THE BEAM ENVELOPE EQUATION — SYSTEMATIC SOLUTION FOR A PERIODIC QUADRUPOLE LATTICE WITH SPACE CHARGE*

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Many approximate solutions for matched beam envelope functions with space charge have been developed; they generally have errors of 2–10% for the parameters of interest and cannot be reliably improved. The new, systematic approach described here provides the K-V envelope functions to high accuracy as a power series in the quadrupole gradient. A useful simplification results from defining the sum and difference of the envelope radii; $S = (a + b)/2$ varies only slightly with distance z along the system axis, and $D = (a - b)/2$ contains most of the envelope oscillations. To solve the coupled equations for S and D , the quadrupole strength $K(z)$ is turned on by replacing K with αK_1 and letting α increase continuously from 0 to 1. It is found that S and D may be expanded in even and odd powers of α , respectively. Equations for the coefficients of powers of α are then solved successively by integration in z . The periodicity conditions and tune integration close the calculation. Simple low order results are typically accurate to 1% or better.

Keywords: Beam transport, collective effects, particle dynamics

1 INTRODUCTION

The matched (periodic) solution of the coupled Kapchinskij-Vladimirskij (K-V) beam envelope equations^{1,2} is used extensively in the design of quadrupole transport systems. Exact results for a specified set of beam and lattice parameters are readily obtained numerically. However to perform scoping studies, scaling, cost optimization, and to gain physical understanding, it is very desirable to have simple analytical formulas for the envelope functions. The general problem may be stated as follows. For specified quadrupole strength $K(z)$ with period (P), beam edge emittance (ε), and beam perveance (Q), find the matched envelope radii $a(z)$ and $b(z)$. The depressed phase advance per period or tune (σ) is then determined from

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the mean of a^{-2} , and the undepressed phase advance (σ_0) is determined in similar fashion in the limit $Q = 0$. Usually the maximum edge radius (a_m), occurring at or near the middle of a focusing quadrupole, is of greatest interest. However, other quantities such as the envelope radius and slope (da/dz) at the midpoint of a drift are often desired. It may also happen that a_m is specified and the corresponding value of Q is to be determined, or some other combination of beam and lattice parameters are to be related.

Although the treatment of the beam envelope presented here is restricted to a drifting beam, the envelope formalism can in general be extended to include effects of acceleration. These include an additional axisymmetric focussing action from acceleration gaps, a scale decrease in the envelope slopes from acceleration, and conservation of normalized emittance, ($\varepsilon_n = \beta\gamma\varepsilon$) instead of ε itself. An approximate matched solution can be maintained in this case by varying P and K with beam energy, and approximate formulas for envelope parameters can be derived. In particular the continuous limit formulas (1) and (2) given below remain a valid lowest order approximation. A full treatment of this subject would go considerably beyond the intended scope of this paper and is therefore not presented here.

The simplest rough design formulas are obtained by assuming that the quadrupoles effectively act like a confining harmonic well with frequency $\sigma_0/2\pi P$ and that the envelope radii are approximated by their mean values ($\bar{a}$). The perveance and emittance are then related to the tunes, σ_0 and σ (expressed in radians) by the "smooth limit" formulas

$$\frac{QP^2}{\bar{a}^2} = \sigma_0^2 - \sigma^2, \quad (1)$$

$$\frac{\varepsilon P}{\bar{a}^2} = \sigma. \quad (2)$$

These formulas are correct in the limit $\sigma_0 \rightarrow 0$, but are in error by up to 40% for the typical design case of $\sigma_0 = 72^\circ$. Furthermore the smooth limit, by itself, does not provide formulas for σ_0 , $a_m/\bar{a}$, or the envelope slopes. The evaluation of the latter quantities requires a definite prescription for $K(z)$ and treatment of the associated envelope oscillations.

In the absence of space charge, the envelope radii and tune may be determined using the transfer matrix formulation of Courant and Snyder,³ with simple analytical formulas obtained for flat-topped $K(z)$. To include space charge, a hybrid of the smooth limit with the transfer matrix method,⁴ developed by Reiser, yields a fairly simple formula replacing Equation (1) with accuracy improved to $\sim 10\%$. Recently

Davidson and Qian⁵ have also treated and extended the smooth limit by an extensive use of Fourier expansions. Another approach,⁶ which does not start from the smooth limit, is to introduce the averaged space charge force directly into the transfer matrix as a series of kicks proportional to Q . The resulting formulas are typically accurate to about 3% in the lowest order of an expansion in Q and K^2 . However, in higher orders of Q , they become messy without any large improvement in accuracy.

A new approach to solving for matched envelope functions is developed here. Essentially, an expansion in powers of $K^2 P^4$ is made for various envelope quantities, and the low order non-trivial formulas are typically accurate to about 1% or better. The main results are summarized in Section 6. The general envelope formulation and its approximate solution are given in Sections 2–5. Specific application to a symmetric FODO lattice with flat topped $K(z)$ is given in Section 7. These results are compared to an exact numerical example in Section 8.

