PHYSICAL REVIEW D 79, 125010 (2009)

Duality, entropy, and ADM mass in supergravity

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(Received 25 March 2009; published 11 June 2009)

We consider the Bekenstein-Hawking entropy-area formula in four dimensional extended ungauged supergravity and its electric-magnetic duality property. Symmetries of both “large” and “small” extremal black holes are considered, as well as the ADM mass formula for \( \mathcal{N} = 4 \) and \( \mathcal{N} = 8 \) supergravity, preserving different fraction of supersymmetry. The interplay between BPS conditions and duality properties is an important aspect of this investigation.

DOI: 10.1103/PhysRevD.79.125010 PACS numbers: 04.65.+e, 04.70.Dy, 12.60.Jv

I. INTRODUCTION

In \( d = 4 \) extended ungauged supergravity theories based on scalar manifolds which are (at least locally) symmetric spaces

\[
M = \frac{G}{H},
\]

it is known that the classification of static, spherically symmetric and asymptotically flat extremal black hole (BH) solutions is made in terms of charge orbits of the corresponding classical electric-magnetic duality group \( G \) [1–6] (later called \( U \)-duality in string theory).

These orbits correspond to certain values taken by a duality invariant\(^2\) combination of the “dressed” central charges and matter charges. Denoting such an invariant by \( I \), the set of scalars parametrizing the symmetric manifold \( M \) by \( \phi \), and the set of “bare” magnetic and electric charges of the (dyonic) BH configuration by the \( 2n \times 1 \) symplectic vector

\[
\mathcal{P} = \begin{pmatrix} b^\Lambda \\ q_\Lambda \end{pmatrix}, \quad \Lambda = 1, \ldots, n,
\]

then it holds that

\[
\partial_\phi I(\phi, \mathcal{P}) = 0 \iff I = I(\mathcal{P}).
\]

In some cases, the relevant invariant \( I \) is not enough to characterize the orbit, and additional constraints are needed. This is especially the case for the so-called small BHs, in which case \( I = 0 \) on the corresponding orbit [3,4,9].

An explicit expression for the \( E_7 \)-invariant [10] was firstly introduced in supergravity in [11], and then adopted in the study of BH entropy in [12]. The additional \( U \)-invariant constraints which specify charge orbits with higher supersymmetry were given in [3]. The corresponding (large and small) charge orbits for \( \mathcal{N} = 8 \) and exceptional \( \mathcal{N} = 2 \) supergravity were determined in [4], whereas the large orbits for all other symmetric \( \mathcal{N} = 2 \) supergravities were obtained in [6], and then in [13] for all \( \mathcal{N} > 2 \)-extended theories. Furthermore, the invariant for \( \mathcal{N} = 4 \) supergravity was earlier discussed in [14,15].

The invariants play an important role in the attractor mechanism [16–20], because the Bekenstein-Hawking BH entropy [8], determined by evaluating the effective black hole potential ([18–20])

\[
V_{\text{BH}}(\phi, \mathcal{P}) = -\frac{1}{2} \mathcal{M}(\phi) \mathcal{P}
\]

at its critical points, actually coincides with the relevant invariant:

\[
\frac{S_{\text{BH}}}{\pi} = V_{\text{BH}}|_{\partial_\phi V_{\text{BH}} = 0} = V_{\text{BH}}(\phi_H(\mathcal{P}), \mathcal{P}) = |I(\mathcal{P})|^{1/2}(\text{or} |I(\mathcal{P})|).
\]

In Eq. (1.4) \( \mathcal{M} \) stands for the \( 2n \times 2n \) real (negative definite) symmetric scalar-dependent symplectic matrix

\[^2\text{By duality invariant, throughout our treatment we mean that such a combination is } G \text{-invariant. Thus, it is actually independent on the scalar fields, and it depends only on “bare” electric and magnetic (asymptotical) charges (defined in Eq. (1.2)).}\]

\[^3\text{Throughout the present treatment, we will, respectively, call small or large (extremal) BHs those BHs with vanishing or nonvanishing area of the event horizon (and therefore with vanishing or nonvanishing Bekenstein-Hawking entropy [8]). For symmetric geometries, they can be } G \text{-invariantly characterized, respectively, by } I = 0 \text{ or by } I \neq 0.\]
defined in terms of the normalization of the Maxwell and topological terms\(^4\)

\[ \mathcal{M}(\phi) = \begin{pmatrix} -\text{Im}\mathcal{N}_\Lambda\Sigma + \text{Re}\mathcal{N}_\Lambda\Sigma (\text{Im}\mathcal{N})^{-1}\mathcal{N}_\Lambda\Sigma & -\text{Re}\mathcal{N}_\Lambda\Xi (\text{Im}\mathcal{N})^{-1}\mathcal{N}_\Xi\Sigma \\ -(\text{Im}\mathcal{N})^{-1}\mathcal{N}_\Xi\Sigma & \text{Re}\mathcal{N}_\Xi\Sigma \\ \end{pmatrix}, \quad (1.6) \]

of the corresponding supergravity theory (see e.g. [21,22] and Refs. therein). Furthermore, in Eq. (1.5) \(\phi_H(\mathcal{P})\) denotes the set of charge-dependent, stabilized horizon values of the scalars, solutions of the criticality conditions for \(V_{\text{BH}}\):

\[ \frac{\partial V_{\text{BH}}(\phi, \mathcal{P})}{\partial \phi} \bigg|_{\phi = \phi_H(\mathcal{P})} = 0. \quad (1.8) \]

For the case of charge orbits corresponding to small BHs, in the case of a single-center solution \(I(\mathcal{P}) = 0\), and thus the event horizon area vanishes, and the solution is singular (i.e. with vanishing Bekenstein-Hawking entropy). However, the charge orbits with vanishing duality invariant play a role for multicare solutions as well as for elementary BH constituents through which large (i.e. with nonvanishing Bekenstein-Hawking entropy) BHs are made [23–25].

In the present investigation, we reexamine the duality invariant and the \(U\)-invariant classification of charge orbits of \(\mathcal{N} = 8, d = 4\) supergravity, we give a complete analysis of the \(\mathcal{N} = 4\) large and small charge orbits, and we also derive a diffeomorphism-invariant expression of the \(\mathcal{N} = 2\) duality invariant, which is common to all symmetric spaces and which is completely independent on the choice of a symplectic basis.

The paper is organized as follows.

In Sec. II we recall some basic facts about electromagnetic duality in \(\mathcal{N}\)-extended supergravity theories, firstly treated in [2]. The treatment follows from the general analysis of [1], and the dictionary between that paper and the present work is given.

In Sec. III we reexamine \(\mathcal{N} = 8, d = 4\) supergravity and the \(U(7)\)-invariant characterization of its charge orbits. This refines, reorganizes and extends the various results of [3–5,9].

In Sec. IV we reconsider matter coupled \(\mathcal{N} = 4, d = 4\) supergravity. The \(SL(2, \mathbb{R}) \times SO(6, M)\)-invariant characterization of all its BPS and non-BPS charge orbits, firstly obtained in [3,9], is the starting point of the novel results presented in this section.

Sec. V is devoted to the analysis of the \(\mathcal{N} = 2, d = 4\) case [3]. Beside the generalities on the special Kähler

\(^4\)Attention should be paid in order to distinguish between the notations of the number \(\mathcal{N}\) of supercharges of a supergravity theory and the kinetic vector matrix \(\mathcal{N}_{\Lambda\Sigma}\) introduced in Eqs. (1.6) and (1.7).
singlet in its 2-fold antisymmetric tensor product
\[ (\mathbf{R}_1 \times \mathbf{R}_1)_p \ni \mathbf{1}. \]  
(2.8)

If the basic requirements (2.5), (2.6), and (2.7) or (2.8) are met, the coset representative of \( M \) in the symplectic representation \( \mathbf{R}_i \) is given by the (scalar-dependent) 2\( n \times 2n \) matrix
\[ S(\phi) \equiv \begin{pmatrix} A(\phi) & B(\phi) \\ C(\phi) & D(\phi) \end{pmatrix} \in \text{Sp}(2n, \mathbb{R}). \]  
(2.9)

A particular role is played by the two (scalar-dependent) complex \( n \times n \) matrices \( f \) and \( h \), which do satisfy the properties
\[ -f^\dagger h + h^\dagger f = i1, \]  
(2.10)
\[ -f^T h + h^T f = 0. \]  
(2.11)

The constraining relations (2.10) and (2.11) are equivalent to require that
\[ S(\phi) = \sqrt{2} \begin{pmatrix} \text{Re}f & -\text{Im}f \\ \text{Re}h & -\text{Im}h \end{pmatrix}. \]  
(2.12)

or equivalently:
\[ f = \frac{1}{\sqrt{2}}(A - iB); \]  
(2.13)
\[ h = \frac{1}{\sqrt{2}}(C - iD). \]  
(2.14)

In order to make contact with the formalism introduced by Gaillard and Zumino in [1], it is convenient to use another (complex) basis, namely, the one which maps an element \( S \in \text{Sp}(2n, \mathbb{R}) \) into an element \( U \in U(n, n) \cap \text{Sp}(2n, \mathbb{C}) \). The change of basis is exploited through the matrix
\[ \mathcal{A} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -i1 & i1 \end{pmatrix}, \quad \mathcal{A}^{-1} = \mathcal{A}^\dagger. \]  
(2.15)

The (scalar-dependent) matrix \( U \) is thus defined as follows:
\[ U(\phi) \equiv \mathcal{A}^{-1} S \mathcal{A} = \frac{1}{\sqrt{2}} \begin{pmatrix} f + ih & \tilde{f} + i\tilde{h} \\ f - ih & \tilde{f} - i\tilde{h} \end{pmatrix} \]  
(2.16)
\[ \in U(n, n) \cap \text{Sp}(2n, \mathbb{C}). \]

This is the matrix named \( S \) in Eq. (5.1) of [1]. Correspondingly, the \( \text{Sp}(2n, \mathbb{R}) \)-covariant vector \( (F^\Lambda, G^\Lambda)^T \) is mapped into the vector
\[ \mathcal{A}^{-1} \begin{pmatrix} F^\Lambda \\ G^\Lambda \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ i1 & -i1 \end{pmatrix} \begin{pmatrix} F^\Lambda \\ G^\Lambda \end{pmatrix} \]  
(2.17)
\[ = \frac{1}{\sqrt{2}} \begin{pmatrix} F^\Lambda + iG^\Lambda \\ F^\Lambda - iG^\Lambda \end{pmatrix}. \]

The kinetic vector matrix \( \mathcal{N}_{\Lambda \Sigma} \) appearing in Eqs. (1.6) and (1.7) is given by (in matrix notation)
\[ \mathcal{N}(\phi) = h f^{-1} = (f^{-1})^T h^T, \]  
(2.18)
and it is named \(-i\tilde{k}\) in [1].

Thus, by introducing the \( 2n \times 1 \) (\( n \times n \) matrix-valued) complex vector
\[ \Xi = \begin{pmatrix} f \\ h \end{pmatrix} \]  
(2.19)

and recalling the definition (1.6), the matrix \( \mathcal{M} \) can be written as
\[ \mathcal{M}(\phi) = -i\Omega + 2i\Xi(\Xi)^\dagger = -i\Omega - 2i\Xi\Xi^\dagger \Omega \]  
\[ = -i\Omega - 2(\begin{pmatrix} h \\ f \end{pmatrix})(h^\dagger, -f^\dagger) \]  
\[ = -i(0 \quad -1) + 2(\begin{pmatrix} h h^\dagger & -f f^\dagger \\ f h^\dagger & -h f^\dagger \end{pmatrix}). \]  
(2.20)

Equations (1.4), (1.6), and (2.20) imply that
\[ V_{\text{BH}}(\phi, \mathcal{P}) = -\frac{1}{2} \mathcal{P}^T \mathcal{M}(\phi) \mathcal{P} = \text{Tr}(Z Z^\dagger) = \text{Tr}(Z^\dagger Z) \]  
\[ = \sum_{\Lambda > \tilde{\Lambda} - 1} Z_{\Lambda \tilde{\Lambda}} Z_{\tilde{\Lambda} \Lambda} + Z_{\tilde{\Lambda} \tilde{\Lambda}} = \frac{1}{2} Z_{AB} \tilde{Z}^{AB} + Z_{\bar{I} \bar{I}} \]  
\[ = \frac{1}{2} \text{Tr}(Z Z^\dagger) + Z_{\bar{I} \bar{I}} = \frac{1}{2} \text{Tr}(Z^\dagger Z) + Z_{\bar{I} \bar{I}}, \]  
(2.21)

where \( (A, B = 1, \ldots, \mathcal{N} \text{ and } I = 1, \ldots, m \text{ throughout; recall } \Lambda = 1, \ldots, n) \)
\[ Z = \mathcal{P}^T \Omega \Xi = q f - p h = (Z_{AB}(\phi, \mathcal{P}), Z_{I}(\phi \mathcal{P}) \mathcal{P}); \]  
(2.22)

\[ Z_{AB}(\phi, \mathcal{P}) = f_{AB}^\Lambda q_{\Lambda} - h_{AB}; \]  
(2.23)
\[ Z_{I}(\phi, \mathcal{P}) = \tilde{f}_{I} q_{\Lambda} - \tilde{h}_{\bar{I} \Lambda} \rho_{I}^\Lambda; \]  
(2.24)
\[ Z_{\Lambda}(\phi, \mathcal{P}) = \tilde{f}_{I} q_{\Lambda} - \tilde{h}_{\bar{I} \Lambda} \rho_{I}^\Lambda. \]  
(2.25)

Thus, Eq. (2.21) yields the “BH potential” \( V_{\text{BH}}(\phi, \mathcal{P}) \) to be nothing but the sum of the squares of the “dressed” charges. It is here worth noticing that \( (f_{AB}^\Lambda, \tilde{f}_{I}) \) and \( (h_{AB}, \tilde{h}_{\Lambda \tilde{\Lambda}}) \) are \( n \times n \) complex matrices, because it holds that
\[ f_{AB}^\Lambda = f_{[AB]}^\Lambda, \quad h_{AB} = h_{[AB]\Lambda}, \]  
thus implying \( Z_{AB} = Z_{[AB]} \), and

\(^5\)Unless otherwise noted, square brackets denote antisymmetrization with respect to the enclosed indices.
\[ n = \frac{\mathcal{N}(\mathcal{N} - 1)}{2} + m , \] (2.26)

where \( \mathcal{N} \) stands for the number of spinorial supercharges (see Footnote 4), and \( m \) denotes the number of matter multiplets coupled to the supergravity multiplet, except for \( \mathcal{N} = 6, d = 4 \) pure supergravity, for which \( m = 1 \).

