</td>
<td>$-\sqrt{1}/3$</td>
<td>$\sqrt{1}/5$</td>
<td>0</td>
</tr>
<tr>
<td>$\Lambda \pi^0$</td>
<td>000</td>
<td>010</td>
<td>$\sqrt{1}/5$</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

**Fig. 69**

ref. 30. Figs. 70 and 71 illustrate the case of the $1/2^-$ nonet. Those branching fractions which have been measured appear to be in rather good agreement with the predictions of a single $d$ and $f$ coupling constant. This is indicated by a common crossing area (the dark spot of fig. 71) in the
\[ \frac{1}{2} \text{- nonet} \]

(baryon-eta resonances)

\[
\begin{align*}
\Lambda (1405) & \quad + \quad \text{OCTET} \\
\Lambda (1570) & \quad \Phi \quad (\Sigma, \Lambda) \\
\Lambda (1670) & \\
\Lambda (1820) & \quad \text{not observed} \\
\Sigma (1770) & \quad \Xi \eta \\
\Xi (1820) & \quad \Xi \gamma
\end{align*}
\]

\[ N^+ \Xi = \frac{3 \Lambda - \Sigma}{2} \]

Fig. 70

\[
\begin{align*}
\frac{1}{2}^- \\
N (1570) \\
\Lambda (1670)
\end{align*}
\]

Fig. 71
$g_d$, $g_f$ plane. The bars across the straight lines indicate the uncertainty due to the experimental input. The convention followed throughout is that the thinner a line is, the more uncertain is the corresponding branching fraction.

Figs. 72 and 73 show the situation of what is currently believed to

\[ \Lambda (1620) + \text{OCTET} \]

\[ \begin{align*}
N (1530) & \quad \rightarrow \quad \Lambda \pi = 65 \% \\
\Lambda (1700) & \quad \rightarrow \quad \Sigma \pi = 46 \%
\end{align*} \]

\[ \begin{align*}
\Sigma (1660) & \quad \rightarrow \quad \Lambda \pi = 10 \% \\
\Xi \pi & = 67 \%
\end{align*} \]

\[ \Xi (1815) \quad \rightarrow \quad \Lambda \Xi = 65 \% \\
\Xi \Xi & = 2 \%
\]

---

**Fig. 72**

**Fig. 73**
be the $\frac{3}{2}^-$ nonet. There are problems here (see ref. 28 for details) but a reasonable crossing area can still be found. A valid question, at this point, is how sure can one be that the crossing area is where it has been drawn and not somewhere else. As it is evident from fig. 73, there are other possibilities which are not necessarily less likely than the one chosen. There is not much of an answer to this, except that, when more and more accurate branching fractions become known, then the above uncertainty is guaranteed to disappear. This, however, brings up another point which is of interest in the choice of the crossing area. The dotted lines labelled CHS or KS in fig. 73 mark the boundaries of the allowed crossing regions; that is, only a part of the $\xi_\Delta$, $\xi_\tau$ plane is available for this region and we can exploit this constraint when trying to decide between alternative choices.

How this comes about is detailed in ref. 31. Until now we have never considered the information conveyed by the phase of an inelastic amplitude at resonance. Remember fig. 25, where we saw that the elastic amplitude at resonance is purely imaginary and positive, whereas the inelastic amplitude can be either positive or negative. This means that the phase of the inelastic amplitude with respect to the elastic can be either $0^\circ$ or $180^\circ$. This can be immediately related to the sign of the product of the elastic and the inelastic coupling constants as shown in fig. 74. Now, if we have a way of measuring the relative phases of a set of resonances (and this comes out from the analyses discussed in the second lecture) and if one of these resonances happens to require a definite sign for the product of its coupling constants, then we also have a useful inequality on the coupling constants for the resonances in question. Referring to fig. 75, we see that from the relative phase of the $\Lambda\pi$ decay mode of certain resonances we can limit the allowed region of the $\xi_\Delta$, $\xi_\tau$ plane to the shaded areas$^{31}$. In the same way, the $\Sigma\pi$ decay mode of these and other resonances contributes the information collected in fig. 76$^{28}$. It is from these considerations that one produces the dotted boundary lines in fig. 73 and following.
What can be obtained from the relative phase at resonance between two resonances:

\[
T_{abcd} = \frac{g_{ab} \cdot g_{cd} \cdot R}{(E - E') - i \frac{\Gamma}{2}}
\]

For example, \( \bar{K}N \rightarrow \bar{K}N \) and \( \bar{K}N \rightarrow \Lambda\pi \)

- Elastic channel amplitude \( \propto g_{2}^{\bar{K}N} \cdot g_{\Lambda\pi} \) - always positive
- \( \Lambda\pi \) channel amplitude \( \propto g_{2}^{\Lambda\pi} \cdot g_{\Lambda\pi}^{*} \) - designates the allowed region of the \( g_{2}^{\Lambda\pi} - g_{\Lambda\pi}^{*} \) plane.