2 THE K-V ENVELOPE EQUATIONS

The x and y radii $a(z)$ and $b(z)$ are assumed to satisfy the coupled, non-linear envelope equations of Kapchinskij and Vladimirkij:

$$\frac{d^2 a}{dz^2} = -K(z)a + \frac{\varepsilon^2}{a^3} + \frac{2Q}{a+b}, \quad (3)$$

$$\frac{d^2 b}{dz^2} = +K(z)b + \frac{\varepsilon^2}{b^3} + \frac{2Q}{a+b}. \quad (4)$$

Here the quadrupole strength $K(z)$ is the ratio of the transverse magnetic field gradient $G(z)$ and particle rigidity $[B\rho] = \beta\gamma mc/q$:

$$K(z) = \frac{G(z)}{[B\rho]}.$$

For electrostatic quadrupoles the electric field gradient is used, with $G \rightarrow G/\beta c$ in the formula for K . The perveance is the dimensionless constant proportional to current as defined by Lawson⁷

$$Q = \frac{2qI}{(\beta\gamma)^3 mc^3 4\pi \varepsilon_0},$$

and the (un-normalized) edge emittance $\varepsilon = \varepsilon_x = \varepsilon_y$ is the occupied $(x, dx/dz)$ phase space area divided by π . For the K-V phase space distribution this is a uniformly filled ellipse.

FIGURE 1: Unsymmetrical Quadrupole Doublet System. The full period length is $P = 2L$. Drifts are centered at $Z = 0, \pm L, \pm 2L, \dots$ and $K(-z) = -K(z)$.

By assumption

$$K(z + P) = K(z). \quad (5)$$

We also assume the mean of K vanishes:

$$\bar{K} = P^{-1} \int_0^P K(z) dz = 0, \quad (6)$$

and K is antisymmetric around $z = 0$; $K(-z) = -K(z)$. Denoting the half period length $L \equiv P/2$, it follows that $K(z)$ is also antisymmetric around $z = \pm L, \pm 2L, \dots$. No additional symmetry is assumed, so a system of unsymmetrical quadrupole doublets is accommodated by the general formulation (see Figure 1).

The matched envelope radii exhibit the periodicity of the lattice:

$$a(z + P) = a(z), \quad b(z + P) = b(z), \quad (7a)$$

and at mid-drift points $z = 0, \pm L, \pm 2L, \dots$, it is easily shown that

$$a = b, \quad (7b)$$

$$da/dz = -db/dz.$$

For the special case in which $K(z)$ is also symmetric around $L/2$, e.g. the FODO system, it is also found that $a(z + L) = b(z)$ and $da/dz = db/dz = 0$ at $z = L/2$.

3 SCALE QUANTITIES

We define the sum and difference of envelope radii

$$\begin{aligned} S(z) &= (a + b)/2, & D(z) &= (a - b)/2, \\ a &= S + D, & b &= S - D. \end{aligned} \quad (8)$$

It will be shown that $S(z)$ is nearly constant; most of the envelope oscillation appears in $D(z)$. Denoting d/dz by a superscript prime, Equations (3) and (4) are added and subtracted to yield

$$S'' = -KD + \frac{\varepsilon^2}{2} \left[\frac{1}{(S+D)^3} + \frac{1}{(S-D)^3} \right] + \frac{Q}{S}, \quad (9)$$

$$D'' = -KS + \frac{\varepsilon^2}{2} \left[\frac{1}{(S+D)^3} - \frac{1}{(S-D)^3} \right]. \quad (10)$$

From the symmetries of a and b [Equations (7)] it follows that S is symmetric and D is antisymmetric around $z = 0, \pm L, \pm 2L, \dots$. Hence

$$S'(0) = S'(L) = 0, \quad (11a)$$

$$D(0) = D(L) = 0. \quad (11b)$$

It is sufficient to integrate Equations (9) and (10) in the interval $0 \leq z \leq L$ using the end conditions (11) to determine the unique matched solution. For the special (FODO) case of $K(z)$ symmetric around $z = L/2$, we also have $D'(L/2) = S'(L/2) = 0$.