Equations (2.24) and (2.25) are the basic relation between the (scalar-dependent) “dressed” charges \( Z_{AB} \) and \( Z_I \) and the (scalar-independent) “bare” charges \( \mathcal{P} \). It is worth remarking that \( Z_{AB} \) is the “central charge matrix function”, whose asymptotical value appears in the right-hand side of the \( \mathcal{N} \)-extended \((d = 4)\) supersymmetry algebra, pertaining to the asymptptical Minkowski space-time background:

\[ \{Q^A_\alpha, Q^B_\beta\} = \epsilon_{\alpha\beta} Z^{AB}(\phi_\infty, \mathcal{P}) , \] (2.27)

where \( \phi_\infty \) denotes the set of values taken by the scalar fields at radial infinity \((r \to \infty)\) within the considered static, spherically symmetric and asymptotically flat dyonic extremal BH background. Notice that the indices \( A, B \) of the central charge matrix are raised and lowered with the metric of the relevant \( R \)-symmetry group of the corresponding supersymmetry algebra.

By denoting the ADM mass [27] of the considered BH background by \( M_{\text{ADM}}(\phi_\infty, \mathcal{P}) \), the BPS bound [28] implies that

\[ M_{\text{ADM}}(\phi_\infty, \mathcal{P}) \geq |Z_I(\phi_\infty, \mathcal{P})| \geq \ldots \geq |Z_{[\mathcal{N}/2]}(\phi_\infty, \mathcal{P})| , \] (2.28)

where \( Z_I(\phi, \mathcal{P}), \ldots, Z_{[\mathcal{N}/2]}(\phi, \mathcal{P}) \) denote the set of skew-eigenvalues of \( Z_{AB}(\phi, \mathcal{P}) \), and here square brackets denote the integer part of the enclosed number. If \( 1 \leq k \leq [\mathcal{N}/2] \) of the bounds expressed by Eq. (2.28) are saturated, the corresponding extremal BH state is named to be \( Z_I \)-BPS. Thus, the minimal fraction of total supersymmetries (pertaining to the asymptotically flat space-time metric) preserved by the extremal BH background within the considered assumptions is \( \frac{1}{\mathcal{N}} \)(for \( k = 1 \)), while the maximal one is \( \frac{1}{2} \)(for \( k = \frac{\mathcal{N}}{2} \)). See Sec. VI for further details.

We end the present Section with some considerations on the issue of duality invariants.

A duality invariant \( I \) is a suitable linear combination (in general with complex coefficients) of \((\phi\text{-dependent})\) \( H \)-invariant combinations of \( Z_{AB}(\phi, \mathcal{P}) \) and \( Z_I(\phi, \mathcal{P}) \) such that Eq. (1.3) holds, i.e. such that \( I \) is invariant under \( G \), and thus \( \phi \)-independent:

\[ I = I(Z_{AB}(\phi, \mathcal{P}), Z_I(\phi, \mathcal{P})) = I(\mathcal{P}) . \] (2.29)

In presence of matter coupling, a charge configuration \( \mathcal{P} \) (and thus a certain orbit of the symplectic representation of the \( U \)-duality group \( G \), to which \( \mathcal{P} \) belongs) is called supersymmetric iff, by suitably specifying \( \phi = \phi(\mathcal{P}) \), it holds that

\[ \partial_\phi I_{4, \mathcal{N}=8}(Z_{AB}(\phi, \mathcal{P})) = 0 , \quad \forall \phi \in \frac{E_{7(7)}}{SU(8)} , \] (3.2)

\[ Z_I(\phi(\mathcal{P}), \mathcal{P}) = 0 , \quad \forall I = 1, \ldots, m . \] (2.30)

Notice that the conditions (2.30) cannot hold identically in \( \phi \), otherwise such conditions would be \( G \)-invariant, which generally are not. Indeed, in order for the supersymmetry constraints (2.30) to be invariant (or covariant) under \( G \), the following conditions must hold identically in \( \phi \):

\[ \partial_\phi Z_I(\phi, \mathcal{P}) = 0 , \quad \forall \phi \in M . \] (2.31)

Therefore, supersymmetry conditions are not generally \( G \)-invariant (i.e. \( U \)-invariant), otherwise extremal BH attractors (which are large) supported by supersymmetric charge configurations would not exist.

Nevertheless, in some supergravities it is possible to give \( U \)-invariant supersymmetry conditions. In light of previous reasoning, such \( U \)-invariant supersymmetric conditions cannot stabilize the scalar fields in terms of charges (by implementing the attractor mechanism in the considered framework), because such \( U \)-invariant conditions are actually identities, and not equations, for the set of scalar fields \( \phi \). Actually, \( U \)-invariant supersymmetry conditions can be given for all supersymmetric charge orbits supporting small BHs (for which the classical attractor mechanism does not hold). This can be seen e.g. in \( \mathcal{N} = 8 \) (pure) and \( \mathcal{N} = 4 \) (matter coupled) \( d = 4 \) supergravities, respectively, treated in Secs. III and IV.

### III. \( \mathcal{N} = 8 \)

The scalar manifold of the maximal, namely \( \mathcal{N} = 8 \), supergravity in \( d = 4 \) is the symmetric real coset

\[ \left( \frac{G}{H} \right)_{\mathcal{N}=8, d=4} = \frac{E_{7(7)}}{SU(8)} , \quad \dim_G = 70 , \] (3.1)

where the usual notation for noncompact forms of exceptional Lie groups is used, with subscripts denoting the difference “# noncompact generators - # compact generators”. This theory is pure, i.e. matter coupling is not allowed. The classical (see Footnote 1) \( U \)-duality group is \( E_{7(7)} \). Moreover, the \( R \)-symmetry group is \( SU(8) \) and, due to the absence of matter multiplets, it is nothing but the stabilizer of the scalar manifold (3.1) itself.

The Abelian vector field strengths and their duals, as well the corresponding fluxes (charges), sit in the fundamental representation \( 56 \) of the global, classical \( U \)-duality group \( E_{7(7)} \). Such a representation determines the embedding of \( E_{7(7)} \) into the symplectic group \( Sp(56, \mathbb{R}) \), which is the largest symmetry acting linearly on charges. The \( 56 \) of \( E_{7(7)} \) admits an unique invariant, which will be denoted by \( I_{4, \mathcal{N}=8} \) throughout. \( I_{4, \mathcal{N}=8} \) is quartic in charges, and it was firstly determined in [11].

More precisely, \( I_{4, \mathcal{N}=8} \) is the unique combination of \( Z_{AB}(\phi, \mathcal{P}) \) satisfying

\[ \partial_\phi I_{4, \mathcal{N}=8}(Z_{AB}(\phi, \mathcal{P})) = 0 , \quad \forall \phi \in \frac{E_{7(7)}}{SU(8)} . \]
Equation (3.2) can be computed by using the Maurer-Cartan Eqs. of the coset $E_{7(7)}/SU(8)$ (see e.g. [29] and Refs. therein):

$$\nabla Z_{AB} = \frac{1}{2} P_{ABCD} Z_{CD},$$

(3.3)

or equivalently by performing an infinitesimal $E_{7(7)}/SU(8)$-transformation of the central charge matrix (see e.g. [29] and Refs. therein):

$$\delta \xi_{ABCD} Z_{AB} = \frac{1}{2} \xi_{ABCD} Z_{CD},$$

(3.4)

where $\nabla$ and $P_{ABCD}$ respectively denote the covariant differential operator and the infinitesimal $E_{7(7)}/SU(8)$-parameters $\xi_{ABCD}$ satisfy the reality constraint

$$\xi_{ABCD} = \frac{1}{4!} \epsilon_{ABCDEFGH} \xi_{EFGH}.$$

(3.5)

As first found in [11] and rigorously reobtained in [29], the unique solution of Eq. (3.2) reads:

$$I_{4,N=8} = \frac{1}{2^5} \left[ 2^2 \text{Tr}((Z_{AC}Z_{BC})^2) - (\text{Tr}(Z_{AC}Z_{BC}))^2 + 2^5 \text{Re}(Pf(Z_{AB})) \right],$$

(3.6)

where the Pfaffian of $Z_{AB}$ is defined as [11]

$$Pf(Z_{AB}) = \frac{1}{2^4!} \epsilon^{ABCDEF} Z_{AB} Z_{CD} Z_{EF} Z_{GH},$$

(3.7)

and it holds that (see e.g. [29])

$$|Pf(Z_{AB})| = |\det(Z_{AB})|^{1/2},$$

(3.8)

In [29] it was indeed shown that, although each of the three terms of the expression (3.6) is $SU(8)$-invariant but scalar-dependent, only the combination given by the expression (3.6) is actually $E_{7(7)}$-independent and thus scalar-independent, satisfying

$$\delta \xi_{ABCD} I_{4,N=8} = 0,$$

(3.9)

with Eqs. (3.4) and (3.5) holding true.

$$\cos \varphi_Z(\phi, P) = \frac{2^2 I_{4,N=8}(P) - 2^2 \text{Tr}((Z_{AC}Z_{BC})^2) + (\text{Tr}(Z_{AB}Z_{AC}))^2}{2^5 (\text{det}(Z_{AC}Z_{BC}))^{1/4}}.$$

(3.13)

Notice that through Eq. (3.13) $(\cos) \varphi_Z$ is determined in terms of the scalar fields $\phi$ and of the BH charges $P$, also along the small orbits where $I_{4,N=8} = 0$. However, Eq. (3.13) is not defined in the cases in which $\det(Z_{AC}Z_{BC}) = 0$, i.e. when at least one of the eigenvalues of the matrix $Z_{AC}Z_{BC}$ vanishes. In such cases, $\varphi_Z$ is actually undetermined.

In $N = 8, d = 4$ supergravity five distinct orbits of the $56$ of $E_{7(7)}$ exist, as resulting from the analyses performed in [4,5]. They can be classified in large and small charge orbits, depending whether they correspond to $I_{4,N=8} \neq 0$ or $I_{4,N=8} = 0$, respectively.

Only two large charge orbits (for which $I_{4,N=8} \neq 0$, and the attractor mechanism holds) exist in $N = 8, d = 4$ supergravity:

1. The large $1/8$-BPS orbit [4,5]

$$O_{(1/8)\text{-BPS large}} = \frac{E_{7(7)}}{E_{6(2)}}, \quad \dim_{\text{R}} = 55,$$

(3.14)

is defined by the $E_{7(7)}$-invariant constraint
\[ I_{4,\mathcal{N}=8} > 0. \]  

(3.15)

At the event horizon of the extremal BH, the solution of the \(\mathcal{N} = 8, d = 4\) Attractor Eqs. yields \([3,9,30]\)

\[ e_1 \in \mathbb{R}^+_0, \quad e_2 = e_3 = e_4 = 0, \]  

(3.16)

implying \(\det(Z_{AB}) = 0 \iff P_f(Z_{AB}) = 0\), and thus \(\varphi_z\) to be undetermined. Thus, at the event horizon, the symmetry of the skew-diagonalized central charge matrix \(Z_{AB,\text{skew-diag}}\) defined in Eq. (3.10) gets enhanced as follows, revealing the maximal compact symmetry of \(O_{(1/8)-\text{BPS,large}}\):

\[ (USp(2))^4 \rightarrow USp(2) \times SU(6) \sim SU(2) \times SU(6). \]  

(3.17)

Indeed, \(SU(2) \times SU(6)\) is the maximal compact subgroup (mcs, with symmetric embedding \([31]\) of \(E_6(2)\) (stabilizer of \(O_{(1/8)-\text{BPS,large}}\)) itself.