Fig. 74

From the relative phase at resonance in \( \bar{K}N \rightarrow \Lambda\pi \):

<table>
<thead>
<tr>
<th>Resonance observed</th>
<th>( \sum(1640) )</th>
<th>( \sum(1760) )</th>
<th>( \sum(1910) )</th>
<th>( \sum(2030) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Phase at resonance</td>
<td>( \uparrow )</td>
<td>( \downarrow )</td>
<td>( \downarrow )</td>
<td>( \uparrow )</td>
</tr>
<tr>
<td>Multiplet assignment</td>
<td>( \frac{3}{2} ) - octet</td>
<td>( \frac{5}{2} ) - octet</td>
<td>( \frac{5}{2} ) + octet</td>
<td>( \frac{3}{2} ) - decuplet</td>
</tr>
<tr>
<td>Value of ( T )</td>
<td>( \sqrt{\frac{3}{50}} g_{d}^{2} \left( -\frac{1}{\sqrt{20}} g_{d}^{2} + \frac{1}{\sqrt{12}} g_{f}^{2} \right) )</td>
<td>( \sqrt{\frac{1}{45}} g_{d}^{2} )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Allowed values of ( g_{f} ) and ( g_{d} )</td>
<td>( g_{f} &gt; \sqrt{\frac{1}{15}} g_{d} )</td>
<td>( g_{f} &lt; \sqrt{\frac{1}{15}} g_{d} )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Allowed region of the ( g_{2}^{\Lambda\pi}, g_{\Lambda\pi}^{*} ) plane</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Fig. 75
From the phases at resonance in $K^- p \rightarrow \Sigma \pi$ -

<table>
<thead>
<tr>
<th>Resonance</th>
<th>$\Lambda$ (1110)</th>
<th>$\Lambda$ (1520)</th>
<th>$\Lambda$ (1750)</th>
<th>$\Lambda$ (1910)</th>
<th>$\Lambda$ (2050)</th>
<th>$\Lambda$ (2100)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Phase</td>
<td>$\uparrow$ $\uparrow$</td>
<td>$\uparrow$ $\uparrow$</td>
<td>$\uparrow$ $\uparrow$</td>
<td>$\uparrow$ $\uparrow$</td>
<td>$\uparrow$ $\uparrow$</td>
<td>$\uparrow$ $\uparrow$</td>
</tr>
<tr>
<td>Multiplet</td>
<td>$\frac{1}{2}^-$ octet</td>
<td>$\frac{1}{2}^-$ octet</td>
<td>$\frac{1}{2}^-$ octet</td>
<td>$\frac{1}{2}^+$ octet</td>
<td>$\frac{1}{2}^+$ octet</td>
<td>$\frac{1}{2}^-$ octet</td>
</tr>
<tr>
<td>Sign</td>
<td>+</td>
<td>+</td>
<td>-</td>
<td>-</td>
<td>+</td>
<td>-</td>
</tr>
</tbody>
</table>

Fig. 76

The last four figures, from 77 to 80 show the status of the remaining octets. The same type of considerations is valid here as in the previous cases. The detailed discussion of these results is left to ref. 26.

The general conclusion that one derives from this exercise is that, remarkably enough, the bulk of the data examined agrees rather well with the model in question. At the same time there are also discrepancies, here and there, which are serious enough to cast doubts on the correctness of either some specific multiplet composition or the experimental information employed. In all cases the exercise is quite useful. It shows where the experimental information needs a more careful treatment and also provides a guide to which could be the most meaningful experiments to perform in the future.
\[
\begin{align*}
\Sigma & \quad \text{(5/2-)} \\
N(1688) & \quad \text{(1688)} \\
\Lambda(1827) & \quad \text{(1827)} \\
\Sigma(1765) & \quad \text{(1765)} \\
\Xi(1933) & \quad \text{(1933)}
\end{align*}
\]
Regge - recurrences?

\[ 1/2^- \text{octet} \quad \text{and} \quad 5/2^- \text{octet} \]

\begin{align*}
\mathbf{N} (940) & \quad \mathbf{N} (1688) \\
\mathbf{\Lambda} (1115) & \quad \mathbf{\Lambda} (1820) \\
\mathbf{\Sigma} (1190) & \quad \mathbf{\Sigma} (1910) \\
\mathbf{\Xi} (1320) & \quad \mathbf{\Xi} (1990)
\end{align*}

\begin{align*}
\mathcal{N}\pi & \geq 65\% \\
\mathcal{N}\rho & \leq 0.1\% \\
\mathcal{N}\eta & \leq 1.5\% \\
\mathcal{L}\pi & \geq 60\% \\
\mathcal{L}\rho & \leq 12\% \\
\mathcal{L}\eta & \leq 1.4\% \\
\mathcal{N}\Xi & \geq 8\% \\
\mathcal{L}\Xi & \leq 17\% \\
\mathcal{L}\pi & \geq 10\% \\
\text{not found}
\end{align*}

Fig. 79

---

(5/2^+) \quad \frac{5/2^+}{\text{N} (1688)} \quad \text{A} (1820) \quad \Sigma (1910)

\[ g_d \]

Fig. 80
References


8. G.A. Smith and J.S. Lindsey; Proc. of Athens Conf. on Resonant Particles (1965) 251.


11. CHS Collaboration, same authors as in ref. 1; Phys. Letters 19 (1965) 338.

12. CHS Collaboration, same authors as in ref. 1; private communication.


PHENOMENOLOGY OF CP VIOLATION IN $K^0$ DECAYS

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PHENOMENOLOGY OF CP VIOLATION IN K⁰ DECAYS