Next we define the dimensionless envelope functions $s(z)$ and $d(z)$:

$$S(z) = \bar{a}(1 + s(z)), \quad (12)$$

$$D(z) = \bar{a}d(z), \quad (13)$$

where the bar always denotes the simple full period average, e.g.

$$\bar{a} = \bar{b} = \bar{S} = P^{-1} \int_0^P dz S(z). \quad (14)$$

Hence, we have the important condition

$$\bar{s} = 0. \quad (15)$$

Equations (9) and (10) yield

$$s'' = -Kd + \frac{\varepsilon^2}{\bar{a}^4} \frac{1}{2} \left[\frac{1}{(1+s+d)^3} + \frac{1}{(1+s-d)^3} \right] + \frac{Q}{\bar{a}^2} \frac{1}{1+s}, \quad (16)$$

$$d'' = -K(1+s) + \frac{\varepsilon^2}{\bar{a}^4} \frac{1}{2} \left[\frac{1}{(1+s+d)^3} - \frac{1}{(1+s-d)^3} \right]. \quad (17)$$

The functions $s(z)$ and $d(z)$ satisfy the same symmetry conditions as S and D respectively. Note that since the envelope radii are periodic, we have the non-trivial condition

$$0 = \overline{s''} = -\overline{Kd} + \frac{\varepsilon^2}{\bar{a}^4} \frac{1}{2} \left[\overline{\frac{1}{(1+s+d)^3}} + \overline{\frac{1}{(1+s-d)^3}} \right] + \frac{Q}{\bar{a}^2} \overline{\frac{1}{(1+s)}}. \quad (18)$$

In general, the rate of phase advance in the x or y plane is the inverse of the respective β function defined by Courant and Snyder³ ($\beta_x = a^2/\varepsilon$). We therefore have an integration for the tune:

$$\sigma = \int_0^P dz \frac{\varepsilon}{a^2} = \frac{\varepsilon}{2} \int_0^P dz \left(\frac{1}{a^2} + \frac{1}{b^2} \right).$$

Using the definitions (12)–(14), we may write

$$\sigma = \left(\frac{P\varepsilon}{\bar{a}^2} \right) \frac{1}{2} \left[\overline{\frac{1}{(1+s+d)^2}} + \overline{\frac{1}{(1+s-d)^2}} \right]. \quad (19)$$

This is the *depressed tune*. To obtain the undepressed tune, σ_0 , a separate calculation using s , d , and $\varepsilon/\bar{a}^2$ derived for $Q=0$ must be made.

Recall that in the initial formulation of the matched envelope calculation $K(z)$, ε and Q are considered to be specified and the matched radii are to be determined. However, in Equations (16)–(19), $\bar{a}$ actually gets absorbed into combinations with ε and Q ; only $K(z)$, $\varepsilon/\bar{a}^2$, $Q/\bar{a}^2$, $s(z)$ and $d(z)$ appear. In fact, if the scaled distance z/P is employed as a variable, the system may be considered to be driven by the three dimensionless “forces” $K P^2$, $\varepsilon P/\bar{a}^2$ and $Q P^2/\bar{a}^2$. However, due to the matched envelope condition (18), these three quantities cannot be specified independently. Intuitively this is clear since, for example, if we set $\varepsilon=0$ then the transport is space-charge-dominated and we would expect the current *density* $J \propto Q/\bar{a}^2$ to be determined by $K(z)$ alone. The relation (18) is the necessary connection between K , $\varepsilon/\bar{a}^2$ and $Q/\bar{a}^2$, which results from the assumption that the envelope is matched. If we did not look for a matched solution, then all three quantities could be specified independently.

4 METHOD OF SOLUTION

Equations (16) and (17) are solved by a standard technique of perturbation theory. We turn on $K(z)$ proportional to a continuous variable (α):

$$K(z) = \alpha K_1(z), \quad (20)$$

where $K_1(z)$ is the full quadrupole strength function and α increases from 0 to 1. For small α , we expect to recover the smooth limit formulas. As α increases the envelope radii become lumpy, i.e. $d(z)$ becomes appreciable (unless the final value of σ_0 is small).

As $K(z)$ turns on it is also necessary that $\varepsilon/\bar{a}^2$ and $Q/\bar{a}^2$ turn on. A natural dependence suggested by the smooth limit formulas (1) and (2) is $\varepsilon/\bar{a}^2 \sim \alpha$ and $Q/\bar{a}^2 \sim \alpha^2$, so that $\sigma_0 \sim \alpha$ and σ_0/σ is independent of α . Due to the condition (18), it cannot be quite this simple; the system would be overdetermined beyond the lowest two orders in α . An equivalent difficulty appears if one tries to solve Equations (16) and (17) by a simple iterative procedure⁸ starting with $d'' = -K$.

A consistent, but not unique, turn-on procedure is as follows. The ratio $(Q\bar{a}^2/\varepsilon^2)$ is held fixed and we define the constant angle (ϕ):

$$\cos^2 \phi = \left(1 + \frac{Q\bar{a}^2}{\varepsilon^2}\right)^{-1}, \quad (21a)$$

$$\sin^2 \phi = \frac{Q\bar{a}^2}{\varepsilon^2} \left(1 + \frac{Q\bar{a}^2}{\varepsilon^2}\right)^{-1}. \quad (21b)$$

Then a convenient form for the turn-on expansions of $\varepsilon/\bar{a}^2$ and $Q/\bar{a}^2$ is

$$\frac{\varepsilon^2}{\bar{a}^4} = \cos^2 \phi (A_2 \alpha^2 + A_4 \alpha^4 + \dots), \quad (22)$$

$$\frac{Q}{\bar{a}^2} = \sin^2 \phi (A_2 \alpha^2 + A_4 \alpha^4 + \dots), \quad (23)$$

where $A_2, A_4, \dots$ are constants to be determined from condition (18). Holding the ratio $Q\bar{a}^2/\varepsilon^2$ constant as K turns on corresponds to holding the degree of space charge dominance approximately constant.