(2) The large non-BPS \((Z_{AB} \neq 0)\) orbit \([4,5]\)

\[ O_{\text{non-BPS,}Z_{ab} \neq 0} = \frac{E_{7(7)}}{E_{6(6)}}, \quad \text{dim}_R = 55, \]  

(3.18)

is defined by the \(E_{7(7)}\)-invariant constraint

\[ I_{4,\mathcal{N}=8} < 0. \]  

(3.19)

At the event horizon of the extremal BH, the solution of the \(\mathcal{N} = 8, d = 4\) Attractor Eqs. yields \([3,9,30]\)

\[ e_1 = e_2 = e_3 = e_4 \in \mathbb{R}^+_0, \]  

(3.20)

so the skew-eigenvalues of \(Z_{ab}\) at the horizon (see Eq. (3.10)) are complex. Thus, at the event horizon, the symmetry of the skew-diagonalized central charge matrix \(Z_{AB,\text{skew-diag}}\) defined in Eq. (3.10) gets enhanced as follows, revealing the maximal compact symmetry of \(O_{\text{non-BPS,}Z_{ab} \neq 0}\):

\[ (USp(2))^4 \rightarrow USp(2). \]  

(3.21)

Indeed, \(USp(8)\) is the mcs (with symmetric embedding \([31]\) of \(E_{6(6)}\) (stabilizer of \(O_{\text{non-BPS,}Z_{ab} \neq 0}\)) itself.

As mentioned above, for such large charge orbits, corresponding to a nonvanishing quartic \(E_{7(7)}\)-invariant \(I_{4,\mathcal{N}=8}\) and thus supporting large BHs, the attractor mechanism holds. Consequently, the computations of the Bekenstein-Hawking BH entropy can be performed by solving the criticality conditions for the “BH potential”:

\[ V_{\text{BH,}\mathcal{N}=8} = \frac{1}{2} Z_{AB} \hat{Z}^{AB}, \]  

(3.22)

the result being

\[ \frac{S_{\text{BH}}}{\pi} = V_{\text{BH,}\mathcal{N}=8} \delta V_{\text{BH,}\mathcal{N}=8} = V_{\text{BH,}\mathcal{N}=8}(\phi_H(\mathcal{P}), \mathcal{P}) \]  

(3.23)

where \(\phi_H(\mathcal{P})\) denotes the set of solutions to the criticality conditions of \(V_{\text{BH,}\mathcal{N}=8}\), namely, the Attractor Eqs. of \(\mathcal{N} = 8, d = 4\) supergravity:

\[ \partial_\phi V_{\text{BH,}\mathcal{N}=8} = 0, \quad \forall \phi \in \frac{E_{7(7)}}{SU(8)}. \]  

(3.24)

expressing the stabilization of the scalar fields purely in terms of supporting charges \(\mathcal{P}\) at the event horizon of the extremal BH. Through Eqs. (3.3) and (3.22), Eqs. (3.24) can be rewritten as follows (notice the strict similarity to Eq. (3.40) further below) \([30]\):

\[ Z_{[\mathcal{A} \mathcal{B} \mathcal{Z}_{\mathcal{C} \mathcal{D}}]} + \frac{1}{4!} \epsilon_{\mathcal{A} \mathcal{B} \mathcal{C} \mathcal{D} \mathcal{E} \mathcal{F} \mathcal{G} \mathcal{H}} \hat{Z}^{\mathcal{E} \mathcal{F}} \hat{Z}^{\mathcal{G} \mathcal{H}} = 0. \]  

(3.25)

Actually, the critical potential \(V_{\text{BH,}\mathcal{N}=8} \delta V_{\text{BH,}\mathcal{N}=8}\) exhibits some “flat” directions, so not all scalars are stabilized in terms of charges at the event horizon \([32,33]\). Thus, Eq. (3.23) yields that the unstabilized scalars, spanning a related moduli space of the considered class of attractor solutions, do not enter in the expression of the BH entropy at all. The moduli spaces\(^6\) exhibited by the Attractor Eqs. (3.24) and (3.25) are \([33]\)

\[ \mathcal{M}_{(1/8)-\text{BPS,large}} = \frac{E_{6(6)}}{SU(2) \times SU(6)}, \quad \text{dim}_\mathbb{R} = 40; \]  

(3.26)

\[ \mathcal{M}_{\text{non-BPS,}Z_{ab} \neq 0} = \frac{E_{6(6)}}{USp(8)}, \quad \text{dim}_\mathbb{R} = 42. \]  

(3.27)

As found in \([33]\), the general structure of the moduli spaces of attractor solutions in supergravities based on symmetric scalar manifolds \(\frac{G}{H}\) is

\[ \frac{\mathcal{H}_\text{nc}}{\mathcal{H}} \]  

(3.28)

where \(\mathcal{H}_\text{nc}\) is the noncompact stabilizer of the charge orbit \(\frac{G}{H}\) (apart from eventual \(U(1)\) factors), \(\mathcal{H}_\text{nc}\) is a noncompact, real form of \(H\), and \(\mathcal{H} = \text{mcs}(\mathcal{H}_\text{nc})\). As justified in \([29]\) and then in \([32]\), \(\mathcal{M}_{(1/8)-\text{BPS,large}}\) is a quaternionic symmetric manifold. Furthermore, \(\mathcal{M}_{\text{non-BPS,}Z_{ab} \neq 0}\) given by Eq. (3.27) is nothing but the scalar manifold of \(\mathcal{N} = 8, d = 5\) supergravity. The stabilizers of \(\mathcal{M}_{(1/8)-\text{BPS,large}}\) and \(\mathcal{M}_{\text{non-BPS,}Z_{ab} \neq 0}\) exploit the maximal compact symmetry of the corresponding charge orbits; this symmetry becomes

\(\text{Results obtained by explicit computations within the } \mathcal{N} = 2, d = 4 \text{ symmetric so-called stu model in } [23,34] \text{ seem to point out that the moduli spaces should be present not only at the event horizon of the considered extremal BH (i.e. for } r \rightarrow r_H^\text{ext}), \text{ but also all along the scalar attractor flow (i.e. } \forall r \geq r_H).\)
fully manifest through the enhancement of the compact symmetry group of $Z_{AB,\text{skew-diag}}$ at the event horizon of the extremal BH, respectively, given by Eqs. (3.17) and (3.21).

It is now convenient to denote with $\lambda_i$ ($i = 1, \ldots, 4$) the four real non-negative eigenvalues of the matrix $Z_{AB}Z_{CB} = (ZZ^\dagger)^A$. By recalling Eq. (3.10), one can notice that

$$\lambda_1 = e^2,$$

and one can order them as $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4$, without any loss of generality. The explicit expression of $\lambda_i$ in terms of $U(8)$-invariants (namely of $\text{Tr}(ZZ^\dagger)$, $\text{Tr}((ZZ^\dagger)^2)$, $\text{Tr}((ZZ^\dagger)^3)$ and $\text{Tr}((ZZ^\dagger)^4)$, and suitable powers) is given by Eqs. (4.74), (4.75), (4.86) and (4.87) of [9], and it will be used in Sec. VI to determine the ADM mass for $\frac{1}{4} - \text{BPS}$ ($k = 1, 2, 4$) extremal BH states.

Three distinct small charge orbits (all with $I_{4, N = 8} = 0$) exist, and they all are supersymmetric:

1. The generic small lightlike orbit is $\frac{1}{8} - \text{BPS}$, it is defined by the $E_{7(7)}$-invariant constraint

$$I_{4, N = 8} = 0,$$

and it reads [4,5]

$$O_{(1/8) - \text{BPS, small}} = \frac{E_{7(7)}}{E_{6(6)} \times T_{26}}, \quad \text{dim}_R = 55.$$

Generally, it yields four different $\lambda_i$'s, and in this case Eq. (3.13) reduces to

$$\cos \varphi_Z(\phi, P) \big|_{I_{4, N = 8} = 0} = -\frac{[2^2 \text{Tr}(Z_{AC}Z_{BC})^2 - (\text{Tr}(Z_{AC}Z_{BC}))^2]}{2^4(\text{det}(Z_{AC}Z_{BC}))^{1/4}} \bigg|_{I_{4, N = 8} = 0}.$$

(3.32)

In agreement with the results of [4,5], the (maximal compact) symmetry of the skew-diagonalized central charge matrix $Z_{AB,\text{skew-diag}}$ all along the $\frac{1}{8} - \text{BPS}$ small flow is the generic one: $(SU(2))^4$. The counting of the parameters of $O_{(1/8) - \text{BPS, small}}$ consistently reads: $55 = 4$ skew-eigenvalues $\lambda_1 + 1$ phase $\varphi_Z + 51(= \text{dim}_R^{(SU(8)/(SU(2)))})$ "generalized angles" – 1 constraining constraint (3.30).

2. The small critical orbit is $\frac{1}{4} - \text{BPS}$. It reads [4,5]

$$O_{(1/4) - \text{BPS}} = \frac{E_{7(7)}}{(SO(6, 5) \times T_{32}) \times T_1}, \quad \text{dim}_R = 45,$$

and it is defined by the following differential constraint on $I_{4, N = 8}$ [3,9]:

$$\frac{\partial I_{4, N = 8}}{\partial Z_{AB}} = 0,$$

(3.34)

which, due to the reality of $I_{4, N = 8}$, is actually $E_{7(7)}$-invariant. Let us also notice that, due to the homogeneity of $I_{4, N = 8}$ of degree four in $P$, Eq. (3.34) implies the constraint (3.30). In particular, along the $\frac{1}{4} - \text{BPS}$ orbit it holds that (the labelling does not yield any loss of generality)

$$\lambda_1 = \lambda_2 > \lambda_3 = \lambda_4 \geq 0.$$

(3.35)

If $P f(Z_{AB}) \neq 0$ then

$$\lambda_1 = \lambda_2 > \lambda_3 = \lambda_4 > 0,$$

and Eq. (3.13) yields $\varphi_Z(\lambda, P) \big|_{I_{4, N = 8} = 0} = 0$.

(3.36)

and $\varphi_Z$ is undetermined. In this case, the (maximal compact) symmetry of the skew-diagonalized central charge matrix $Z_{AB,\text{skew-diag}}$ is $USp(4) \times SU(4) \sim SO(5) \times SO(6)$, which is the mcs of the nontranslational part of the stabilizer of $O_{(1/4) - \text{BPS}}$, expressing the maximal compact symmetry of $O_{(1/4) - \text{BPS}}$ itself. In agreement with the results of [4,5], the maximal compact symmetry of the skew-diagonalized central charge matrix $Z_{AB,\text{skew-diag}}$ along the $\frac{1}{4} - \text{BPS}$ small flow (fully manifest in the particular solution (3.37)) is $USp(4) \times SU(4)$.

The counting of the parameters of $O_{(1/4) - \text{BPS}}$ consistently reads: $45 = 2$ skew-eigenvalues $\lambda_1$ and $\lambda_2$ + 43(= dim$_R^{(SU(8)/(USp(4)))}$) "generalized angles".

3. The small doubly-critical orbit is $\frac{1}{8} - \text{BPS}$, and it reads [4,5]

$$O_{(1/2) - \text{BPS}} = \frac{E_{7(7)}}{E_{6(6)} \times T_{27}}, \quad \text{dim}_R = 28.$$

(3.38)

It can be defined in an $E_{7(7)}$-invariant way by performing the following two-step procedure [9]. One starts by considering the requirement that the second derivative of $I_{4, N = 8}$ (with respect to $Z_{AB}$) projected along the adjoint representation $\text{Adj}(SU(8)) = 63$ of $SU(8)$ vanishes, yielding [9]

$$\frac{\partial^2 I_{4, N = 8}}{\partial Z_{AB} \partial Z_{BC}} \bigg|_{\text{Adj}(SU(8))} = 0 \Leftrightarrow Z_{AC}Z_{BC} = \frac{1}{2} \delta^A_B Z_{DE}Z^{DE}.$$

(3.39)

This is a mixed rank-2 $SU(8)$-covariant condition. By further differentiating with respect to the scalars $\phi$ parametrizing $E_{7(7)}/SU(8)$ and using the Maurer-Cartan Eqs. (3.3), one obtains another $SU(8)$-covariant relation [notice the strict similarity to the $\mathcal{N} = 8$, 125010-7.
As given by the analysis of [3], the classification of large and small orbits of the 56 of $E_{7(7)}$ can be performed also considering the symplectic basis composed by the fluxes $q_A$ ($A = 1, \ldots, 56$). In general, the symplectic basis of charges is useful in order to determine, through constraints imposed on the relevant $U$-invariant, the number and topology of orbits of the relevant representation of the $U$-duality group. On the other hand, using the manifestly $H$-covariant basis of central charges and matter charges one can achieve a symplectic-invariant characterization of charge orbits, and also study the related supersymmetry-preserving features.

Finally, it is worth pointing out once again that there is a crucial difference among the various constraints defining the two large and the three small charge orbits of $\mathcal{N} = 8$, $d = 4$ supergravity listed above:

- **The large charge orbits** $O_{(1/8)-\text{BPS, large}}$ and $O_{\text{non-\text{-BPS, large}}}$, respectively, given by Eqs. (3.18) and (3.38), are in order defined by the $E_{7(7)}$-invariant conditions $I_{4, N^\rightarrow 8} > 0$ and $I_{4, N^\rightarrow 8} < 0$. Because of their $E_{7(7)}$-invariance, these conditions are **identities** for the scalar fields $\phi$ spanning $E_{7(7)}$. However, the classical attractor mechanism does hold for large extremal BHs, and the scalars $\phi$ are stabilized purely in terms of charges $P$ at the event horizon ($r \rightarrow r_H^+$) through the only two independent solutions (3.16) and (3.20) to the $\mathcal{N} = 8$, $d = 4$ Attractor Eqs. (3.24) and (3.25).