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1. SUPERPOSITION PRINCIPLE AND SUPERSELECTION RULES

One of the fundamental principles of quantum mechanics is the superposition principle. It states that if \( |A\rangle \) and \( |B\rangle \) are two states of a quantum mechanical system, the quantity \( C_1|A\rangle + C_2|B\rangle \), where \( C_1 \) and \( C_2 \) are two arbitrary complex numbers, is also a state of the system. In other words, the superposition principle expresses the basic axiom that the quantum mechanical states of a system can be considered as vectors in a certain Hilbert space. The question that arises, therefore, is the following: given two physical states, any linear combination between them will give a vector of the Hilbert space. Do all these vectors represent physically realizable states? We know, by experience, that the answer is no! For example, if \( |p\rangle \) and \( |n\rangle \) represent the states of a proton and a neutron, respectively, no physical state can be represented by a linear superposition of these two. Notice that a state of the form \( C_1|p\rangle + C_2|n\rangle \) should represent a one-particle state and should not be confounded with \( |p,n\rangle \) which contains two particles, a proton and a neutron. Thus we arrive at the conclusion that, although the states of a system form a Hilbert space, not all of its vectors represent physically realizable states. This result can be expressed precisely by introducing the idea of a superselection rule: there exist certain observables with the property that every physical state is an eigenstate of them. Classical examples of such observables are the electric charge \( Q \), the baryon number \( B \), etc. Physically speaking, this means that every physical state must have well-defined electric charge, baryon number, etc. Therefore, we cannot superpose states with different electric charges as in the above example of a proton and a neutron. In the same way, a state that is a superposition of a nucleon
and a pion is not physically realizable since it has no definite baryon number.

This implies that the only parts of the Hilbert space which are spanned by vectors representing physically realizable states are the subspaces which correspond to definite eigenvalues of all the observables that define the superselection rules of the theory. It is obvious that these operators commute with all the other observables; therefore there is no observable which has non-zero matrix elements between states belonging to different subspaces. Thus, the relative phase between two states having, say, different electric charges, is undetermined since there exists no possibility of transitions between them. Notice that every superselection rule defines a conservation law, but the inverse is not true. Angular momentum is an absolutely conserved quantity, but all physical states are not necessarily eigenstates of it.

2. PHENOMENOLOGY OF THE $K^0 - \bar{K}^0$ SYSTEM

2.1 Generalities

Let me start by recalling some well-known principles. The time evolution of a quantum mechanical state $|a\rangle$ is governed by the Schrödinger equation:

$$i \frac{d}{dt} |a\rangle = H |a\rangle$$

(1)

where $H$ is the Hamiltonian of the system. If $|a\rangle$ has a definite energy, the solutions of (1) have the general form $|a\rangle = |a\rangle_0 e^{-iE_0 t}$. Let us now assume that $|a_1\rangle$ and $|a_2\rangle$ are two eigenstates of the Hamiltonian $H_0$ corresponding to the same energy $E_0$ which, without loss of generality, we can take equal to zero. Furthermore, $|a_1\rangle$ and $|a_2\rangle$ are chosen orthogonal to each other, i.e., $\langle a_1 | a_2 \rangle = 0$. We now introduce a small perturbation $H$. As is well known from elementary quantum mechanics, the eigenstates in zero order will be linear combinations of the form: $|\psi\rangle = C_1 |a_1\rangle + C_2 |a_2\rangle$ which we
can denote, in a convenient form, by the spinor \( \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \). Equation (1) becomes now

\[
i \frac{d}{dt} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \mathcal{H} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}
\]

(2)

where \( \mathcal{H} \) is a 2x2 matrix whose elements are given by

\[
\mathcal{H}_{ij} = \langle a_i | H | a_j \rangle
\]

(3)

The solutions of (2) have the form \( |\psi\rangle = |\psi_0\rangle e^{-i\lambda t} \), where \( |\psi_0\rangle \) and \( \lambda \) are the eigenstates and eigenvalues of the matrix \( \mathcal{H} \).

We thus find the familiar result that the stationary states, in the case of a degenerate level of the unperturbed Hamiltonian, are the ones that diagonalize the matrix of the perturbation.

2.2 The mass matrix

Let us now try to apply the above reasoning to the \( K^0 - \bar{K}^0 \) system. In the absence of weak interactions, the conservation of strangeness defines a superselection rule. Since neutral kaons are produced by strong interactions, they are produced in two mutually orthogonal states \( |K^0\rangle \) and \( |\bar{K}^0\rangle \) with strangeness +1 and -1, respectively. The relative phase of these two states is not determined by the strong interactions but, if we assume the invariance under CPT, their masses are rigorously equal.

The weak interactions violate strangeness and allow for transitions between \( K \rightarrow \bar{K} \). Therefore perturbation theory tells us that the states which have a simple exponential time dependence are certain linear superpositions of \( K^0 \) and \( \bar{K}^0 \) of the form \( |\psi\rangle = p|K^0\rangle + q|\bar{K}^0\rangle \) with \( |\psi\rangle = |\psi_0\rangle e^{-i\lambda t} \). However, the weak interactions allow also for decays of kaons into other states (2\( \pi \), 3\( \pi \), leptonic, etc.). Therefore the norm of the state \( |\psi\rangle \) cannot be constant in time since the probability of finding a \( |K^0\rangle \) or \( |\bar{K}^0\rangle \) must decrease in time. It follows that \( \lambda \) must be a complex number.

Equation (2) can now be written in the general form
\[ i \frac{d}{dt} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \mathcal{M} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} \]  

(4)

where \((C_1, C_2)\) denotes the state \(C_1|K^0\rangle + C_2|\bar{K}^0\rangle\) and \(\mathcal{M}\) is a 2x2 matrix.