The consistent expansions for $s(z)$ and $d(z)$ are found to be of the form

$$s(z) = s_2(z)\alpha^2 + s_4(z)\alpha^4 + \dots, \quad (24)$$

$$d(z) = d_1(z)\alpha + d_3(z)\alpha^3 + \dots \quad (25)$$

Setting $\alpha = 1$ gives the unique matched envelope functions for the full strength values of K , $\varepsilon/\bar{a}^2$ and $Q/\bar{a}^2$. However, all intermediate values of α also correspond to matched solutions provided Equation (18) and the symmetry conditions are satisfied for all α . Since α is varied continuously, the coefficients s_n and d_n must individually satisfy the symmetry conditions, with $\bar{s}_n = 0$ and $\bar{s}_n'' = 0$.

Inserting the expansions (20)–(25) into Equations (16) and (17), expanding all expressions in powers of α , and equating coefficients of each power of α , we get

$$d_1'' = -K_1(z), \quad (26)$$

$$s_2'' = -K_1 d_1 + A_2 \cos^2 \phi + A_2 \sin^2 \phi, \quad (27)$$

$$d_3'' = -K_1 s_2 - 3A_2 d_1 \cos^2 \phi, \quad (28)$$

$$s_4'' = -K_1 d_3 + [A_4 - A_2(3s_2 - 6d_1^2)] \cos^2 \phi + [A_4 - A_2 s_2] \sin^2 \phi, \quad (29)$$

and so forth. We are now able to solve sequentially for d_1, s_2, d_3, s_4 , etc., by a straightforward program of integration. The associated constants (A_n) are determined from averages ($\bar{s}_n'' = 0$) and can always be evaluated using lower order s and d functions. The first two, and most useful, constants are

$$A_2 = \overline{K_1 d_1}, \quad (30)$$

$$A_4 = \overline{K_1 d_1 s_2} - 3\overline{K_1 d_1 d_1^2} \cos^2 \phi, \quad (31)$$

where we have used Equation (28) to calculate $\overline{K_1 d_3}$:

$$\overline{K_1 d_3} = \overline{-d_1'' d_3} = \overline{-d_1 d_3''} = \overline{K_1 d_1 s_2} + 3A_2 \overline{d_1^2} \cos^2 \phi. \quad (32)$$

The most useful results are Equations (26) and (30), which allow most quantities of interest to be calculated with reasonable accuracy. To evaluate the envelope functions with an accuracy better than 1%, we also need to evaluate $s_2(z)$ and $d_3(z)$. Eliminating A_2 from Equations (27) and (28), these functions satisfy

$$s_2'' = \overline{K_1 d_1} - K_1 d_1, \quad (33)$$

$$d_3'' = -K_1 s_2 - 3\overline{K_1 d_1} d_1 \cos^2 \phi. \quad (34)$$

For most applications $\sigma_0 \leq 90^\circ$, and it is found that $|d_5| \ll 0.01$ and $|s_4| \ll 0.001$, so they are not included in further calculations here. Typically $|s_2| \approx 0.01$, and $|d_3| \leq 0.01$ if $\cos \phi = 0$. However $|d_3|$ may be as large as 0.05 for $\cos \phi = 1$.

The exact functions $s(z)$ and $d(z)$ are given by power series with α set equal to unity. Therefore, strictly speaking, α is not a small parameter to justify truncation of these series. However, it is found that the powers of α correspond to powers of K_1 in these expansions; the true small parameter is a dimensionless quantity proportional to $K_1^2 P^4$. For example, for the simple case $K_1(z) = k \cos(\pi z/L)$

$$d_1 = \frac{1}{\pi} \left(\frac{kL^2}{\pi} \right) \cos(\pi z/L),$$

$$\sigma_0 = \sqrt{2} \left(\frac{kL^2}{\pi} \right) \left[1 + \left(\frac{25}{32\pi^2} \right) \left(\frac{kL^2}{\pi} \right)^2 + \dots \right]. \quad (35)$$

The typical range of interest is $\sigma_0 = 1.0 - 1.5$ radians/period, so $k^2 L^4 / \pi^2$ ranges from about 0.7 to 1.0. The effective expansion parameter for terms beyond lowest order appears to be

$$\delta = \frac{1}{\pi^2} \left(\frac{kL^2}{\pi} \right)^2 \leq 0.1. \quad (36)$$

However, the coefficients of δ^n in the various expansions may be greater than unity, and in the case of $d(z)$ are found to increase with n . Although the first few terms of the expansions give an excellent approximation to the exact solution, the range of convergence has not been established.