- **The small charge orbits** $O_{(1/8)-\text{BPS, small}}$, $O_{(1/4)-\text{BPS}}$ and $O_{(1/2)-\text{BPS}}$, respectively, given by Eqs. (3.31), (3.33), and (3.38), are in order defined by the $E_{7(7)}$-invariant conditions (3.30), (3.34), and (3.42). Because of their $E_{7(7)}$-invariance, these conditions are **identities** for the scalars $\phi$, which thus are not stabilized along such orbits. Indeed, the classical attractor mechanism does not hold for small BHs.

### IV. $\mathcal{N} = 4$

In $\mathcal{N} = 4$, $d = 4$ supergravity, unlike the $\mathcal{N} = 8$ case, matter (vector) multiplets appear (see e.g. [35,36]). By denoting their number with $M$, the related scalar manifold is the symmetric coset

$$\begin{align*}
\left( \frac{G}{H} \right)_{\mathcal{N} = 4, d = 4} & = \frac{SL(2, \mathbb{R}) \times SO(6, M)}{U(1) \times SO(6) \times SO(M)}, \\
\dim_{\mathbb{R}} & = 6M + 2.
\end{align*}$$

The Abelian vector field strengths and their duals, as well the corresponding fluxes (charges), sit in the bifundamental of $\mathbf{2}, \mathbf{6} + \mathbf{M}$ representation of the global, classical (see Footnote 1) $U$-duality group $SL(2, \mathbb{R}) \times SO(6, M)$ [37]. Such a representation determines the embedding of $SL(2, \mathbb{R}) \times SO(6, M)$ into the symplectic group $Sp(12 + 2M, \mathbb{R})$. The representation $\mathbf{2}, \mathbf{6} + \mathbf{M}$ is en-
dowed with a natural symplectic metric
\[ \Omega = \epsilon_{\alpha\beta} \eta_{\Lambda\Sigma}, \tag{4.2} \]
where \( \epsilon_{\alpha\beta} \) (\( \alpha, \beta = 1, 2 \)) is the (inverse of the) \( SL(2, \mathbb{R}) \)
skew-symmetric metric defined in Eq. (3.11), and \( \eta_{\Lambda\Sigma} \) \( (\Lambda, \Sigma = 1, \ldots, 6 + M = n) \); recall Eq. (2.26)) is the Lorentzian metric of \( SO(6, M) \). Moreover, the \( R \)-symmetry group is \( U(4) \).

Furthermore, \( (2, 6 + M) \) admits an unique invariant, which will be denoted by \( I_{4, N - 4} \) throughout. \( I_{A, N - 4} \) is quartic in charges, and it was firstly determined in [14,19,38].

More precisely, \( I_{4, N - 4} \) is the unique combination of “dressed” charges \( Z_{AB} = Z_{AB}(\phi, \mathcal{P}) \) (central charge matrix, \( A, B = 1, \ldots, 4 \) and \( Z_I(\phi, \mathcal{P}) \) (matter charges, \( I = 1, \ldots, M \)) satisfying
\[
\partial_{\phi} I_{4, N - 4}(Z_{AB}(\phi, \mathcal{P}), Z_I(\phi, \mathcal{P})) = 0, 
\tag{4.3}
\]
for all \( \phi \in \left( \frac{G}{H} \right)_{N - 4, d = -4} \).

Equation (4.3) can be computed by using the Maurer-Cartan Eqs. of the coset \( \frac{SL(2, \mathbb{R})}{U(1)} \times \frac{SO(6, M)}{SO(6) \times SO(M)} \) (see e.g. [29], and Refs. therein):
\[
\nabla Z_{AB} = \frac{1}{2} P e_{ABCD} \xi_{CD} + P A_{B} \hat{Z}^{I}; \tag{4.4}
\]
\[
\nabla Z_I = \frac{1}{2} P A_{B} \xi_{B}^{AB} + P \eta_{I} \hat{Z}^{I}, \tag{4.5}
\]
or equivalently by performing an infinitesimal \( \frac{SL(2, \mathbb{R})}{U(1)} \times \frac{SO(6, M)}{SO(6) \times SO(M)} \)-transformation of the central charge matrix and of matter charges (see e.g. [29], and Refs. therein):
\[
\delta_{\hat{\xi}} Z_{AB} = \frac{1}{2} \hat{\xi} e_{ABCD} \xi_{CD} + \hat{\xi} A_{B} Z^{I}; \tag{4.6}
\]
\[
\delta_{\hat{\xi}} Z_I = \hat{\xi} \eta_{I} \hat{Z}^{I} + \frac{1}{2} \hat{\xi} A_{B} \hat{Z}^{AB}, \tag{4.7}
\]
where \( \nabla \) stands for the covariant differential operator in \( \frac{SL(2, \mathbb{R})}{U(1)} \times \frac{SO(6, M)}{SO(6) \times SO(M)} \). \( P \) and \( P A_{B} \) respectively are the Vielbein 1-forms of \( \frac{SL(2, \mathbb{R})}{U(1)} \) and \( \frac{SO(6, M)}{SO(6) \times SO(M)} \), with \( P A_{B} \) satisfying the reality condition:
\[
P A_{B} = \frac{1}{2} \eta_{IJ} e_{ABCD} \xi_{CD}^{IJ}. \tag{4.8}
\]
Moreover, \( \hat{\xi} \) is the infinitesimal \( \frac{SL(2, \mathbb{R})}{U(1)} \)-parameter and \( \hat{\xi} A_{B} \) are the infinitesimal \( \frac{SO(6, M)}{SO(6) \times SO(M)} \)-parameters, satisfying the reality condition
\[
\hat{\xi} A_{B} = \frac{1}{2} \eta_{IJ} e_{ABCD} \xi_{CD}^{IJ}. \tag{4.9}
\]
As found in [14,19,38] and rigorously reobtained in [29], in terms of \( Z_{AB} \) and \( Z_I \) the unique solution of Eq. (4.3) reads:
\[
I_{4, N - 4} = S_{1}^{2} - |S_{2}|^{2}, \tag{4.10}
\]
where one can identify \( S_{1} = L_{0}, S_{2} = L_{1} + i l_{2} \), with \( L = (L_{0}, L_{1}, L_{2}) \) being an \( SL(2, \mathbb{R}) \sim SO(1, 2) \)-vector with square norm
\[
L^{2} = L_{0}^{2} - L_{1}^{2} - L_{2}^{2} = S_{1}^{2} - |S_{2}|^{2}. \tag{4.11}
\]
\( S_{1} \) and \( S_{2} \) are defined as \([29]\)
\[
S_{1} = \frac{1}{2} Z_{AB} \hat{Z}^{AB} - Z_{I} \hat{Z}^{I} \in \mathbb{R}; \tag{4.12}
\]
\[
S_{2} = \frac{1}{4} e^{ABCD} Z_{AB} Z_{CD} - Z_{I} \hat{Z}^{I} \in \mathbb{C}. \tag{4.13}
\]
In [29] it was indeed shown that \( I_{4, N - 4} \) given by Eq. (4.10) is the unique combination of \( SO(6, M) \)-invariant and scalar-dependent quantities, which is actually also \( SL(2, \mathbb{R}) \)-independent and thus scalar-independent, satisfying
\[
\delta_{\hat{\xi}} I_{4, N - 4} = 0; \tag{4.14}
\]
\[
\delta_{\hat{\xi}, A_{B}} I_{4, N - 4} = 0, \tag{4.15}
\]
with Eqs. (4.6), (4.7), and (4.9) holding true.

On the other hand, the expression of \( I_{4, N - 4} \) in terms of the “bare” charges \( \mathcal{P} \) reads [14,15,18,19]
\[
I_{4, N - 4} = p^{2} - (p \cdot q)^{2} = \frac{1}{2} \left( p \hat{\Sigma} q \hat{\Sigma} - p \hat{\Sigma} q \Lambda \right) \left( p \hat{\Sigma} q \Omega - p \hat{\Sigma} q \Xi \right) \eta^{\Lambda \Xi} \eta^{\Sigma \Omega}; \tag{4.16}
\]
where
\[
p^{2} = p \cdot p = p \Lambda p_{\Sigma} \eta^{\Lambda \Xi}, \quad q^{2} = q \cdot q = q_{\Lambda} q_{\Xi} \eta^{\Lambda \Xi}, \tag{4.17}
\]
and the tensor
\[
T^{(a)}_{\Lambda \Xi} = p \Lambda q_{\Xi} - p_{\Lambda} q \Xi = T^{(a)}_{[\Lambda \Xi]} \tag{4.18}
\]
has been introduced (the superscript “(a)” stands for “antisymmetric”).

The classification of charge orbits, in particular, the BPS ones, was performed in [3,9]. By performing a suitable \( U(1) \times SO(6)(-U(4)) \)-transformation, the central charge matrix \( Z_{AB} \) can be skew-diagonalized in the normal frame (recall definition (3.11)):
\[
Z_{AB} U(4) \rightarrow Z_{AB, \text{skew-diag}} = \left( \begin{array}{cc}
\hat{z}_{1} & 0 \\
0 & \hat{z}_{2}
\end{array} \right), \quad \hat{z}_{1}, \hat{z}_{2} \in \mathbb{R}^{+}, \tag{4.19}
\]
where the ordering \( \hat{z}_{1} \geq \hat{z}_{2} \) does not imply any loss of generality. Furthermore, by performing a suitable
$SO(M)$-transformation, the vector $Z_l$ of matter charges can be reduced to have only two nonvanishing entries, one real positive and the other one complex, say (without loss of generality, with the subscript “red,” standing for “reduced”)

\[ Z_{l,\text{red}} = (\rho_1 e^{i\theta}, \rho_2, 0, \ldots, 0), \]  

(4.20)

\[ \rho_1, \rho_2 \in \mathbb{R}^+, \quad \theta \in \mathbb{R}. \]

For nonvanishing (in general different) skew-eigenvalues $z_1$ and $z_2$, the symmetry group of $Z_{AB,\text{skew-diag}}$ is $(USp(2))^2 \sim (SU(2))^2$. Analogously, for nonvanishing (in general different) $\rho_1$ and $\rho_2$ (and nonvanishing phase $\theta$) the symmetry group of $Z_{l,\text{red}}$ is $SO(M - 2)$. Thus, beside $z_1, z_2, \rho_1, \rho_2$ and $\theta$ the generic $Z_{AB}$ and $Z_l$ are described by $7 + 2M = \dim_{\mathbb{R}}(U(1) \times SO(M))$ “generalized angles”.

Consistently, the total number of parameters is $2 + 2 + 1 + 7 + 2M = 12 + 2M$, which is the real dimension of the bi-fundamental representation $(2, 6 + M)$, defining the embedding of $SL(2, \mathbb{R}) \times SO(6, M)$ into $Sp(12 + 2M, \mathbb{R})$.

In $\mathcal{N} = 4, d = 4$ matter coupled supergravity three distinct large charge orbits of the $(2, 6 + M)$ of $SL(2, \mathbb{R}) \times SO(6, M)$ (for which $I_{4,\mathcal{N}=4} \neq 0$, and the attractor mechanism holds) exist, as resulting from the analysis performed in $^7$ [13]:

1. The large $\frac{1}{4}$ BPS orbit

\[ O_{(1/4)-\text{BPS,large}} = SL(2, \mathbb{R}) \times \frac{SO(6, M)}{SO(4, M) \times SO(2)}, \]

(4.21)

\[ \dim_{\mathbb{R}} = 11 + 2M, \]

is defined by the $SL(2, \mathbb{R}) \times SO(6, M)$-invariant constraint

\[ I_{4,\mathcal{N}=4} > 0. \]  

(4.22)

Thus, the corresponding horizon solution of the $\mathcal{N} = 4, d = 4$ Attractor Eqs. yields $[3, 9, 13]$

\[ z_1 \in \mathbb{R}^+_0, \quad z_2 = 0, \]

(4.23)

\[ \rho_1 = \rho_2 = 0, \theta \quad \text{undetermined}; \]

\[ S_1 = z_1^2 > 0, \quad S_2 = 0. \]  

(4.24)

Therefore, at the event horizon, the symmetry group of $Z_{AB,\text{skew-diag}}$ defined in Eq. (4.19) does not get enhanced, while the symmetry group of $Z_{l,\text{red}}$ defined in Eq. (4.20) gets enhanced as follows:

\[ SO(6, M) \xrightarrow{\text{large}} SO(M). \]  

(4.25)

As a consequence, the horizon attractor solution exploits the maximal compact symmetry $SU(2) \times SU(2) \times SO(M) \times SO(2)$, which is the mcs $[31]$ of the stabilizer of $O_{(1/4)-\text{BPS,large}}$ itself.

