Presentation 13

Simulation Studies of Synchro-Betatron Resonances

By J. Hagel / J. Jowett

In the absence of J. Hagel, J. Jowett presented the following paper.

13.1 Theory

13.2 Introduction

In every accelerator and storage ring the moving charge experiences at the same time transverse oscillations mainly caused by the focusing magnets and synchrotron oscillations generated by the energy transfer in the accelerating structures. In the ideal case these motions are decoupled. However due to various mechanisms coupling terms are present. They cause an increase of the unperturbed synchrotron and betatron amplitudes. In addition if the betatron and synchrotron tune satisfy rational relations of the form

\[ Q_\beta - mQ_\sigma = p \]  

(13.1)

where \( m \) and \( p \) are integers, resonance denominators drive the motion to very high amplitudes thus causing possible instabilities.

One important mechanism is the presence of vertical or horizontal finite dispersion in the RF cavities [1] - [4]. Since there exists in LEP a relatively large vertical dispersion in the cavities (up to 20 cm) this effect is expected to be rather important.

In addition it has been found (originally at SPEAR) that longitudinal wakefields caused by the longitudinal impedance of the cavities excite synchro betatron coupling [5]. However it has been shown in [6] that these excitations are only possible if a finite dispersion already exists in the accelerating cavities.

Furthermore excitations of coupling is caused by transverse wakefields in the presence of a finite closed orbit in the cavities. This mechanism is independent of the presence of a finite dispersion as has been demonstrated in [6]. A detailed evaluation of the influence of transverse wakes on synchro betatron motion can be found in [7].

Based on the analytic formalism in [6] the program SYBILLE has been created [8] which evaluates the amplitude growth between and in the neighbourhood of synchro betatron sidebands and takes into account the coupling terms generated by dispersion as well as by longitudinal and transverse wakefields. Since SYBILLE is based on analytic expressions giving the amplitude increase after an arbitrary number of turns it is very quick and many turns can be 'simulated' using little computing time even with the full cavity structure of LEP.

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13.3 Formulae used by SYBILLE

We give a very short review of the theory of T. Suzuki and J. Hagel [6]. In its present state this is a single particle theory in the sense that it deals with the effect a single particle experiences due to the various coupling mechanisms listed above. This contains nonlinear kicks the electron experiences due to the presence of dispersion in the cavities but also the displacements created by the wakefields of the entire (Gaussian) bunch. Since the nonlinear structure of the synchrotron motion is taken into account the effect of any order side band \( m > 1 \) is described. The input parameters are

- General accelerator parameters like momentum compaction \( \alpha \), energy deviation \( \frac{dE}{E_0} \), circumference, \( Q_\beta, Q_\sigma \) etc.
- Twiss Parameters \( \alpha, \beta, D, D' \) at the cavity positions and at the observation point (for LEP at the low beta insertion)
- Cavity parameters like \( V, f_{RES}, R_L, R_T \) (longitudinal and transverse impedances)
- Parameters concerning the electron bunch: \( I_B, \sigma_L \)

The nonlinear coupled equations for synchro-betatron coupled motion are solved using a perturbation technique. The entire motion is regarded as consisting of the unperturbed decoupled transverse and longitudinal oscillations and perturbations caused by the (weak) coupling mechanisms. Please look in [6] for all details. Here we just list the results. \( Y(\phi) \) is the transverse coordinate in Courant and Snyder notation.

\[
Y(\phi) = Y_0 \cos(Q_\beta \phi) + \frac{Y'_0}{Q_\beta} \sin(Q_\beta \phi) + \sum_{i=1}^{N_c} \frac{\beta_i D'_i + \alpha_i \eta_i}{2\sqrt{\beta_i}} B_{1i} - \frac{\eta_i}{2\sqrt{\beta_i}} B_{2i}
\]

(13.2)

\[
\phi = \int_0^\phi \frac{R d\theta}{Q_\beta \beta}
\]

(13.3)