5 TUNE FORMULAS

Although the formalism developed so far is self-contained, it is of interest to calculate the tunes σ and σ_0 associated with the matched envelopes. For example, they determine the stable ranges of envelope parameters.⁹ The expansion of Equation (19) in powers of α yields

$$\frac{\sigma \bar{a}^2}{P\varepsilon} = 1 + \overline{3d_1^2} \alpha^2 + \left(\overline{3s_2^2} + \overline{6d_1 d_3} - 12 \overline{s_2 d_1^2} + \overline{5d_1^4} \right) \alpha^4 + \dots \quad (37)$$

The quantity $\varepsilon/\bar{a}^2$ may be eliminated using Equation (22) to obtain

$$\begin{aligned} \sigma^2 &= P^2 \left(\frac{\varepsilon^2}{\bar{a}^4} \right) \left(\frac{\sigma \bar{a}^2}{P\varepsilon} \right)^2 = P^2 \cos^2 \phi (A_2 \alpha^2 + A_4 \alpha^4 + \dots) \left(1 + \overline{3d_1^2} \alpha^2 + \dots \right)^2 \\ &= P^2 \cos^2 \phi \left[\overline{K_1 d_1} \alpha^2 + \overline{K_1 d_1 s_2} \alpha^4 + 3 \overline{K_1 d_1 d_1^2} (2 - \cos^2 \phi) \alpha^4 + \dots \right]. \end{aligned} \quad (38)$$

Improved convergence is obtained for the expansion of $\cos \sigma$ as compared with σ^2 . Similarly, the expansion of $(\bar{a}^2 \sin \sigma)/P\varepsilon$ converges more rapidly than that of $\bar{a}^2 \sigma/P\varepsilon$. This behavior of expansions is not surprising because the trigonometric functions of σ appear in the full period transfer matrix.³ They are connected with the single particle orbital instability and blowup of the matched envelope radius as σ approaches 180° . The simple form of the transfer matrix obtained for thin lens quadrupoles without space charge also suggests the more rapid convergence of the trigonometric functions. For a FODO system of thin lenses of focal length f centered at $z = \pm L/2, \pm 3L/2, \dots$:

$$\cos \sigma_0 = 1 - \frac{P^2}{8f^2}, \quad (39)$$

$$\frac{a^2(0) \sin \sigma_0}{P\varepsilon} = 1 - \frac{P^2}{32f^2}; \quad (40)$$

there are no higher order terms in $(P/f)^2$.

Motivated by these observations, we evaluate the expansions

$$\begin{aligned}\cos \sigma &= 1 - \frac{\sigma^2}{2} + \frac{\sigma^4}{24} - \dots \\ &= 1 - \left(\frac{P^2 \overline{K_1 d_1} \cos^2 \phi}{2} \right) \alpha^2 + \left[\frac{P^4 (\overline{K_1 d_1})^2 \cos^4 \phi}{24} \right. \\ &\quad \left. - \frac{P^2}{2} \overline{K_1 d_1 s_2} \cos^2 \phi - \frac{3}{2} P^2 \overline{K_1 d_1 d_1^2} (2 - \cos^2 \phi) \cos^2 \phi \right] \alpha^4 + \dots, \quad (41)\end{aligned}$$

$$\frac{\bar{a}^2 \sin \sigma}{P \varepsilon} = \left(\frac{\bar{a}^2 \sigma}{P \varepsilon} \right) \left(1 - \frac{\sigma^2}{6} + \dots \right) = 1 + \left(\frac{\overline{d_1^2}}{3} - \frac{P^2 \overline{K_1 d_1}}{6} \cos^2 \phi \right) \alpha^2 + \dots. \quad (42)$$

The undepressed tune (σ_0) is obtained from Equation (41) by setting $\cos \phi$ equal to unity. Note that d_1 and s_2 do not depend on ϕ , so that terms of $\cos \sigma_0$ through (α^4) may be immediately written down:

$$\cos \sigma_0 = 1 - \frac{P^2 \overline{K_1 d_1}}{2} \alpha^2 + \left[\frac{P^4 (\overline{K_1 d_1})^2}{24} - \frac{P^2 \overline{K_1 d_1 s_2}}{2} - \frac{3 P^2 \overline{K_1 d_1 d_1^2}}{2} \right] \alpha^4 + \dots. \quad (43)$$