2. The large non-BPS $Z_{AB} = 0$ orbit (existing for $M \geq 2$) $[13]$

\[ O_{\text{non-BPS},Z_{AB}=0,\text{large}} = SL(2, \mathbb{R}) \times \frac{SO(6, M)}{SO(6, M - 2) \times SO(2)}, \]

(4.26)

\[ \dim_{\mathbb{R}} = 11 + 2M, \]

is defined by the $SL(2, \mathbb{R}) \times SO(6, M)$-invariant constraint

\[ I_{4,\mathcal{N}=4} > 0. \]  

(4.27)

Thus, the corresponding attractor solution of the $\mathcal{N} = 4, d = 4$ Attractor Eqs. yields (for $M \geq 2$) $[3, 9, 13]$

\[ z_1 = z_2 = 0, \]

(4.28)

\[ \rho_1^2 e^{2i\theta} + \rho_2^2 = 0 \Leftrightarrow \rho_1 = \rho_2 \in \mathbb{R}^+_0, \]

\[ \theta = \frac{\pi}{2} + k\pi, \quad k \in \mathbb{Z}; \]

\[ S_1 = 2 \rho_1^2 < 0, \quad S_2 = 0. \]  

(4.29)

Therefore, at the event horizon, the symmetry group of $Z_{AB,\text{skew-diag}}$ defined in Eq. (4.19) gets enhanced as follows:

\[ (SU(2))^2 \xrightarrow{\text{large}} SU(4). \]  

(4.30)

and the symmetry group of $Z_{l,\text{red}}$ defined in Eq. (4.20) does not get enhanced. Consequently, the horizon attractor solution exploits the maximal compact symmetry $SU(4) \times SO(M - 2) \times SO(2)$, which is the mcs $[31]$ of the stabilizer of $O_{\text{non-BPS},Z_{AB} \neq 0,\text{large}}$ itself.

3. The large non-BPS $Z_{AB} \neq 0$ orbit (existing for $M \geq 1$) $[13]$

\[ O_{\text{non-BPS},Z_{AB} \neq 0,\text{large}} = SL(2, \mathbb{R}) \times \frac{SO(6, M)}{SO(5, M - 1) \times SO(1, 1)}, \]

(4.31)

\[ \dim_{\mathbb{R}} = 11 + 2M, \]

is defined by the $SL(2, \mathbb{R}) \times SO(6, M)$-invariant constraint

\[ I_{4,\mathcal{N}=4} < 0. \]  

(4.32)

At the event horizon of the extremal BH, the solution of the $\mathcal{N} = 4, d = 4$ Attractor Eqs. yields (for $M \geq 1$) $[3, 9, 13]$

\[ \text{consistent with the analysis of} \ [13], \text{Eqs. (4.21), (4.26), and (4.31), fix a slightly misleading notation for the large charge orbits of} \mathcal{N} = 4, d = 4 \text{ matter coupled supergravity, as given by Table 1 of} \ [39]. \]
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\[ z_1 = z_2 = \frac{\rho_1}{\sqrt{2}} \in \mathbb{R}_+^+; \quad p_2 = 0, \]
\[ \theta = \frac{\pi}{2} + k\pi, \quad k \in \mathbb{Z}; \]
\[ S_1 = 0, \quad S_2 = 3z_1^2 > 0. \] (4.33)

Thus, at the event horizon, the symmetry group of \( Z_{AB, \text{skew-diag}} \) defined in Eq. (4.19) gets enhanced as follows:

\[ (SU(2))^2 \rightarrow USp(4), \] (4.35)

and the symmetry group of \( Z_{I, \text{red}} \) defined in Eq. (4.20) gets also enhanced as

\[ SO(M-2) \rightarrow SO(M-1). \] (4.36)

As a consequence, the horizon attractor solution exploits the maximal compact symmetry \( USp(4) \times SO(M-1) \) which, due to the isomorphism \( USp(4) \sim SO(5) \), is the mcs \([31]\) of the stabilizer of \( O_{\text{non-BPS,} Z_{AB} \neq 0, \text{large}} \) itself.

As mentioned above, for such large charge orbits, corresponding to a nonvanishing quartic \( SL(2, \mathbb{R}) \times SO(6, M) \)-invariant \( I_{4, N=4} \) and thus supporting large BHs, the attractor mechanism holds. Consequently, the computations of the Bekenstein-Hawking BH entropy can be performed by solving the criticality conditions for the \( \text{“BH potential”} \)

\[ V_{BH, \mathcal{N}=4} = \frac{1}{2} Z_{AB} \tilde{Z}^A + Z_I \tilde{Z}^I, \] (4.37)

the result being

\[ \frac{S_{BH}}{\pi} = V_{BH, \mathcal{N}=4} |_{\delta V_{BH, \mathcal{N}=4} = 0} = V_{BH, \mathcal{N}=4}(\phi_H(\mathcal{P}), \mathcal{P}) = \frac{1}{2} I_{4, \mathcal{N}=4}^{1/2}, \] (4.38)

where \( \phi_H(\mathcal{P}) \) denotes the set of solutions to the criticality conditions of \( V_{BH, \mathcal{N}=4} \), namely, the Attractor Eqs. of \( \mathcal{N} = 4, d = 4 \) matter coupled supergravity:

\[ \partial_\phi V_{BH, \mathcal{N}=4} = 0, \quad \forall \phi \in \frac{SL(2, \mathbb{R})}{U(1)} \times \frac{SO(6, M)}{SO(6) \times SO(M)}; \] (4.39)

expressing the stabilization of the scalar fields purely in terms of supporting charges \( \mathcal{P} \) at the event horizon of the extremal BH. Through Eqs. (4.4), (4.5), and (4.37), Equations (4.39) can be rewritten as follows \([13]\):

\[ \left( Z_{AB} + \frac{1}{2} \epsilon_{ABCD} \tilde{Z}^C \right) Z^D = 0; \] (4.40)

\[ Z^I Z^I \delta_{IJ} + \frac{1}{4} \epsilon_{ABCD} Z_{AB} \tilde{Z}^C = 0. \]

Actually, the critical potential \( V_{BH, \mathcal{N}=4} \) exhibits some \( \text{“flat”} \) directions, so not all scalars are stabilized in terms of charges at the event horizon \([39]\). Thus, Eq. (4.38) yields that the unstabilized scalars, spanning a related moduli space of the considered class of attractor solutions, do not enter in the expression of the BH entropy at all. The moduli spaces exhibited by the Attractor Eqs. (4.39) and (4.40) are \([39]\)

\[ \mathcal{M}_{(1/4)-\text{BPS, large}} = \frac{SO(4, M)}{SU(2) \times SU(2) \times SO(M)}, \quad \text{dim}_{\mathbb{R}} = 4M; \]
(4.41)

\[ \mathcal{M}_{\text{non-BPS, } Z_{AB} = 0, \text{large}} = \frac{SO(6, M - 2)}{SU(4) \times SO(M - 2)}, \quad \text{dim}_{\mathbb{R}} = (6M - 2); \]
(4.42)

\[ \mathcal{M}_{\text{non-BPS, } Z_{AB} \neq 0, \text{large}} = \frac{SO(5, M - 1)}{USp(4) \times SO(M - 1)}, \quad \text{dim}_{\mathbb{R}} = 5(M - 1) + 1. \]
(4.43)

As justified in \([29]\) and then in \([39]\), \( \mathcal{M}_{(1/4)-\text{BPS, large}} \) is a quaternionic symmetric manifold. Furthermore, \( \mathcal{M}_{\text{non-BPS, } Z_{AB} = 0, \text{large}} \) given by Eq. (4.43) is nothing but the scalar manifold of \( \mathcal{N} = 4, d = 5 \) matter coupled supergravity. The stabilizers of \( \mathcal{M}_{(1/4)-\text{BPS, large}}, \mathcal{M}_{\text{non-BPS, } Z_{AB} = 0, \text{large}} \) and \( \mathcal{M}_{\text{non-BPS, } Z_{AB} \neq 0, \text{large}} \) exploit the maximal compact symmetry of the corresponding charge orbits; this symmetry becomes fully manifest through the enhancement of the compact symmetry group of \( Z_{AB, \text{skew-diag}} \) and \( Z_{I, \text{red}} \) at the event horizon of the extremal BH, respectively, given by Eqs. (4.25), (4.30), (4.35), and (4.36).

Let us now analyze the small charge orbits of the \( (2, 6 + M) \) of \( SL(2, \mathbb{R}) \times SO(6, M) \), associated to \( I_{4, \mathcal{N}=4} = 0 \), for which the attractor mechanism does not hold. The analysis performed below completes the one given in \([3,9]\).

While in \( \mathcal{N} = 8, d = 4 \) supergravity all three small charge orbits are BPS (with various degrees of supersymmetry-preservation), in the considered \( \mathcal{N} = 4, d = 4 \) theory there are five small charge orbits, two of them being \( \frac{1}{2} \) BPS one \( \frac{1}{2} \) BPS, and the other two non-BPS (one with \( Z_{AB} = 0 \) and the other with \( Z_{AB} \neq 0 \)). Such an abundance of different charge orbits can be traced back to the factorized nature of the \( U \)-duality group \( SL(2, \mathbb{R}) \times SO(6, M) \). Furthermore, it should be remarked that in \( \mathcal{N} = 4, d = 4 \) supergravity the \( \mathcal{M}_{(1/4)-\text{BPS, charge}} \) charge orbit exists only in its large version, differently from the \( d = 4 \) maximal theory, in which both large and small \( \frac{1}{(\mathcal{N} - 4)} \) BPS charge orbits exist.

It is now convenient to denote with \( \alpha_1 \) and \( \alpha_2 \) the two real non-negative eigenvalues of the matrix \( Z_{AB} = Z^{CB}_{AB} = (ZZ^\dagger)_A^C \). By recalling Eq. (4.19), one can notice that
and one can order them as $\alpha_1 \geq \alpha_2$, without any loss of generality. The explicit expression of $\alpha_i$ in terms of $U(4) \times SO(M)$-invariants (namely of $\text{Tr}(\mathbb{Z}^4)$, $\text{Tr}((\mathbb{Z}^4)^2)$, and suitable powers) is given by Eqs. (5.108) and (5.109) of [9].

Firstly, let us observe that from Eqs. (4.16) and (4.11) the $SL(2, \mathbb{R}) \times SO(6, M)$-invariant “degeneracy” condition can be written in the “dressed” ($R$-symmetry- and $SO(M)$-covariant) and “bare” (symplectic-, i.e. $Sp(12 + 2M, \mathbb{R})$-covariant) charges’ bases, respectively, as follows:

$$I_{4, N=4} = 0 \iff S^2_2 = |S_2|^2 \iff p_2^2 q^2 = (p \cdot q)^2 \geq 0.$$  \hfill (4.45)

Then, in order to determine the number and typology of small orbits, it is convenient to start differentiating $I_{4, N=4}$ in the symplectic “bare” charges’ basis $\mathcal{P} = (\rho, \eta, q_{\Lambda})^T$ (recall definition (1.2)). Equations (4.16) and (4.18) yield the constraints defining the small critical orbits to read

$$\frac{\partial I_{4, N=4}}{\partial \rho_{\Lambda}} = 2[q^2 \rho^2 - (q \cdot p) \rho^\Lambda] = 2T^{(a)}_{\Lambda \Sigma} q_{\Sigma} = 0;$$ \hfill (4.46)

$$\frac{\partial I_{4, N=4}}{\partial q_{\Lambda}} = 2[p^2 q^2 - (q \cdot p) q^\Lambda] = -2T^{(a)}_{\Lambda \Sigma} p_{\Sigma} = 0.$$ \hfill (4.47)

Because of the definition (4.18), or equivalently to the homogeneity (of degree four) in charges of $I_{4, N=4}$, it is worth noticing that the “criticality” constraints (4.46) and (4.47) imply the “degeneracy” condition (4.45).

Beside the trivial one ($p_{\Lambda} = 0 = q_{\Lambda} \forall \Lambda$), all the solutions to the “criticality” constraints (4.46) and (4.47) list as follows:

$$(A) \begin{cases} T^{(a)}_{\Lambda \Sigma} = 0; \\ A.1 \ [p^2 > 0, q^2 > 0; \text{aut}] \\ A.2 \ [p^2 < 0, q^2 < 0; \text{aut}] \\ A.3 \ [p^2 q^2 = (p \cdot q)^2 = 0; \text{p}^2 = 0, q^2 = 0; \end{cases}$$ \hfill (4.48)

$$(B) \begin{cases} T^{(a)}_{\Lambda \Sigma} \neq 0; \\ p^2 = q^2 = p \cdot q = 0 \iff T^{(a)} = 0 \end{cases}.$$ \hfill (4.49)

Notice that each set (A.1, A.2, A.3 and B) of constraints is $SL(2, \mathbb{R}) \times SO(6, M)$-invariant, but formulated in terms of the symplectic charge basis $\mathcal{P}$.