Among all states which satisfy Eq. (4) there exist only two which have simple exponential time dependence of the form \(|\psi\rangle = |\psi_0\rangle e^{-i\lambda t}\). They obviously are the eigenstates of the matrix with corresponding eigenvalues \(\lambda\):

\[ \mathcal{U} \begin{pmatrix} \psi \\ \eta \end{pmatrix} = \lambda \begin{pmatrix} \psi \\ \eta \end{pmatrix} \]  

(5)

The matrix \(\mathcal{U}\) is called the mass matrix. Due to the presence of the decay channels, \(\mathcal{U}\) is not Hermitian (otherwise it would have had real eigenvalues) and its matrix elements are not given by the simple Eq. (3). It is convenient to separate the Hermitian and antihermitian parts of \(\mathcal{U}\) and write:

\[ \mathcal{M} = M - \frac{i}{2} \mathcal{\Gamma} \quad M = \frac{1}{2}(\mathcal{U}^* \mathcal{U} + \mathcal{U} \mathcal{U}^*) \quad \mathcal{\Gamma} = i(\mathcal{U} \mathcal{U}^* - \mathcal{U}^* \mathcal{U}) \]  

(6)

where \(M\) and \(\mathcal{\Gamma}\) are Hermitian 2x2 matrices. Thus, the problem of the \(K^0-\bar{K}^0\) system has been reduced to that of determining the matrix elements of \(M\) and \(\mathcal{\Gamma}\).

2.3 The unitarity relations

Let \(|\psi\rangle\) denote an arbitrary superposition of \(|K^0\rangle\) and \(|\bar{K}^0\rangle\) of the form \(C_1|K^0\rangle + C_2|\bar{K}^0\rangle\). Kaons being unstable under weak interactions, the probability — as a function of time — for the decay \(|\psi\rangle \rightarrow |F\rangle\), where \(|F\rangle\) is a specified final state, will be given by the square of the transition matrix element:

\[ |\langle F | T | \psi \rangle|^2 = |\langle F | T | K^0 \rangle C_1 + \langle F | T | \bar{K}^0 \rangle C_2 |^2 \]  

(7)
The total decay rate is given by the sum over all possible final states with the same energy momentum as the state \( |\Phi\rangle \):

\[
\Gamma_{\text{tot}} = |C_1|^2 \sum_F |\langle F|T|K^o\rangle|^2 + |C_2|^2 \sum_F |\langle F|T|\bar{K}^o\rangle|^2
\]

\[+ C_1^* C_2 \sum_F (\langle F|T|K^o\rangle)^* \langle F|T|K^o\rangle \]

\[+ C_1 C_2^* \sum_F (\langle F|T|\bar{K}^o\rangle)^* \langle F|T|\bar{K}^o\rangle \]

(8)

Conservation of probabilities implies that \( \Gamma_{\text{tot}} \) must be equal to the corresponding decrease in the norm of the state \( |\Phi\rangle \). Using Eq. (4), we find:

\[-\frac{d}{dt} |\langle \Phi | \Phi \rangle| = -\frac{d}{dt} \left( |C_1|^2 + |C_2|^2 \right) =
\]

\[-2 Re \left( C_1^* \frac{dC_1}{dt} + C_2^* \frac{dC_2}{dt} \right) =
\]

\[-2 |m| \left( C_1^* \mathcal{U}_1 C_1 + C_2^* \mathcal{U}_2 C_2 + C_1 \mathcal{U}_1^* C_1 + C_2 \mathcal{U}_2^* C_2 \right) \]

\[-\sum_{s=1,2} |C_s|^2 \Gamma_s + C_1^* C_2 \Gamma_1 + C_2^* C_1 \Gamma_2 \]

(9)

The equality \( \Gamma_{\text{tot}} = -(d/dt) |\langle \Phi | \Phi \rangle| \) must hold for any state \( |\Phi\rangle \), i.e., for any pair \( C_1, C_2 \). Therefore we obtain:

\[\Gamma_1 = \sum_F |\langle F|T|K^o\rangle|^2\]

\[\Gamma_2 = \sum_F |\langle F|T|\bar{K}^o\rangle|^2\]

(10)

\[\Gamma_1 = \Gamma_2^* = \sum_F \left( \langle F|T|K^o\rangle \right)^* \left( \langle F|T|\bar{K}^o\rangle \right)\]

These equations are called "unitarity relations" because they follow from the conservation of probabilities. They relate the matrix elements of the antihermitian part of \( \mathcal{U} \) with decay rates. We remark that they do not give any information on the elements of \( M \).
The reason is that the summations in (10) extend only over all physical
transitions, i.e., those which conserve the energy and momentum. The
elements of \( \eta \) depend also on all possible virtual transitions like
\( N \bar{N} \) pairs, arbitrary numbers of pions, etc., which do not contribute
to the conservation of probabilities.

2.4 The TCP relations

Theoreticians have good reasons to believe that all interactions
are invariant under the operation \( \Theta \) which is defined as the product
of the operations \( P, C \) and \( T \) (parity, charge conjugation and time
reversal) taken in any order. The so-called TCP theorem can be
shown to follow from the general axioms of local quantum field theory.
In a simple form, it states the following: let \( H \) be a local
Hermitian Hamiltonian, invariant under proper Lorentz transformations,
which describes the interaction between a number of fields \( \phi_{\pm}(x) \).
Under the action of the operators \( P, C \) and \( T \), \( \phi_{\pm}(x) \) transform in
the usual way. For example, for spin zero particles, we have:

\[
P \phi(\vec{x}, t) P^* = \eta_P \phi(-\vec{x}, t)
\]

\[
C \phi(\vec{x}, t) C^{-1} = \eta_C \phi^*(\vec{x}, t)
\]

\[
T \phi(\vec{x}, t) T^{-1} = \eta_T \phi(\vec{x}, -t)
\]

where \( + \) means Hermitian conjugate.