The summation in Eq. 13.2 extends over all the cavities. The \( l \) dependent term \( B_{1l} \) is given by

\[
B_{1l} = \left[ \frac{eV}{E} \sin \psi_a[J_0(a) - 1] + A_0 \right] \cos \left( Q_\beta \tilde{\phi} - \pi Q_\beta \right) - \cos Q_\beta \pi \sin Q_\beta
\]

\[
\sum_{m=1}^{\infty} \left[ \frac{eV}{E} J_m(a) + \frac{1}{2} A_m \right] \times \left[ \frac{\cos[Q_\beta \tilde{\phi} - \pi (Q_\beta - mQ_\sigma) + m\theta_0] - \cos[mQ_\sigma \tilde{\phi} - \pi (Q_\beta - mQ_\sigma) + m\theta_0]}{\sin \pi (Q_\beta - mQ_\sigma)} \right] \left[ \frac{\cos[Q_\beta \tilde{\phi} - \pi (Q_\beta + mQ_\sigma) + m\theta_0] - \cos[mQ_\sigma \tilde{\phi} + \pi (Q_\beta + mQ_\sigma) + m\theta_0]}{\sin \pi (Q_\beta + mQ_\sigma)} \right]
\]

(13.4)

\( B_{2l} \) has a very similar form. We see that resonances \( Q_\beta \pm mQ_\sigma \) are excited. In practice it turned out to be sufficient to truncate the infinite sum over the synchrotron sidebands \( m \) in Eq. after \( m = 5 \). By \( \tilde{\phi} \) we denote \( \phi - \phi_l \) while \( a \) stands for the synchrotron amplitude. The excitation constants \( A_m \) account for the wake field effects and are all proportional to the cavity impedance and the bunch current. If we replace \( \phi \) by \( 2\pi N \), \( Y(N) \) represents the perturbed betatron displacement after \( N \) turns. It is evaluated by the program SYBILLE.

13.4 Simulation of Synchro-Betatron Motion in LEP using SYBILLE

Using the program code SYBILLE the effect of synchro betatron motion in LEP has been computed. The goal was to understand the importance of synchro betatron coupling for the performance of
LEP with the present machine parameters. The computations have been executed on the CERN IBM under the operating system VM in interactive mode.

Since the vertical oscillation amplitude is much smaller than the horizontal one we restricted the study to the vertical plane. The amplitude increase (increase in vertical beam size) has been computed at the low beta interaction point. The program took into account effects of dispersion in the RF cavities as well as longitudinal and transverse wakefields acting on the Gaussian electron bunch. The input parameters chosen are listed in Table 13.1.

<table>
<thead>
<tr>
<th>Machine Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Energy</td>
<td>20 GeV</td>
</tr>
<tr>
<td>frac(Q_y)</td>
<td>0.24</td>
</tr>
<tr>
<td>Q_s</td>
<td>0.082</td>
</tr>
<tr>
<td>β_v</td>
<td>0.21 m</td>
</tr>
<tr>
<td>y(0)</td>
<td>20 μ m</td>
</tr>
<tr>
<td>δE</td>
<td>0.003</td>
</tr>
<tr>
<td>ΔE</td>
<td>0.2 m</td>
</tr>
<tr>
<td>I_B</td>
<td>0.5 mA</td>
</tr>
<tr>
<td>R_L</td>
<td>10^5Ω</td>
</tr>
</tbody>
</table>

Table 13.1: Input Parameters for LEP at injection energy

All the 128 RF-cavities are considered to sit at their proper positions although they have been modelled by δ-function pulses. The evolution of the coupled motion has been followed over 2000 turns which represents one damping time in LEP at injection energy.

The first result of the computations is that for the nominal parameters listed in Table 13.1 and for zero current bunches the unperturbed betatron amplitude is increased by about 20% with respect to its unperturbed value. However increasing the current has a big effect and at the maximum reproducible current obtained so far (I_{max} / bunch = 0.54 mA) the amplitude increase due to synchro betatron coupling by dispersion and wakefields is already a factor of 2. For the exact behaviour of the amplitude growth as function of current see Figure 13.1.

In the next example we place ourselves in the close neighbourhood of the 4-th synchrotron sideband by changing Q_y to 0.328. We follow the vertical betatron amplitude over 1000 turns. The bunch current is set to 0.5 mA. As can be seen from Figure 13.2 the fourth side band has still a big effect and drives the amplitude growth to unacceptably large values.

The growth rate is about 200 turns which is 10 times faster than the damping rate, thus indicating the degree of importance that synchro-betatron resonances might have for the LEP performance.

From Figure 13.1 we may draw an interesting conclusion: If we believe that synchro betatron coupling is the main limitation for the present LEP performance then it follows from SYBILLE that an amplitude increase of about 90% (which occurs at the maximum obtained current of 0.54 mA) is the maximum tolerable one.