At this stage, the tune calculations may be summarized as follows. The quadrupole strength $K_1(z)$ is specified and the ratio $Q\bar{a}^2/\varepsilon^2$ is also *formally regarded* as specified. The functions $d_1, s_2, d_3, \dots$, are evaluated sequentially using the formalism of Section 4. Tunes σ and σ_0 are then evaluated using Equations (41) and (43), and $\varepsilon/\bar{a}^2$ is then evaluated from Equation (42). Finally, we calculate

$$\frac{Q}{\bar{a}^2} = \left(\frac{Q\bar{a}^2}{\varepsilon^2} \right) \bullet \left(\frac{\varepsilon}{\bar{a}^2} \right)^2. \quad (44)$$

Unfortunately, the expansion of tunes in powers of α becomes very cumbersome beyond the lowest non-trivial order. Some simplification is achieved by combining formulas in such a way that some of the higher order terms cancel. Subtracting Equation (43) from (41) gives

$$\begin{aligned}2(\cos \sigma - \cos \sigma_0) &= \left(P^2 \overline{K_1 d_1} \sin^2 \phi \right) \alpha^2 + (\sin^2 \phi) \left[\frac{P^4 (\overline{K_1 d_1})^2}{12} (\sin^2 \phi - 2) \right. \\ &\quad \left. + P^2 \overline{K_1 d_1 s_2} + 3 P^2 \overline{K_1 d_1 d_1^2} \sin^2 \phi \right] \alpha^4 + \dots. \quad (45)\end{aligned}$$

All terms on the right in Equation (45) are multiplied by at least one power of $\sin^2 \phi$, which vanishes as $Q \rightarrow 0$, leaving $\sigma = \sigma_0$. A much more spectacular cancellation of terms may be verified for the combination

$$2(\cos \sigma - \cos \sigma_0) - \left(\frac{QP^2}{\bar{a}^2} \right) + \left(\frac{QP^2}{\bar{a}^2} \right) \left[\frac{\cos \sigma - \cos \sigma_0}{6} - \left(\frac{\bar{a}^2 \sin \sigma}{P\varepsilon} - 1 \right) \right] = O(\alpha^6). \quad (46)$$

The α^2 and α^4 terms of the expansions combined in Equation (46) both cancel. It is recommended that this equation be used in place of Equations (41) or (45).

6 SUMMARY OF FORMULAS

For rough design calculations, it is generally sufficient to determine $d_1(z)$ and compute the averages $\overline{k_1 d_1}$ and $\overline{d_1^2}$. The resulting first order expressions for σ_0 and σ are accurate to $\sim 1\%$. The maximum of $a/\bar{a}$ is also accurate to $\sim 1\%$ if $\cos \phi \rightarrow 0$, i.e. the beam is space-charge dominated. More generally, $a/\bar{a}$ can be in error by up to $\sim 5\%$ in lowest order; this is corrected to $\sim 1\%$ by the inclusion of d_3 and to $\sim 0.3\%$ by also including s_2 .

To carry out this program, the necessary general formulas are summarized here:

$$\frac{a}{\bar{a}} = 1 + d_1 + s_2 + d_3 + \dots, \quad (47)$$

$$\frac{b}{\bar{a}} = 1 - d_1 + s_2 - d_3 + \dots, \quad (48)$$

$$d_1'' = -K_1, \quad (49)$$

$$s_2'' = \overline{K_1 d_1} - K_1 d_1, \quad (50)$$

$$d_3'' = -K_1 s_2 - 3\overline{K_1 d_1} d_1 \cos^2 \phi, \quad (51)$$

$$\cos^2 \phi = (1 + Q\bar{a}^2/\varepsilon^2)^{-1}, \quad (52)$$

$$2(1 - \cos \sigma_0) = P^2 \overline{K_1 d_1} - \left[\frac{P^4 (\overline{K_1 d_1})^2}{12} - P^2 \overline{K_1 d_1 s_2} - 3P^2 \overline{K_1 d_1} \overline{d_1^2} \right] + \dots, \quad (53)$$

$$\left(\frac{\bar{a}^2 \sin \sigma}{P\varepsilon} - 1 \right) = \left(3\bar{d}_1^2 - \frac{P^2}{6} \overline{K_1 d_1} \cos^2 \phi \right) + \dots, \quad (54)$$

$$2(\cos \sigma - \cos \sigma_0) - \left(\frac{QP^2}{\bar{a}^2} \right) + \left(\frac{QP^2}{\bar{a}^2} \right) \left[\frac{\cos \sigma - \cos \sigma_0}{6} - \left(\frac{\bar{a}^2 \sin \sigma}{P\varepsilon} - 1 \right) \right] = O(KP^2)^6. \quad (55)$$

7 FLAT TOP QUADRUPOLE FIELDS

A very useful set of design formulas is derived for the simple FODO lattice with drifts of length $(1 - \eta)L$ centered at $z = 0, \pm L, \pm 2L, \dots$, and flat-topped quadrupoles of strength $\pm k$ and length ηL centered at $z = \pm L/2, \mp 3L/2, \dots$.