Analogous formulae hold for any spin. The phases \( \eta_P, \eta_C \)
and \( \eta_T \) have absolute values equal to unity and characterize the
intrinsic transformation properties of the fields. For example,
\( \eta_P = -1 \) for a pseudoscalar particle. The TCP theorem then states
that there always exists a choice of phases \( \eta_P, \eta_C \) and \( \eta_T \)
for the various fields (usually in more than one way) with the following
properties: a) \( H \) commutes with \( \Theta \); and b) if with this
choice \( H \) does not commute with \( P \), it will not do so with any other
choice and the theory will not conserve parity. The same is true for
\( C \) and \( T \).
In the following we shall state some of the main consequences of the TCP theorem:

a) The mass of a particle is equal to the mass of its antiparticle. The best evidence for such an equality is given by

$$m_{K^c} = \langle K^c | H | K^c \rangle = m_{K^c} = \langle K^c | H | K^c \rangle$$  \hspace{1cm} (11)$$

which holds to an accuracy of:

$$\sim \frac{m_S - m_L}{m_K} \sim 6.5 \times 10^{-15}$$

where $m_S - m_L$ is the mass difference between the short- and long-lived neutral kaons. Note, however, that this equality tells nothing about the strangeness changing weak interaction which does not contribute to Eq. (11).

b) The lifetime of any particle is equal to the lifetime of its antiparticle. This property allows us to test the $|\Delta S| = 1$ part of the weak interactions. The best evidence is the lifetime equality between $K^+$ and $K^-$

$$\frac{\tau_{K^+}}{\tau_{K^-}} = 1 - 0.0002 \pm 0.0007$$

c) More generally, the TCP theorem shows that the amplitudes for the processes $A \rightarrow B$ and $\bar{B} \rightarrow \bar{A}$ are equal. The bar denotes interchange of particles and antiparticles and the prime denotes reversal of spin directions.

Let us apply this result to the $K^0 - \bar{K}^0$ system: Eq. (4) shows that a state which at $t = 0$ was a pure $|K^0\rangle$ ($C_2(t=0) = 0$) will evolve at $t = \frac{\tau}{2} t$:

$$|K^c\rangle_t = |K^c\rangle_0 \exp \left\{ \frac{i}{\hbar} t \left\{ \mathcal{H}_1 |K^c\rangle + \mathcal{H}_2 |\bar{K}^c\rangle \right\} \right\}$$  \hspace{1cm} (12)$$

Similarly, for an initially $|\bar{K}^0\rangle$ state:
\[ |\bar{K}^o_\tau\rangle = |\bar{K}^o_\tau\rangle - i \delta t \left( M_{12} |K^o\rangle + M_{22} |\bar{K}^o\rangle \right) \]  

(12b)

The property c) above, applied to the transitions $K^0 \rightarrow \bar{K}^0$ and $\bar{K}^0 \rightarrow K^0$, yields:

\[ M_{11} = M_{22} \quad \text{or} \quad M_{11} = M_{22} , \quad \Gamma_{11} = \Gamma_{22} \]  

(13)

The relations (13) are consequences of the TCP theorem which we shall assume in the following.

2.5 Eigenstates - Long- and short-lived kaons

Let us now solve formally the eigenvalue problem, Eq. (5), to obtain the two eigenstates which will constitute the "true particles" in the presence of the weak interactions, i.e., the ones that have well-defined lifetimes. Since Eq. (5) is linear in $p$ and $q$, we can only determine the ratio $p/q$. This leaves the possibility of an over-all arbitrary factor with which we can multiply both $p$ and $q$. We shall use the following convention:

\[ \rho = i \mathcal{U}_{12} , \quad q = i \mathcal{U}_{21} \]  

(14)

One can then verify that the eigenstates and eigenvalues are

\[ \mathcal{U} \left( \pm \rho \right) = \lambda_\pm \left( \pm \rho \right) \]  

(15)

with

\[ \lambda_\pm = \mathcal{U}_{11} \mp i p q \]  

(16)

We therefore define the two states
\[ |L\rangle = \frac{\rho |k^0\rangle + q |\bar{k}^0\rangle}{\sqrt{|\rho|^2 + |q|^2}} \]  
\[ |s\rangle = \frac{\rho |k^0\rangle - q |\bar{k}^0\rangle}{\sqrt{|\rho|^2 + |q|^2}} \]  
\[ \langle L|s\rangle = \frac{|\rho|^2 - |q|^2}{|\rho|^2 + |q|^2} \]  

(17a)  
(17b)  
(17c)

with the time evolution

\[ |L\rangle \rightarrow e^{-i\lambda_+ t} |L\rangle, \quad |s\rangle \rightarrow e^{-i\lambda_- t} |s\rangle \]

\[ \lambda_+ = \frac{m_L}{2} \gamma_L, \quad \lambda_- = \frac{m_S}{2} \gamma_S \]  

(18)

where \( m_L (m_S) \) and \( \gamma_L (\gamma_S) \) are the mass and decay rate of the long- (short) lived component. We remark that if CP is conserved, \( p = -q \), so that - with a suitable change of phases - \( |L\rangle \) and \( |S\rangle \) become the familiar \( |k^0_1\rangle \) and \( |k^0_2\rangle \) states.