The solutions (4.48) and (4.49) can be rewritten by noticing that $\frac{\partial^2 I_{4, N=4}}{\partial \mathcal{P}^2}$, i.e. the tensor of second derivatives of $I_{4, N=4}$ with respect to $\mathcal{P}$, sits in the symmetric product representation $((2, 6 + M) \times (2, 6 + M))_s$ of the $U$-duality group $SL(2, \mathbb{R}) \times SO(6, M)$, which decomposes as follows [9]:

$$((2, 6 + M) \times (2, 6 + M))_s \rightarrow (3, \text{TrSym}(SO(6, M))) + (1, \text{Adj}(SO(6, M))).$$ \hfill (4.50)

The antisymmetric tensor

$$T^{(a)}_{\Lambda \Sigma} = \frac{\partial^2 I_{4, N=4}}{\partial \mathcal{P}^2} \bigg|_{(3, \text{TrSym}(SO(6, M)))}$$ \hfill (4.51)

was already introduced in Eq. (4.18), $\text{TrSym}$ and $\text{Adj}$ respectively denote the traceless symmetric and adjoint representations, and [9]

$$T^{(a)}_{\Lambda \Sigma} = \frac{\partial^2 I_{4, N=4}}{\partial \mathcal{P}^2} \bigg|_{(1, \text{Adj}(SO(6, M)))}.$$ \hfill (4.52)

The definition (4.53) of $T^{(a)}$ implies that (recall Eq. (4.16))

$$I_{4, N=4} = \text{det}(T^{(a)}) = \text{det}\left(\frac{\partial^2 I_{4, N=4}}{\partial \mathcal{P}^2} \bigg|_{(3, \text{Sym}(SO(6, M)))}\right).$$ \hfill (4.54)

in turn yielding another, equivalent $SL(2, \mathbb{R}) \times SO(6, M)$-invariant characterization of the “degeneracy” condition (4.45):

$$\text{det}(T^{(a)}) = \text{det}\left(\frac{\partial^2 I_{4, N=4}}{\partial \mathcal{P}^2} \bigg|_{(3, \text{Sym}(SO(6, M)))}\right) = 0.$$ \hfill (4.55)

Thus, Eqs. (4.48) and (4.49) can be recast as follows:

$$(A) \begin{cases} T^{(a)}_{\Lambda \Sigma} = 0; \\ \text{det}(T^{(a)}) = 0, \text{det}(\mathcal{P}) > 0; \\ A.1 \ [\text{Tr}(T^{(a)}) > 0; \text{aut}] \\ A.2 \ [\text{Tr}(T^{(a)}) < 0; \text{aut}] \\ A.3 \ [\text{Tr}(T^{(a)}) = 0 \iff T^{(a)} = 0. \end{cases}$$ \hfill (4.56)
As mentioned above, each set \((A.1, A.2, A.3)\) and \(B\) of constraints is \(SL(2, \mathbb{R}) \times SO(6, M)\)-invariant, but formulated in terms of the symplectic charge basis \(P\).

It is interesting to point out that, differently from \(\mathcal{N} = 8, d = 4\) supergravity treated in Sec. III, in \(\mathcal{N} = 4, d = 4\) supergravity there are no \textit{small doubly-critical} (or with higher degree of criticality) charge orbits \textit{independent} from the \textit{small critical} ones. This can be easily seen by noticing that the solutions (4.56) and (4.57) to the \textit{criticality} constraints (4.46) and (4.47) can actually be rewritten in a \textit{doubly-critical} fashion, \textit{i.e.} through \(\frac{\partial^2 I_{4,N-4}}{\partial q \partial p}\) and related projections (according to decomposition (4.50)).

For completeness’ sake, we report here the second-order derivatives of \(I_{4,N-4}\) with respect to the \textit{bare} symplectic charges:

\[
\begin{align*}
\frac{\partial^2 I_{4,N-4}}{\partial p \partial p} &= 2(q^2 \eta^\Lambda \Sigma - q^\Lambda q^\Sigma); \\
\frac{\partial^2 I_{4,N-4}}{\partial q \partial q} &= 2(p^2 \eta^\Lambda \Sigma - p^\Lambda p^\Sigma); \\ \\
\frac{\partial^2 I_{4,N-4}}{\partial q \partial p} &= 4T^{(o)}|_{\Sigma} .
\end{align*}
\]

In order to determine the \textit{small} orbits of the bi-fundamental representation \((2, 6 + \mathbf{M})\) of the \(U\)-duality group \(SL(2, \mathbb{R}) \times SO(6, M)\) and to study their supersymmetry-preserving properties, it is now convenient to switch to the basis of \textit{dressed} \textit{charges} (recall Eqs. (2.22) and (2.23))

\[
U \equiv (Z, \bar{Z})^T = (Z_{AB}, Z^I, \bar{Z}_{AB}, Z^I)^T .
\]

From the analysis of [9], one obtains the following equivalence:

\[
T^{(o)}|_{\Sigma} \rightarrow \frac{\partial^2 I_{4,N-4}}{\partial q \partial p}|_{\text{Adj}(SO(6, M))} = 0 .
\]

The \(SL(2, \mathbb{R}) \times SO(6, M)\)-invariant constraint (4.62) is common to the \textit{small critical} charge orbits determined by the solutions \(A.1, A.2\) and \(A.3\) of Eqs. (4.56). It also implies that \(\alpha_1 = \alpha_2\) [9]. Then, the further \(SL(2, \mathbb{R}) \times SO(6, M)\)-invariant constraints \(\text{Tr}(T^{(0)}) = 0\) can equivalently be rewritten as (recall definition (4.12))

\[
\text{Tr}(T^{(0)}) = S_1 = 0 .
\]

Therefore, one can characterize the \textit{small critical} orbits \(A.1, A.2,\) and \(A.3\) of Eqs. (4.48) and (4.56) as follows:

\[
\begin{align*}
\text{A.1}) S_1 &> 0; \\
\text{A.2}) S_1 &< 0; \\
\text{A.3}) S_1 &= 0 \Rightarrow S_2 = 0 .
\end{align*}
\]

Notice that each set \((A.1, A.2, A.3)\) and \(B\) of constraints is \(SL(2, \mathbb{R}) \times SO(6, M)\)-invariant but, differently from Eqs. (4.48) and (4.56), it is also independent from the symplectic basis eventually considered.

On the other hand, the \(SL(2, \mathbb{R}) \times SO(6, M)\)-invariant constraints (4.49) and (4.57) defining the \textit{small critical} orbit \(B\) can be recast in a form which (differently from Eqs. (4.49) and (4.57)) is independent from the symplectic basis eventually considered, as follows:

\[
B \rightarrow \frac{\partial^2 I_{4,N-4}}{\partial q \partial p}|_{\text{Adj}(SO(6, M))} \neq 0 .
\]

Thus, five distinct small charge orbits (all with \(I_{4,N-4} = 0\)) exist:

1. The \textit{critical} orbit \(A.1\) is defined by the \(SL(2, \mathbb{R}) \times SO(6, M)\)-invariant constraints (4.48) (or (4.56), or (4.64)). Such constraints are solved by the following flow solution (exhibiting maximal symmetry):

\[
\begin{align*}
z_1 &= z_2 \in \mathbb{R}^+, \\
\rho_1 &= \rho_2 = 0, \theta \quad \text{undetermined}.
\end{align*}
\]

Thus, from the reasoning performed at the end of Sec. II and the analysis of [9], the considered \textit{small critical} orbit is \(\frac{1}{2}\) BPS. Along the corresponding \textit{small critical} \(\frac{1}{2}\) BPS flow, the (maximal compact) symmetry of the \textit{skew-diagonalized} central charge matrix \(Z_{AB,\text{skew-diag}}\) defined in Eq. (4.19) is \(USp(4)\), whereas the one of \(Z_{\text{red}}\) defined in Eq. (4.20) is \(SO(M)\). Therefore, the resulting maximal compact symmetry of the \textit{critical} orbit \(A.1\) is \(USp(4) \times SO(M)\).

2. The \textit{critical} orbit \(A.2\) is defined by the \(SL(2, \mathbb{R}) \times SO(6, M)\)-invariant constraints (4.48) (or (4.56), or (4.64)). Such constraints are solved by the following flow solution, existing for \(M \geq 1\) (and exhibiting maximal symmetry):

\[
\begin{align*}
z_1 &= z_2 = 0, \\
\rho_1 \in \mathbb{R}^+, \quad \rho_2 = 0.
\end{align*}
\]

Thus, the considered \textit{small critical} orbit is non-BPS \(Z_{AB} = 0\). Along the corresponding \textit{small critical} non-BPS \(Z_{AB} = 0\) flow, the (maximal compact) symmetry of the \textit{skew-diagonalized} central charge matrix \(Z_{AB,\text{skew-diag}}\) defined in Eq. (4.19) is \(SU(4)\), whereas the one of \(Z_{\text{red}}\) defined in Eq. (4.20) is \(SO(M - 1)\). Therefore, the resulting maximal com-
The critical orbit \( A.2 \) is \( SU(4) \times SO(M - 1) \).

(3) The critical orbit \( A.3 \) is defined by the \( SL(2, \mathbb{R}) \times SO(6, M) \)-invariant constraints \((4.48)\) (or \((4.56)\), or \((4.64)\)). Such constraints are solved by the following flow solution, existing for \( M \geq 1 \) (and exhibiting maximal symmetry)

\[
z_1 = z_2 = \frac{\rho_2}{\sqrt{2}} \in \mathbb{R}_0^+, \quad \rho_1 = 0, \theta \text{undetermined.}
\]

\[
(4.68)
\]

This small critical orbit is \( \frac{1}{2} \)-BPS. Along the corresponding small critical non-BPS \( Z_{AB} \neq 0 \) flow, the (maximal compact) symmetry of the skew-diagonalized central charge matrix \( Z_{AB, \text{skew-diag.}} \) defined in Eq. \((4.19)\) is \( USp(4) \), whereas the one of \( Z_{\text{1reg}} \) defined in Eq. \((4.20)\) is \( SO(M - 1) \). Therefore, the resulting maximal compact symmetry of the critical orbit \( A.3 \) is \( USp(4) \times SO(M - 1) \).

(4) The critical orbit \( B \) is defined by the \( SL(2, \mathbb{R}) \times SO(6, M) \)-invariant constraints \((4.49)\) (or \((4.57)\), or \((4.65)\)). Such constraints are solved by the following flow solution, existing for \( M \geq 2 \) (and exhibiting maximal symmetry)

\[
z_1 \in \mathbb{R}_0^+, \quad z_2 = 0, \quad \rho_1 = \rho_2 = \frac{z_1}{\sqrt{2}}; \quad \theta = \frac{\pi}{2} + k\pi, \quad k \in \mathbb{Z}.
\]

\[
(4.69)
\]

This small critical orbit is \( \frac{1}{4} \)-BPS. Along the corresponding small critical non-BPS \( Z_{AB} \neq 0 \) flow, the (maximal compact) symmetry of the skew-diagonalized central charge matrix \( Z_{AB, \text{skew-diag.}} \) defined in Eq. \((4.19)\) is \( SU(2) \), whereas the one of \( Z_{\text{1reg}} \) defined in Eq. \((4.20)\) is \( SO(M - 2) \). Therefore, the resulting maximal compact symmetry of the critical orbit \( B \) is \( SU(2)^2 \times SO(M - 2) \).

(5) The generic small lightlike case is defined by the \( SL(2, \mathbb{R}) \times SO(6, M) \)-invariant constraints \((4.45)\) (or \((4.55)\)). In this case, it is more convenient to consider the symplectic basis of "bare" charges \( P \) and, in order to determine the maximal compact symmetry of the flow solution(s), one can consider the saturation of the bound \((4.45)\), namely:

\[
p^2 q^2 = (p \times q)^2 = 0.
\]

\[
(4.71)
\]

This is in general solved by \( p^2 = 0, \quad p \cdot q = 0 \) and \( q^2 \neq 0 \) (or equivalently by \( q^2 = 0, \quad p \cdot q = 0 \) and \( p^2 \neq 0 \)). It is easy to realize that the maximal compact symmetry of the flow solution is \( SO(4) \times SO(M - 1) \) in the case \( q^2 > 0 \), and \( SO(5) \times SO(M - 2) \) in the case \( q^2 < 0 \). In the first case the solution exists for \( M \geq 1 \), whereas in the second case the solution exists for \( M \geq 2 \). Thus, one actually gets two generic small lightlike orbits, both non-BPS \( Z_{AB} \neq 0 \), with maximal compact symmetry, respectively, given by \( SO(4) \times SO(M - 1) \) and \( SO(5) \times SO(M - 2) \).

**Mutatis mutandis**, the same considerations made at the end of Sect. III for \( \mathcal{N} = 8 \), \( d = 4 \) supergravity also hold for \( \mathcal{N} = 4 \), \( d = 4 \) matter coupled supergravity.

Notice that in pure \( \mathcal{N} = 4 \), \( d = 4 \) supergravity only the small \( \frac{1}{2} \)-BPS orbit \( A.1 \) and the large \( \frac{1}{2} \)-BPS orbit exist. Indeed, the non-BPS \( Z_{AB} \neq 0 \) and non-BPS \( Z_{AB} = 0 \) large orbits and the small orbits \( A.2, A.3 \), and \( B \) cannot be realized, and the small lightlike orbit(s) of point 5 above coincide with small orbit \( A.1 \).

Finally, it is worth noticing that the \( U(1) \) (stabilizer of the factor \( SU(2) / U(1) \) of the scalar manifold \((4.1)\)) is broken both in large and small charge orbits, because both the central charge matrix \( Z_{AB} \) and the matter charges \( Z_j \) are charged with respect to it.

**V. \( \mathcal{N} = 2 \)**

In \( \mathcal{N} = 2 \), \( d = 4 \) supergravity one can repeat the analysis of [1,40] (see also [41]), by using the properties of special Kähler geometry (SKG, see e.g. [22], and Refs. therein). Indeed, in SKG one can define an \( Sp(2n, \mathbb{R}) \) matrix over the scalar manifold (as in Eq. \((2.9)\)), as well complex matrices \( f \) and \( h \) (as in Eqs. \((2.10), (2.11), (2.12), (2.13), \) and \((2.14)\)), without the need for the manifold to be necessarily a \((n \text{ at least locally})\) symmetric space (see e.g. [13,21]).