In the interval $0 < z < L/2$, we have

$$d_1'' = \begin{cases} 0 & 0 < z < (1 - \eta)\frac{L}{2}, \\ -k & (1 - \eta)\frac{L}{2} < z < \frac{L}{2}. \end{cases} \quad (56)$$

Integrating Equation (56) twice yields

$$d_1' = \begin{cases} \frac{k\eta L}{2} & 0 < z < (1 - \eta)\frac{L}{2}, \\ k\left(\frac{L}{2} - z\right) & (1 - \eta)\frac{L}{2} < z < \frac{L}{2}, \end{cases} \quad (57)$$

$$d_1 = \begin{cases} \frac{k\eta L}{2} z & 0 < z < (1 - \eta)\frac{L}{2}, \\ \frac{\eta k L^2}{4} \left(1 - \frac{\eta}{2}\right) - \frac{k}{2} \left(\frac{L}{2} - z\right)^2 & (1 - \eta)\frac{L}{2} < z < \frac{L}{2}. \end{cases} \quad (58)$$

The maximum value of d_1 is

$$d_{1m} = d_1(L/2) = \frac{\eta k L^2}{4} \left(1 - \frac{\eta}{2}\right), \quad (59)$$

and the useful averages over d_1 are

$$\overline{K_1 d_1} = \frac{\eta^2 k^2 L^2}{4} \left(1 - \frac{2}{3} \eta \right), \quad (60)$$

$$\overline{d_1^2} = \frac{\eta^2 k^2 L^4}{48} \left(1 - \eta^2 + \frac{2}{5} \eta^3 \right). \quad (61)$$

Equation (50) is also integrated twice, making use of the condition $\overline{s_2} = 0$, to obtain the minimum and maximum values

$$s_2(0) = -\frac{\eta^2 k^2 L^4}{24} \left(\frac{1}{4} - \frac{\eta}{6} - \frac{\eta^2}{4} + \frac{\eta^3}{5} \right), \quad (62)$$

$$s_2(L/2) = \frac{\eta^2 k^2 L^4}{24} \left(\frac{1}{2} - \frac{13}{12} \eta + \frac{13}{16} \eta^2 - \frac{\eta^3}{5} \right). \quad (63)$$

Intermediate values of $s_2(z)$ are not displayed here. A useful fourth order averaged quantity is

$$\overline{K_1 d_1 s_2} = \frac{\eta^4 k^4 L^6}{12} \left(\frac{1}{16} - \frac{5}{24} \eta + \frac{191}{720} \eta^2 - \frac{16}{105} \eta^3 + \frac{\eta^4}{30} \right). \quad (64)$$

A double integration of Equation (51) gives the maximum of $d_3(z)$:

$$\begin{aligned} d_3(L/2) &= \frac{\eta^3 k^3 L^6}{192} \left(1 - \frac{19}{6} \eta + \frac{463}{120} \eta^2 - \frac{511}{240} \eta^3 + \frac{9}{20} \eta^4 \right) \\ &\quad + \frac{\eta^3 k^3 L^6}{64} \left(1 - \frac{2}{3} \eta \right) \left(1 - \frac{\eta^2}{2} + \frac{\eta^3}{8} \right) \cos^2 \phi. \end{aligned} \quad (65)$$

The third order contribution to the envelope slope at the origin is

$$\begin{aligned} d'_3(0) &= \frac{\eta^3 k^3 L^5}{48} \left(\frac{1}{2} - \frac{4}{3} \eta + \frac{6}{5} \eta^2 - \frac{11}{30} \eta^3 \right) \\ &\quad + \frac{\eta^3 k^3 L^5}{64} \left(3 - 2\eta - \eta^2 + \frac{2}{3} \eta^3 \right) \cos^2 \phi. \end{aligned} \quad (66)$$

Using the averages $\overline{k_1 d_1}$, $\overline{d_1^2}$, and $\overline{k_1 d_1 s_2}$, we get the tune formulas from Equations (53) and (54):

$$2(1 - \cos \sigma_0) = \eta^2 k^2 L^4 \left(1 - \frac{2\eta}{3}\right) - \eta^4 k^4 L^8 \left(\frac{\eta^2}{90} - \frac{\eta^3}{63} + \frac{\eta^4}{180}\right) + \dots, \quad (67)$$

$$\left(\frac{\bar{a}^2 \sin \sigma}{2L\varepsilon} - 1\right) = \eta^2 k^2 L^4 \left[\frac{1}{16} \left(1 - \eta^2 + \frac{2}{5}\eta^3\right) - \frac{\cos^2 \phi}{6} \left(1 - \frac{2}{3}\eta\right)\right]. \quad (68)$$

Equations (52) and (55), relating Q to other parameters, do not depend on the specific form of $K_1(z)$ in the order of approximation included here and are therefore not repeated in this section.