Using equations (17) and (18), we can write explicitly the time evolution of any arbitrary state of the form \( |\Phi\rangle = c_1 |k^0_1\rangle + c_2 |\bar{k}^0\rangle \). We first invert the relations (17) to obtain:

\[ 2\rho |k^c\rangle = (|L\rangle + |S\rangle) \sqrt{|\rho|^2 + |q|^2} \]

\[ 2q |\bar{k}^c\rangle = (|L\rangle - |S\rangle) \sqrt{|\rho|^2 + |q|^2} \]  

(19)

We then use (18) for the time evolution of \( |L\rangle \) and \( |S\rangle \) and finally we re-express \( |L\rangle \) and \( |S\rangle \), using (17), on the basis of \( |k^0\rangle \) and \( |\bar{k}^0\rangle \). The result is:
\[ |\psi\rangle \rightarrow \frac{C_1 + C_2}{2} \left( e^{-i\lambda_1 t} + e^{-i\lambda_2 t} \right) |K^0\rangle + \frac{1}{2} \left( C_1 \frac{q}{p} + C_2 \frac{p}{q} \right) \left( e^{-i\lambda_1 t} - e^{-i\lambda_2 t} \right) |\bar{K}^0\rangle \] (20)

2.6 Phenomenology of CP violation in \( K^0 \) decay

It has been discovered that the long-lived component \( |L\rangle \) decays into \( 2\eta \) which, in the s wave, have \( CP = +1 \). Therefore, if \( CP \) invariance holds, both \( |L\rangle \) and \( |S\rangle \) cannot decay into two pions, in contradiction with experiment. Therefore we conclude that \( CP \) is violated in \( K^0-\bar{K}^0 \) decay. In this Section we shall attempt to analyze the experimental situation from the phenomenological viewpoint.

We shall assume that the final states which contribute in the sums of Eq. (10) are the \( \pi\pi \) (I=0), \( \pi\bar{\pi} \) (I=2), \( 3\pi \) and \( \pi\pi\nu \) (leptonic). We shall neglect contributions coming from the rare decay modes such as \( 2\eta\pi\nu \), radiative, etc. We remind that the phase of the state \( |\bar{K}^0\rangle \) relative to that of \( |K^0\rangle \) was not fixed by the strong interactions, since \( |K^0\rangle \) and \( |\bar{K}^0\rangle \) have different values of strangeness, which, as we already mentioned, defines a superselection rule for strong and electromagnetic interactions. We therefore adopt the following convention: we fix the phase of \( |\bar{K}^0\rangle \) such that:

\[ A_c = \langle 2\pi, I=0 | T | K^0 \rangle = \langle 2\pi, I=0 | T | \bar{K}^0 \rangle = \text{real} \] (21)

In Eq. (21) \( \langle 2\pi, I=0 \rangle \) denotes the \( I=0, 2\pi \) standing wave state, i.e., we do not include the phase coming from the final state strong interaction which is equal to \( e^{i\delta_0} \) with \( \delta_0 \) being the s wave \( I=0 \) \( \pi^- \pi^- \) phase shift. Notice that the TCP invariance gives, for any final state \( F \),

\[ \langle F | T | K^0 \rangle = \left( \langle \bar{F}' | T | \bar{K}^0 \rangle \right)^* \] (22)
Using Eqs. (10), (14), (22) and the phase convention (21), we can write:

\[ 2p^2 = A_0^2 + A_2^* A_2 + X_\ell i \gamma_\ell + X_{3n} i \gamma_{3n} + i M_r - M_i \]  

\[ 2q^2 = A_0^2 + A_2^* A_2 + X_\ell i \gamma_\ell + X_{3n} i \gamma_{3n} + i M_r + M_i \]  

where:

\[ A_2 = \langle 2n, I=2, \text{standing wave} | T | k^0 \rangle \]

\[ X_\ell + i \gamma_\ell = \sum_{\ell \nu} \left( \langle \ell \nu | T | k^0 \rangle \right)^* \langle \ell \nu | T | k^0 \rangle \]

\[ X_{3n} + i \gamma_{3n} = \sum_{3\ell n} \left( \langle 3\ell n | T | k^0 \rangle \right)^* \langle 3\ell n | T | k^0 \rangle \]

\[ M_r + i M_i = \frac{1}{2} M_{i2} \]

The summations in (24) extend over all possible leptonic and \( 3/7 \) final states. We shall now analyze every final state.

a) Two-pion decays

We shall denote with \( |0\rangle \) and \( |2\rangle \) the \( I=0 \) and \( I=2 \) standing wave states. Without assuming the \( |\Delta I| = \frac{1}{2} \) rule, there are four amplitudes involved:

\[ \sqrt{|p|^2 + |q|^2} \langle 0 | T | L \rangle = (p+q) A_0 \]

\[ \sqrt{|p|^2 + |q|^2} \langle 0 | T | S \rangle = (p-q) A_0 \]

\[ \sqrt{|p|^2 + |q|^2} \langle 2 | T | L \rangle = p A_2 + q A_2^* \]

\[ \sqrt{|p|^2 + |q|^2} \langle 2 | T | S \rangle = p A_2 - q A_2^* \]

Experimentally, one measures the ratios:
\[ \eta_{+-} = \frac{\langle n^+ n^- | T | L \rangle}{\langle n^+ n^- | T | S \rangle} \]  

(26a)

\[ \eta_{00} = \frac{\langle n^0 n^0 | T | L \rangle}{\langle n^0 n^0 | T | S \rangle} \]  

(26b)

where \( | n^+ n^- \rangle \) and \( | n^0 n^0 \rangle \) denote the outgoing waves, i.e., they include the phases of the final state interaction. Using (25), we can write (26) in the form:

\[ \eta_{+-} = \frac{\frac{e^{i\delta_2}}{\sqrt{6}} \langle 2 | T | L \rangle + \frac{e^{i\delta_0}}{\sqrt{3}} \langle 0 | T | L \rangle}{\frac{e^{i\delta_2}}{\sqrt{6}} \langle 2 | T | S \rangle + \frac{e^{i\delta_0}}{\sqrt{3}} \langle 0 | T | S \rangle} = \frac{\epsilon + \epsilon'}{1 + \omega} \approx \epsilon + \epsilon' \]  

(27)

where we have defined

\[ \epsilon = \frac{P + q}{P - q} = \frac{\langle 0 | T | L \rangle}{\langle 0 | T | S \rangle} \]  