The basic identities of SKG applied to the (covariantly homomorphic) \( \mathcal{N} = 2 \), \( d = 4 \) central charge section

\[
Z = e^{k/2}(X^\lambda q_\lambda - F_\lambda P^\lambda)
\]

of the \( U(1) \) Kähler-Hodge bundle (with Kähler weights \((1, -1)\)) read as follows [20] \((i, j = 1, \ldots, n - 1, \text{with } n - 1 \text{ denoting the number of Abelian vector multiplets coupled to the supergravity one})

\[
D_i Z = 0;
\]

\[
D_i D_j Z = iC_{ijk} g^{ik} \delta_k Z;
\]

\[
\tilde{D}_j D_i Z = g_{ij} Z;
\]

where \((X^\lambda, F_\lambda)\) are the holomorphic symplectic sections of the \( U(1) \) Kähler-Hodge bundle (with Kähler weights \((2, 0)\)), and \( K \) denotes the Kähler potential of the Abelian vector multiplets’ scalar manifold, with metric \( g_{ij} = \delta_j \delta_i K \). \( C_{ijk} \) is the rank-3 symmetric and covariantly holomorphic \( C\)-tensor of SKG (see e.g. [22], and Refs. therein):

\[
\tilde{D}_j C_{ijk} = 0;
\]
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\[ D_i C_{ijk} = 0. \]

Thus, in \( \mathcal{N} = 2, d = 4 \) supergravity coupled to \( n - 1 \) Abelian vector multiplets, the “BH potential” is given by [18,19]

\[ V_{\text{BH}}(\phi, \mathcal{P}) = Z \mathcal{Z} + g^{ij}(D_i Z)\bar{D}_j \bar{Z}, \]

and the Attractor Eqs. (5.7) and the Non-BPS, \( Z \neq 0 \) class of solutions to (5.8) is determined by

\[ \partial_i V_{\text{BH}} = 0 \iff 2\bar{Z}D_i Z + iC_{ijk}g^{ij}g^{kk}(\bar{D}_j \bar{Z})\bar{D}_k \bar{Z} = 0. \]

(5.8)

(1) The \( (\frac{1}{2} - \text{BPS}) \) supersymmetric solution to Attractor Eqs. (5.8) is determined by

\[ (D_i Z)_{(1/2)-\text{BPS}} = 0, \quad \forall i, \]

and therefore Eq. (5.7) yields

\[ V_{\text{BH}}(1/2)-\text{BPS} = |Z|^2_{(1/2)-\text{BPS}}, \]

and the corresponding Hessian matrix of \( V_{\text{BH}} \) has block components given by [20]

\[ (D_i \partial_j V_{\text{BH}})(1/2)-\text{BPS} = (\partial_i \partial_j V_{\text{BH}})(1/2)-\text{BPS} = 0; \]

(5.11)

\[ (\partial_i \partial_j V_{\text{BH}})(1/2)-\text{BPS} = 2g_{ij}Z^2_{(1/2)-\text{BPS}}. \]

(5.12)

showing that there are no “flat” directions for such the \( (\frac{1}{2} - \text{BPS} \) class of solutions to Attractor Eqs. (5.8) [33].

(2) Nonsupersymmetric (non-BPS) solutions to Attractor Eqs. (5.8) have \( D_i Z \neq 0 \) (at least) for some \( i \in \{1, \ldots, n - 1\} \). Generally, such solutions fall into two class [6], and they exhibit “flat” directions of \( V_{\text{BH}} \) itself [33]. The non-BPS, \( Z = 0 \) class is defined by the following constraints:

\[ D_i Z = \partial_i Z = 0, \quad \text{for some } i, Z = 0. \]

(5.13)

thus yielding (from Eqs. (5.8))

\[ [C_{ijk}g^{ij}(\partial_j Z)\bar{D}_k]_{\text{non-BPS}, Z = 0} = 0. \]

(14)

Thus, Eqs. (5.7) and (5.13) yield

\[ V_{\text{BH}}(\text{non-BPS}, Z = 0) = [g^{ij}(D_i Z)\bar{D}_j \bar{Z}]_{\text{non-BPS}, Z = 0} = [g^{ij}(\partial_j Z)\bar{D}_j \bar{Z}]_{\text{non-BPS}, Z = 0}. \]

(5.15)

(3) The non-BPS, \( Z \neq 0 \) class is defined by the following constraints:

\[ D_i Z \neq 0, \quad \text{for some } i, Z \neq 0. \]

(5.16)

It is worth remarking that Eqs. (5.8) and the non-BPS, \( Z \neq 0 \) defining constraints (5.16) imply the following relations to hold at the non-BPS \( Z \neq 0 \) critical points of \( V_{\text{BH}} \) [13]:

\[ [g^{ij}(D_i Z)\bar{D}_j \bar{Z}]_{\text{non-BPS}, Z \neq 0} = \frac{i}{2} \left[ \frac{N_3(\bar{Z})}{\bar{Z}} \right]_{\text{non-BPS}, Z \neq 0} \]

(5.17)

where the cubic form \( N_3(\bar{Z}) \) is defined as [13]

\[ N_3(\bar{Z}) = \bar{C}_{ijk}Z^i \bar{Z}^j \bar{Z}^k \iff \bar{N}_3(Z) = \bar{C}_{ijk}Z^i \bar{Z}^j \bar{Z}^k. \]

(5.18)

For an arbitrary SKG, it is in general hard to compute

\[ \frac{S_{\text{BH}}}{\pi} = V_{\text{BH}}|_{\partial_a V_{\text{BH}} = 0} = V_{\text{BH}}(\phi_H(\mathcal{P}), \mathcal{P}), \]

(5.19)

where \( \phi_H(\mathcal{P}) \) are the horizon scalar configurations solving the Attractor Eqs. (5.8). However, the situation dramatically simplifies for symmetric SK manifolds

\[ \frac{G_4}{H_4}, \]

(5.20)

in which case a classification, analogous to the one available for \( \mathcal{N} > 2 \)-extended, \( d = 4 \) supergravities (see e.g. [13] and Refs. therein; see also Secs. III and IV) can be performed [6].

In the treatment below, we are going to give a remarkable general topological formula for \( V_{\text{BH}}(\phi_H(\mathcal{P}), \mathcal{P}) \) for symmetric SKG, which is manifestly invariant under diffeomorphisms of the SK scalar manifold, and which holds for any choice of symplectic basis of “bare” charges \( \mathcal{P} \) and of special coordinates (see e.g. [22] and Refs. therein) of the SK manifold itself. Indeed, such a formula by no means does refer to special coordinates, which may not even exist for certain parametrizations of \( G_4 \)

It should be pointed out that a general formula for the \( G_4 \)-invariant \( I_{4,\mathcal{N}=2} \) is known for the so-called \( d \)-SK homogeneous symmetric manifolds [26], and it reads (\( a = 1, \ldots, n - 1 \)) [4]:

\[ I_{4,\mathcal{N}=2}(\mathcal{P}) = -(p^0 q_0 + p^aq_a)^2 + 4[q_0 I_{3,\mathcal{N}=2}(p) - p^0 I_{3,\mathcal{N}=2}(q) + \{I_{3,\mathcal{N}=2}(p), I_{3,\mathcal{N}=2}(q)\}], \]

(5.21)

where

\[ I_{3,\mathcal{N}=2}(p) = \frac{1}{3!}d_{abc}p^ap^bp^c; \]

(5.22)
in which the constant (number) rank-3 symmetric tensor \( d_{abc} \) has been introduced (and \( d_{abc} \) is its suitably defined completely contravariant form). However, such a formula holds for a particular symplectic basis (namely the one inherited from the \( \mathcal{N} = 2, d = 5 \) theory, i.e. the one of special coordinates), in which the holomorphic prepotential \( F(X) \) of SKG can be written as

\[
F(X) = \frac{1}{3!} d_{abc} \frac{X^a X^b X^c}{X^0}.
\]

(5.25)

In such a symplectic basis, the manifest symmetry is the \( d = 5 \) U-duality \( G_4 \), under which \( G_4 \) branches as \( G_4 \rightarrow G_5 \times SO(1,1) \). Indeed, \( I_{3,\mathcal{N}=2}(p) \) and \( I_{3,\mathcal{N}=2}(q) \) are nothing but, respectively, the magnetic and electric invariants (both cubic in \( \mathcal{P} \)) of the relevant symplectic representations of \( G_5 \).

Equation (5.21) excludes the so-called quadratic (or minimally coupled [42]) sequence of symmetric SK manifolds (particular complex Grassmannians)

\[
SU(1, n - 1) \quad SU(n - 1) \times U(1), \quad n \in \mathbb{N}
\]

(5.26)

(not upliftable to \( d = 5 \)), for which \( F(X) \) is given by (in the symplectic basis exhibiting the maximal noncompact symmetry \( SU(1, n - 1) \))

\[
F(X) = -i \left[ (X^0)^2 - \sum_{i=1}^{n-1} (X^i)^2 \right].
\]

(5.27)

and the invariant of the symplectic representation of \( G_4 = SU(1, n - 1) \) reads as follows (notice it is quadratic in \( \mathcal{P} \) [29]:

\[
I_{2,\mathcal{N}=2}(\mathcal{P}) = (p^0)^2 + q_0^2 - \sum_{i=1}^{n-1} (p^i)^2 + q_i^2
\]

\[= |Z|^2 - g^{ij}(D_i Z)\bar{D}_j\bar{Z}.
\]

(5.28)

Because of the quadratic nature of the \( G_4 \)-invariant \( I_{2,\mathcal{N}=2}(\mathcal{P}) \) given by Eq. (5.28), the quadratic sequence of symmetrical SK manifolds (5.26) exhibits only one small charge orbit, namely, the lightlike one, beside the two large charge orbits determined in [6].

The symmetric SK manifolds whose geometry is determined by the holomorphic prepotential function (5.25) and the minimally coupled ones determined by Eq. (5.27) are all the possible symmetric SK manifolds. After [43], from the geometric perspective of SKG, symmetric SK manifolds can be characterized in the following way.

In SKG the Riemann tensor obeys to the following constraint (see e.g. [22] and Refs. therein):

\[
R_{ijkl} = -g_{ij}g_{kl} - g_{il}g_{kj} + C_{ikm}\tilde{C}_{ljmn}g^{mn}.
\]

(5.29)

The requirement that the manifold to be symmetric demands the Riemann to be covariantly constant:

\[
D_m R_{ijkl} = 0.
\]

(5.30)

Because of the SKG constraint (5.29) and to covariant holomorphicity of the C-tensor (expressed by Eq. (5.5)), Eq. (5.30) generally implies (for nonvanishing \( C_{ijk} \))

\[
D_j C_{ijk} = D_i C_{ijk} = 0,
\]

(5.31)

where in the last step Eq. (5.6) was used. Thus, in a SK symmetric space both the Riemann tensor and the C-tensor are covariantly constant. Equation (5.31) implies the following relation [6]

\[
C_{j[lm}C_{p]qk} \tilde{C}_{ij]kk}g^{jk} = \frac{4}{3} C_{(imp}g_{q)}i,
\]

(5.32)

which is nothing but the “dressed” form of the analogous relation holding for the \( d \)-tensor itself [43,44]

\[
d_{j[lm}d_{p]qk}d^{jk} = \frac{4}{3} d_{(imp}d_{q)}i.
\]

(5.33)

The quadratic sequence of symmetric manifolds (5.26) whose SKG is determined by the prepotential (5.27) has

\[
C_{ijk} = 0,
\]

(5.34)

whereas the remaining symmetric SK manifolds, whose prepotential in the special coordinates is given by Eq. (5.25) (with \( d_{abc} \) constrained by Eq. (5.33)), correspond to

\[
C_{abc} = e^k d_{abc}.
\]

(5.35)

By using Eqs. (5.31) and (5.32), as well as the SKG identities (5.2), (5.3), and (5.4) (which, for symmetric SKG, are equivalent to the Maurer-Cartan Eqs., as Eqs. (3.3), (4.4), and (4.5) for \( \mathcal{N} = 8 \) and \( \mathcal{N} = 4 \), \( d = 4 \) supergravities, respectively; see e.g. [21,29]), one can prove that the following quartic expression is a duality invariant for all symmetric SK manifolds:

\[
I_{4,\mathcal{N}=2,\text{symm}}(\phi, \mathcal{P}) = (Z\bar{Z} - Z_i\bar{Z}_i)^2
\]

\[+ \frac{2}{3} i (Z\mathcal{N}_3 \bar{Z} - \bar{Z}\mathcal{N}_3 (Z))
\]

\[- g^{ij} C_{ijk} \tilde{C}_{i[m}Z_l^m Z^i Z^j,
\]

(5.36)

where the matter charges have been renoted as \( Z_i = D_i Z, \bar{Z}_i = g^{ij} Z_j \), and definition (5.18) was recalled.

As claimed above, \( I_{4,\mathcal{N}=2,\text{symm}} \) given by Eq. (5.36) is \( \phi \)-dependent only apparently, i.e. it is topological, merely charge-dependent.
Thus, by recalling Eq. (1.5), the general entropy-area formula [8] for extremal BHs in $\mathcal{N} = 2, d = 4$ supergravity coupled to Abelian vector multiplets whose scalar manifold is a symmetric (SK) space reads as follows:

$$
\frac{S_{BH}}{\pi} = V_{BH}(\phi_H(P), P) = |I_{4,\mathcal{N}=2,\text{symm}}(P)|^{1/2}. 
$$  (5.38)

Notice that through Eq. (5.40) (cos) $\theta$ is determined in terms of the scalar fields $\phi$ and of the BH charges $P$, also along the small orbits where $I_{4,\mathcal{N}=2,\text{symm}} = 0$. However, Eq. (5.40) is not defined in the cases in which $Z_{3n}(\vec{Z}) = 0$. In such cases, $\theta$ is actually undetermined. It should be clearly pointed out that the phase $\theta$ has nothing to do with the phase of the $U(1)$ bundle over the SK-Hodge vector multiplets’ scalar manifold (see e.g. [22] and Refs. therein).