8 NUMERICAL COMPARISON

The accuracy of the flat-top quadrupole formulas given in Section 7 is compared here with a typical numerical (exact) calculation. Parameters regarded as specified are

$$\begin{aligned} \sigma_0 &= 72^\circ, & \eta &= 1/3, & \varepsilon &= 10^{-4} \text{ m-r}, \\ L &= 1.0 \text{ m}, & KE &= 10.00 \text{ MeV}, & q &= 1, \\ \sigma &= 50.553455^\circ, & m &= 38.96316 \text{ amu}. \end{aligned}$$

Here the mass is selected to be that of the isotope K^{39} less one electron. Two natural constants complete the input:

$$\begin{aligned} c &= 2.99792458 \times 10^8 \text{ m/s}, \\ 1.0 \text{ amu} &= 931.49432 \text{ MeV}/c^2. \end{aligned}$$

Numerically computed ($\sim$ exact) quantities are

$$\beta\gamma = \sqrt{\left(\frac{KE}{mc^2}\right)^2 + 2\left(\frac{KE}{mc^2}\right)} = 0.0234761956$$

$$[B\rho] = \beta\gamma mc = 2.8421134 \text{ T-m}$$

$$G = 11.37468073 \text{ T/m}$$

$$k = G/[B\rho] = 4.00219102 \text{ m}^{-2}$$

$$Q = 0.450123228 \times 10^{-4}$$

$$I = 0.35253177 \text{ Ampere}$$

$$\varepsilon_n = \beta\gamma\varepsilon = 2.347619561 \times 10^{-6}$$

$$\eta k L^2 = 1.334063673$$

$$a(L/2) = 0.02098363962 \text{ m}$$

$$b(L/2) = 0.01148592675 \text{ m}$$

$$a(0) = 0.0157629845 \text{ m}$$

$$\bar{a} = 0.0159693 \text{ m}$$

$$a'(0) = 0.01154141594.$$

The accuracy of Equations (59)–(68) is verified by inserting exact values of σ_0 , k , etc. into the approximate expressions of Section 7, with all terms grouped on the left. If the expressions were exact they would sum to zero. The values of individual terms is displayed to indicate their relative importance:

$$\begin{aligned} & 2(1 - \cos \sigma_0) - (\eta k L^2)^2 \left(1 - \frac{2}{3}\eta\right) + (\eta k L^2)^4 \left(\frac{\eta^2}{90} - \frac{\eta^3}{63} + \frac{\eta^4}{180}\right) \\ &= 1.381,966,012 - 1.384,231,242 + 0.002,265,550 \\ &= 0.000,000,320, \end{aligned}$$

$$\begin{aligned} & \left[\frac{a(L/2)}{\bar{a}} - 1 \right] - d_1(L/2) - s_2(L/2) - d_3(L/2) \\ &= 0.313,996 - 0.277,930 - 0.016,445 - 0.016,457 \\ &= 0.003,164, \end{aligned}$$

$$\begin{aligned}
& \left[\frac{b(L/2)}{\bar{a}} - 1 \right] + d_1(L/2) - s_2(L/2) - d_3(L/2) \\
& = -0.280,750 + 0.277,930 - 0.016,445 + 0.016,457 \\
& = -0.002,808,
\end{aligned}$$

$$\begin{aligned}
& \left[\frac{a(0)}{\bar{a}} - 1 \right] - d_1(0) - s_2(0) - d_3(0) \\
& = -0.012,919 - 0.0 + 0.012,908 - 0.0 = -0.000,011,
\end{aligned}$$

$$\begin{aligned}
& 2(\cos \sigma - \cos \sigma_0) - \left(\frac{4QL^2}{\bar{a}^2} \right) + \left(\frac{4QL^2}{\bar{a}^2} \right) \left[\frac{\cos \sigma - \cos \sigma_0}{6} - \left(\frac{\bar{a}^2 \sin \sigma}{2L\epsilon} - 1 \right) \right] \\
& = 0.652,682 - 0.706,024 + [0.038,400 + 0.010,838] = -0.004,104,
\end{aligned}$$

$$\begin{aligned}
& \left(\frac{\bar{a}^2 \sin \sigma}{2L\epsilon} - 1 \right) - \left[\frac{(\eta k L^2)^2}{16} \left(1 - \eta^2 + \frac{2}{5} \eta^3 \right) - \frac{(\eta k L^2)^2}{6} \left(1 - \frac{2}{3} \eta \right) \cos^2 \phi \right] \\
& = -0.015,351 - [0.100,522 - 0.107,410] = -0.008,463,
\end{aligned}$$

$$\begin{aligned}
& \frac{La'(0)}{\bar{a}} - d'_1(0) - d'_3(0) \\
& = 0.722,725 - 0.667,032 - 0.047,479 = 0.008,214.
\end{aligned}$$

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