(28)

\[ \epsilon' = \frac{i}{\sqrt{2}} e^{i(\delta_2 - \delta_0)} \frac{\langle 2 | T | L \rangle}{\langle 0 | T | S \rangle} \approx \frac{i}{\sqrt{2}} e^{i(\delta_2 - \delta_0)} \frac{m}{A_2} \frac{A_2}{A_0} \]  

(29)

\[ \omega = \frac{i}{\sqrt{2}} e^{i(\delta_2 - \delta_0)} \frac{\langle 2 | T | S \rangle}{\langle 0 | T | S \rangle} \]  

(30)

Notice that the quantities \( \epsilon, \epsilon' \) and \( \omega \), as well as \( \eta_{+-} \) and \( \eta_{00} \), are very small because they all vanish in the limit of CP conservation (\( \epsilon, \epsilon' \)) or \( | \Delta \bar{\Gamma} | = \frac{1}{2} (\epsilon', \omega) \). In other words, the amplitude \( \langle 0 | T | S \rangle \) is much larger than any one of the others. This is the reason of the last step in Eq. (27). In a similar way we
obtain for $\eta_{oo}$:

$$\eta_{oo} \propto \epsilon - 2 \epsilon'$$

(31)

Before the end of this paragraph we shall give some alternative forms for the parameter $\epsilon$ which will be very useful later. We shall exploit the fact that $p + q$ is a small number

$$\epsilon = \frac{p + q}{p - q} = \frac{p^2 - q^2}{p^2 q^2} \propto \frac{p^2 - q^2}{4pq} \approx \frac{\gamma}{\gamma_s} \frac{\gamma_s}{\gamma} = \frac{\hat{m} + i \gamma}{\gamma_L - \gamma_s + 2i \gamma_s (\gamma_L - \gamma_s)}$$

(28')

where Eqs. (16), (18) and (23) have been used. Another useful form is:

$$\epsilon = \frac{p + q}{p - q} \approx \frac{|p|^2 |q|^2 - 2i \gamma |pq|^*}{|p - q|^2}$$

from which it follows

$$R \epsilon \approx \frac{|p|^2 |q|^2}{|p - q|^2} \approx \frac{|p|^2 |q|^2}{4 |p|^2} \approx \frac{|\gamma|}{2}$$

(28'')

In deriving (28'') we have used (17c).

Equations (26) to (31) form the basis of the phenomenological analysis. Notice that one of the parameters to determine, the phase of $\epsilon'$, is a strong interaction parameter, namely the difference of $I = 2$ and $I = 0$ s wave $\pi^- - \pi^+$ phase shifts. We postpone the discussion of the experimental situation until the end of this Section and we proceed by analyzing the other decay modes.
b) Leptonic decay modes

Let us define the amplitudes for the four leptonic decay modes:

\[ A_{\mu \rho} (K^0 \to n^- e^+ \nu) = f \]  \hspace{1cm} (32a)

\[ A_{\mu \rho} (\bar{K}^0 \to n^+ e^- \bar{\nu}) = f^* \]  \hspace{1cm} (32b)

\[ A_{\mu \rho} (K^0 \to n^+ e^- \bar{\nu}) = g^* \]  \hspace{1cm} (32c)

\[ A_{\mu \rho} (\bar{K}^0 \to n^- e^+ \nu) = g \]  \hspace{1cm} (32d)

If the \( \Delta S = \Delta Q \) rule holds, the amplitude \( g \) is equal to zero. Furthermore we remark that in this case \( K^0 \)'s and \( \bar{K}^0 \)'s decay into different final states, so that the contribution of the leptonic decays to the off-diagonal elements of the \( \Gamma \) matrix, Eq. (10), vanishes. Let \( x = g/f \) be the ratio of \( \Delta S = -\Delta Q \) to \( \Delta S = \Delta Q \) amplitudes. We then have that:

\[ \text{if } x = 0 \quad \Rightarrow \quad \chi_{\ell} = \eta_{\ell} = 0 \]  \hspace{1cm} (33)

Experimentally, we can put an upper limit on \( |x| \) of the order of 0.2. Further experiments are now in progress to reduce this limit.

An interesting quantity, directly accessible to experiment, is the charge asymmetry between the leptons in the decay of \( |L\rangle \):

\[ \mathcal{C} = \frac{\Gamma_{e}^- - \Gamma_{e}^+}{\Gamma_{e}^+ + \Gamma_{e}^-} = \frac{|P f + q g|^2 - |P g^* + q f^*|^2}{|P f + q g|^2 + |P g^* + q f^*|^2} \]

\[ = \frac{(|P|^2 - |q|^2)(1 - |x|^2)}{(|P|^2 + |q|^2)(1 + |x|^2)} \cdot \frac{2 P q g^* x + 2 q^* x q}{2 P q g^* x + 2 q^* x q} \]

\[ \approx \frac{(|P|^2 - |q|^2)(1 - |x|^2)}{(|P|^2 + |q|^2)(1 + |x|^2)} \cdot \frac{\langle L | s \rangle}{|\langle L | s \rangle|} = \frac{1 - |x|^2}{1 + |x|^2} \]  \hspace{1cm} (34)
where we have used again the fact that $p+q$ is a small number and, in the last step, we replaced $|\langle L|S \rangle|$ by Eq. (17c). We see immediately that a charge asymmetry different from zero implies the non-orthogonality of the long- and short-lived components and hence CP violation. Furthermore, Eqs. (28") and (34) relate the charge asymmetry with $\text{Re} \epsilon$. The experimental situation will be examined at the end of this lecture.