From Figure 13.2 we see that even close to the resonance condition a lot of frequency components occur which contribute strongly to the solution, besides the strong fundamental frequency belonging to the single resonance.

In the next example we vary Q_y in the interval 0.22 < frac(Q_y) < 0.78 in steps of ΔQ_y = 0.001. For every value of Q_y we follow the motion over 2000 turns (one damping time) and compute the maximum amplitude growth which occurs during this time. Besides seeing the clear effect of the synchrotron side bands m = 3, 4 we realise that the effect of amplitude growth is also very important in between the resonances (e.g. 60% at Q_y = 0.3, well outside the influence of the 4-th side band).

The most interesting result is that there exists an asymmetry of amplitude growth with respect to Q_y = 0.5. Clearly the amplitude growth at Q_y = 0.5 + Δ is less than the
Amplitude Growth over 2000 turns as function of iB

\[ Q_y = 0.240 \quad Q_s = 0.082 \]

\[ \gamma_0 = 0.002 \quad \Delta \gamma = 0.003 \]

\[ \text{ETAY} = 20.00 \text{ cm} \]

Figure 13.1: Vertical amplitude growth in LEP as function of bunch current

one at \( Q_y = 0.5 \) \( - \Delta \).

The explanation for this effect comes from the analytic relations 13.2 and 13.3 describing the amplitude evolution. We realize that the excitation term driving the resonance family \( Q_s - mQ_s \) differs from the one driving the family \( Q_s + mQ_s \) which explains the apparent asymmetry.

We performed also some computations with transverse wakefields for finite values of the closed orbit in the cavities. Up to a closed orbit of 7 cm practically no difference in the observed growth has been seen.

13.5 Conclusions

Using SYBILL, the computations of synchro-betatron coupling in LEP which are presented here indicate that coupling between the transverse (vertical) and the longitudinal phase space strongly influences the transverse motion, and that synchro-betatron coupling can be regarded as one of the most important factors limiting LEP performance. From the results obtained using SYBILL we may derive two main possible cues:

• Reducing the vertical dispersion function in the RF cavities

• Shifting the vertical tune to the upper half integer

Of course any means to reduce the relatively large vertical dispersion in the cavities is extremely welcome. As can be seen from Eq. 13.2 the transverse amplitude distortions scale linearly with the dispersion values at the cavity positions. So for example, reducing the average dispersion to 10 cm should improve the situation by a factor of 2.

As mentioned above changing the working point for \( Q_y \) to the upper half integer should improve the situation by a factor of two even with the same dispersion function. It has already been observed in TRISTAN that moving the vertical betatron tune to fractional values above 0.5 resulted in significantly more current stored during injection although a careful re-correction of the closed orbit was required to obtain this result [9]. Such a working point seems worth trying and probably requires less effort than correcting the dispersion. From Figure 13.3
Amplitude as function of number of turns N

\[
\begin{align*}
Q_y &= 0.331 \\
Q_x &= 0.082 \\
y_0 &= 0.002 \\
\Delta E &= 0.003 \\
\eta &= 20.00 \text{ cm}
\end{align*}
\]

Figure 13.2: Vertical amplitude as function of number of turns close to the 4-th synchrotron side band

\[
\frac{\sqrt{Q}}{\sqrt{Q_0}} \leq 0.6
\]

is the most favorable choice. From the point of view of the beam-beam effect a choice of \( Q_y > 0.7 \) seems preferable [10] but at injection energy the beam-beam effect is not very important and \( Q_y = 0.6 \) is just located at the minimum amplitude growth due to synchro betatron coupling.

13.6 Acknowledgements

The author would like to thank E. Keil, A. Verdier and B. Zotter, and all other members of the SL/AP group for many useful discussions and advice during his work on synchro betatron motion and its application to LEP. In addition the author thanks T. Suzuki devoting a big fraction of his time to the common development of the theory of synchro betatron resonances and their sources. He also thanks K. Hirata and K. Kamada (KEK) for interesting discussions concerning a better choice of the working point in TRISTAN.

References

Amplitude Growth over 2000 turns as function of $Q_y$

$Q_y = 0.240$ $Q_s = 0.082$

$y_0 = 0.002$ $DE_0 = 0.003$ $ETAY = 20.00$ cm

Figure 13.3: Amplitude growth as function of the vertical tune $Q_y$ in the limits $0.22 < Q_y < 0.78$