c) 3/$\pi$ decay modes

Nothing exciting concerning the CP violation has been found in these decays. Their contribution to the CP violating parameter $p+q$ comes from the term $y_{3\eta}$ of Eqs. (23). Its value is not precisely known, but order of magnitude estimations show that it is quite small ($|y_{3\eta}/A_0^2| < 10^{-3}$). The presence of such a term has not yet been detected.

d) Other decay modes

Little can be said about the rare decay modes and their connection to CP violation. One can prove the following statement: for any possible non-leptonic decay mode of the $K^0$ meson the observation of an interference effect between $|L\rangle$ and $|S\rangle$ decays in the partial decay rate of this mode is clear evidence of CP violation. The proof is quite easy. In order to have an interference effect after integration over the relative momenta of the final particles, the interfering final states coming from an $|L\rangle$ and an $|S\rangle$ decay have the same parity. If CP is conserved, these states must have opposite C. The most general non-leptonic final state will be of the form: $m \chi + n \eta^0 + r(\eta^+ + \eta^-)$ with $r \leq 1$, $2r+n \leq 3$. For this mode $C = (-)^{m+1} \epsilon^r$, where $\epsilon$ is the orbital angular momentum of the system of $\eta^+ - \eta^-$. Therefore states with opposite C have different values of $\epsilon$ and hence integration over the relative momentum of $\eta^+ - \eta^-$ destroys the interference effect. Up to now this interference has only been observed in the mode $\eta^+ - \eta^-$. Experiments are now in progress to detect the same effect in the $\eta^0 - \eta^0$ mode. Other interesting possibilities are the $2\chi$ and $\eta^+ \eta^- \chi$ decay modes.
e) **Analysis of the experimental situation**

The present experimental situation can be summarized as follows:

i) $|\eta_+|$ is fairly well known and little progress can be expected. We have

$$|\eta_+| = (1.96 \pm 0.04) \times 10^{-3}$$

ii) The phase of $\eta_+$ has been fluctuating during the past months. The present value is:

$$\phi_\eta \approx 51^\circ \pm 11^\circ$$

iii) $|\eta_{oo}|$ is still controversial. The CERN value, which has been published, is

$$|\eta_{oo}| = (3.6 \pm 0.6) \times 10^{-3}$$

iv) The charge asymmetry in the leptonic mode has also been reported. Its value in $K_{e3}$ decay is:

$$\xi \approx (2.24 \pm 0.35) \times 10^{-3}$$

and in $K_{\mu3}$ decays

$$\xi \approx (4.0 \pm 1.5) \times 10^{-3}$$

All these results, with the exception of $|\eta_+|$, need further confirmation. Therefore the conclusions of this lecture may change radically in the near future.

Equations (23) show that if $\text{Im} A_2 = y_\xi y_\eta = M_4 = 0$, then $p = -q$ and therefore $\xi = \xi' = \eta_+ = \eta_{oo} = 0$ in contradiction with experiment. Furthermore, we have seen that $y_\xi$ and $y_\eta$ are small and cannot account, alone, for the observed CP violating effects. We conclude that the main part of CP violation comes from the non-vanishing of $\text{Im} A_2$ and/or $M_4$. On the other hand, if
\[ \text{Im } A_2 = 0, \text{ Eq. (29) shows that } \epsilon' = 0 \text{ and then } \eta_+ = \eta_{oo} \]

[Eqs. (27) and (31)]. Therefore a theory in which the CP violation comes entirely from the off-shell contributions to the mass matrix (e.g., a superweak theory) seems to be in contradiction with experiment.

Let us now use Eq. (28') to estimate the phase of \( \epsilon \). The imaginary part of the numerator is very small compared to the value of \( \text{Re } \epsilon \), as given by (28''). Therefore we can write approximately:

\[
\varphi_\epsilon \approx \omega c \left[ 2 \frac{M_s - M_L}{Y_L - Y_s} \right]
\]

(35)

Inserting experimental values, Eq. (35) gives

\[
\varphi_\epsilon \approx 43^\circ - 45^\circ
\]

It is hard to estimate the possible errors in the above value. If we take the upper limits of all presently known quantities, such as \( Y_L \) or \( y_3 \), we can arrive at an error as large as \( 8^\circ \). However, this is most probably a huge overestimation and is introduced only at the expense of large \( \Delta S = - \Delta Q \) amplitudes in leptonic decays and large CP violating effects in 3\( \eta \) decays. None of these possibilities is attractive from the theoretical point of view. In the following, we shall take \( \varphi_\epsilon = 45^\circ \), bearing in mind that some errors are possible.

Therefore the fundamental quantities \( \eta_+, \eta_{oo}, \epsilon, \epsilon' \), \( |<L|S>| \), \( \delta \) are related by Eqs. (27), (28''), (31) and (34), and we have information about five real quantities among them. Hence the problem is overdetermined. The graphical solution of the above equations, taking into account the experimental values, is shown in the Figure.

We see that the only solution, consistent with all the above quoted experimental results (actually it corresponds to a somehow larger value of \( \varphi_+ \)), gives a \( \eta_{oo} \sim 0 \) and \( \delta_2 - \delta_0 \) large and positive. This is very unlikely to be true because the information
that we have from strong interactions concerning the $\Omega - \Omega$ phase shifts seems to indicate that $\xi_0$ is substantial and positive and $\xi_2$ small and probably negative. One could still obtain an acceptable solution by choosing $\xi_\ell \sim 52^\circ$ but, as we mentioned already, the price we have to pay for it is too high. We therefore conclude that, unless something is fundamentally wrong concerning our theoretical prejudice, at least one of the above experimental numbers must be reviewed.
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