1. For $\frac{1}{2}$ -- BPS attractors (defined by the constraints (5.9)), Eq. (5.36) yields

$$
I_{4,\mathcal{N}=2,\text{symm}}|_{\text{BPS}} = (Z_{\vec{Z}})^2|_{(1/2)\text{--BPS}} = |Z|^{4}|_{(1/2)\text{--BPS}} 
$$  (5.41)

as in turn also implied by Eqs. (1.5) and (5.10) (or equivalently (5.38)). Notice that Eqs. (5.10) and (5.41) are general, i.e. they hold for any SKG, regardless the symmetric nature of the SK vector multiplets’ scalar manifold. Furthermore, the constraints (5.9) imply that at the event horizon of $\frac{1}{2}$ -- BPS extremal BHs it holds

$$
[N_{3}(\vec{Z})]^{|_{(1/2)\text{--BPS}}} = 0 \Rightarrow \theta|_{(1/2)\text{--BPS}} \text{ undetermined}. 
$$  (5.42)

2. For non-BPS $Z = 0$ attractors (defined by the constraints (5.13) which, through Eqs. (5.8), imply Eq. (5.14)), Eq. (5.36) yields

$$
I_{4,\mathcal{N}=2,\text{symm}}|_{\text{non--BPS,Z=0}} = (Z_{\vec{Z}})^2|_{\text{non--BPS,Z=0}} = [g^{\hat{z}\hat{z}}(\partial_{\hat{z}}Z\partial_{\hat{z}}Z)]^{|_{\text{non--BPS,Z=0}}}. 
$$  (5.43)

Notice that Eqs. (5.15) and (5.43) are general, i.e. they hold for any SKG, regardless the symmetric nature of the SK vector multiplets’ scalar manifold. Furthermore, the constraints (5.9) imply that at the event horizon of non-BPS $Z = 0$ extremal BHs it holds

$$
Z_{\text{non--BPS,Z=0}} = 0 \Rightarrow \theta|_{\text{non--BPS,Z=0}} \text{ undetermined}. 
$$  (5.44)

3. For non-BPS $Z \neq 0$ attractors (defined by the constraints (5.16) as well as by Eqs. (5.8)), Eqs. (5.17) and (5.36) yield

$$
I_{4,\mathcal{N}=2,\text{symm}}|_{\text{non--BPS,Z}\neq0} = -16|Z|^4|_{\text{non--BPS,Z}\neq0}. 
$$  (5.45)

Thus implying, through Eq. (5.7) [6,13,30,45]

$$
Z_{\vec{Z}}|_{\text{non--BPS,Z}\neq0} = 3|Z|^2|_{\text{non--BPS,Z}\neq0} \equiv V_{\text{BH,non--BPS,Z}\neq0} = 4|Z|^2|_{\text{non--BPS,Z}\neq0}. 
$$  (5.46)

By plugging Eqs. (5.8), (5.16), (5.17), and (5.45) into Eq. (5.40), it follows that at the event horizon of non-BPS $Z \neq 0$ extremal BHs it holds that

$$
\theta|_{\text{non--BPS,Z}\neq0} = \pi + 2k\pi, \quad k \in \mathbb{Z}. 
$$  (5.47)

It should be remarked that, differently from the results (5.10), (5.11), (5.12), (5.41), and (5.42) (holding for $\frac{1}{2}$ -- BPS attractors) and from the results (5.14), (5.15), (5.43), and (5.44) (holding for non-BPS $Z = 0$ attractors), Eqs. (5.45), (5.46), and (5.47) are not general; i.e. they hold at the event horizon of extremal non-BPS $Z \neq 0$ BHs for symmetric SK manifolds, but they do not hold true for generic SKG. However, when going beyond the symmetric SK case (and thus encompassing both homogeneous nonsymmetric [26,46] and nonhomogeneous SK
spaces), one can compute both \( V_{\text{BH,non-BPS}, Z \neq 0} \) and \( I_{4, N-2, \text{symm}|_{\text{non-BPS}, Z \neq 0}} \) and express the deviation from the symmetric case considered above in terms of the complex quantity [13]

\[
\Delta = -\frac{3}{4} E_{ij \hat{k} \hat{l} \hat{m}} Z^i Z^j Z^k \bar{Z}^l \bar{Z}^m \frac{N_3(Z)}{N_3(Z)},
\]

(5.48)

where the tensor \( E_{ij \hat{k} \hat{l} \hat{m}} \) was firstly introduced in [26] (see also [13]). The results of straightforward computations read as follows:

\[
V_{\text{BH,non-BPS}, Z \neq 0} = 4|Z|^4_{\text{non-BPS}, Z \neq 0} + \Delta_{\text{non-BPS}, Z \neq 0},
\]

(5.49)

\[
I_{4, N-2, \text{symm}|_{\text{non-BPS}, Z \neq 0}} = \left[ -16|Z|^4 + \Delta^2 \right. \frac{-8}{3} \Delta |Z|^2 \right]_{\text{non-BPS}, Z \neq 0}.
\]

(5.50)

Notice that, as yielded e.g. by Eq. (5.49), \( \Delta \) is real at the non-BPS \( Z \neq 0 \) critical points of \( V_{\text{BH}} \). For symmetric SK manifolds \( E_{ij \hat{k} \hat{l} \hat{m}} = 0 \) globally, and thus Eqs. (5.49) and (5.50) respectively reduce to Eqs. (5.45) and (5.46). On the other hand, the results (5.45) and (5.46) hold also for those nonsymmetry SK spaces \( (E_{ij \hat{k} \hat{l} \hat{m}} \neq 0) \) such that

\[
\Delta_{\text{non-BPS}, Z \neq 0} = 0 \quad \Rightarrow \quad (E_{ij \hat{k} \hat{l} \hat{m}} Z^i Z^j Z^k \bar{Z}^l \bar{Z}^m)_{\text{non-BPS}, Z \neq 0},
\]

(5.51)

where in the implication “\( \Rightarrow \)” the assumption \( [N_3(Z)]_{\text{non-BPS}, Z \neq 0} \neq 0 \) was made. The condition (5.51) might explain some results obtained for generic \( (d)-\text{SKGs} \) in some particular supporting BH charge configurations in [45] (see also the treatment in [13,39]).

Consistently, for the quadratic minimally coupled sequence (5.26), for which Eq. (5.34) holds, Eq. (5.36) formally reduces to

\[
\begin{align*}
I_{4, N-2, \text{symm}}|_{c_{ijk}=0} & = (Z \bar{Z} - Z_i \bar{Z}^i)^2; \\
|I_{4, N-2, \text{symm}}|_{c_{ijk}=0}^2 & = |I_{2, N-2}|.
\end{align*}
\]

(5.52)

where \( I_{2, N-2} \) is given by Eq. (5.28).

Remarkably, Eq. (5.36) turns out to be directly related to the quantity \(-h \) given by Eq. (2.31) of [26] (see also the treatment of [47]). This is seen by noticing that Eq. (4.42) of [26] coincides with Eq. (5.21) (along with definitions (5.22), (5.23), and (5.24)). Note that the mapping of quaternionic coordinates \((A^k, B_A)^T \) into the charges \( p^k = (p^k, q_A)^T \) (in special coordinates) is related to the \( d = 3 \) attractor flows (see e.g. [48–50]).

For symmetric SK manifolds, small charge orbits of the symplectic representation of \( G_A \) are known to exist since [4,5].

(i) Small lightlike charge orbits are defined by the \( G_A \)-invariant constraint

\[
I_{4, N-2, \text{symm}} = 0;
\]

(5.53)

\[
\partial Z = 0.
\]

(5.54)

In this case, Eq. (5.40) reduces to

\[
\cos \theta(\phi, P)|_{I_{4, N-2, \text{symm}} = 0} = \frac{-3[(Z \bar{Z} - Z_i \bar{Z}^i)^2 - g^{ij} C_{ijk} \bar{C}_{i \hat{k} \hat{l} \hat{m}} Z^j \bar{Z}^k \bar{Z}^l \bar{Z}^m]}{2^2 |Z N_3(Z)|} I_{4, N-2, \text{symm}} = 0.
\]

(5.55)

where the second-order differential operators \( D_{ij} \) and \( D_i \) have been introduced:

\[
D_{ij} \equiv R_{ijk} \frac{\partial}{\partial Z_k} \frac{\partial}{\partial Z^i}; \quad D_i \equiv C_{ijk} \frac{\partial}{\partial Z_j} \frac{\partial}{\partial Z^k}.
\]

(5.56)

(5.57)

Notice that, through the definitions (5.58) and (5.59), the constraints (5.57) are \( G_A \)-invariant, because they are equivalent to the following constraint:

\[
\frac{\partial^2 I_{4, N-2, \text{symm}}}{\partial Z_{\text{sympl} (G_A)}} \frac{\partial Z_{\text{sympl} (G_A)}}{\partial Z_{\text{sympl} (G_A)}} \bigg|_{\text{Adj} (G_A)} = 0.
\]

(5.59)
where
\[ Z_{\text{sympl}(G_4)} \equiv (Z, \tilde{Z}, \tilde{Z}^T), \] (5.61)
and the change of charge basis between the manifestly \( H_4 \)-covariant (in "flat" local coordinates) basis \( Z_{\text{sympl}(G_4)} \) and the manifestly \( SP(2n, \mathbb{R}) \)-covariant basis \( \mathcal{P} \) (defined by Eq. (1.2)) is expressed by the fundamental identities of the SKG (see e.g. [22,51] and Refs. therein). Indeed, by considering the Cartan decomposition of the Lie algebra of \( G_4 \):
\[ \mathcal{G}_4 = \tilde{\mathcal{G}}_4 + f_4, \] (5.62)
and switching to "flat" local coordinates in the scalar manifold (here denoted by capital Latin indices), it holds that \( \mathcal{D}_4 \) ("flat" version of the operator defined in Eq. (5.59)) is \( f_4 \)-valued. Furthermore, in symmetric manifolds \( R_{JK}^L \) is a twoform (in the first two "flat" local indices) which is Lie algebra-valued in \( \tilde{\mathcal{G}}_4 \) and thus \( \mathcal{D}_4 \) ("flat" version of the operator defined in Eq. (5.58)) turns out to be \( \tilde{\mathcal{G}}_4 \)-valued. Notice that Eq. (5.60), \( G_4 \)-invariantly defining the small doubly-critical charge orbit(s) of the \( \mathcal{N} = 2, d = 4 \) vector multiplets' symmetric SK scalar manifolds, is the analogue of Eq. (3.42), which defines in an \( E_7(7) \)-invariant way the small doubly-critical charge orbit of \( \mathcal{N} = 8, d = 4 \) pure supergravity. It should be also recalled that in \( \mathcal{N} = 4, d = 4 \) matter coupled supergravity smalldoublycritical (or higher-order-critical) charge orbits (independent from the small critical ones) are absent. As treated in Sec. IV, all small critical charge orbits of the \( \mathcal{N} = 4 \) theory actually are doubly-critical, and the analogues of Eqs. (3.42) and (5.60) are given, through Eq. (4.50) and definitions (4.51) and (4.53), by the rich case study exhibited by Eqs. [(4.48), (4.49), (4.56), and (4.57)].

The classification of small charge orbits of the relevant symplectic representation of \( G_4 \) for \( \mathcal{N} = 2, d = 4 \) supergravity coupled to Abelian vector multiplets whose scalar manifold \( \tilde{M}_4 \) is (SK) symmetric, performed in accordance to their order of criticality (lightlike, critical, doubly-critical), will be given elsewhere.

VI. ADM MASS FOR BPS EXTREME BLACK HOLE STATES

For BPS BH states in \( d = 4 \) ungauged\(^8\) supergravity theories, the ADM mass \([27]\) \( M_{\text{ADM}}(\phi_\infty, \mathcal{P}) \) is defined as the largest (of the absolute values) of the skew-eigenvalues of the (spatially asymptotically) central charge matrix \( Z_{AB}(\phi_\infty, \mathcal{P}) \) which saturate the BPS bound (2.28). The skew-diagonalization of \( Z_{AB} \) is made by performing a suitable transformation of the \( R \)-symmetry, and thus by going to the so-called normal frame. In such a frame, the skew-eigenvalues of \( Z_{AB} \) can be taken to be real and positive (up to an eventual overall phase). By saturating the BPS bound (2.28), it therefore holds that
\[ M_{\text{ADM}}(\phi_\infty, \mathcal{P}) = |Z_1(\phi_\infty, \mathcal{P})| \geq \ldots \geq |Z_{[\mathcal{N}/2]}(\phi_\infty, \mathcal{P})|, \] (6.1)
where \( Z_1(\phi, \mathcal{P}), \ldots, Z_{[\mathcal{N}/2]}(\phi, \mathcal{P}) \) denote the set of skew-eigenvalues of \( Z_{AB}(\phi, \mathcal{P}) \), and square brackets denote the integer part of the enclosed number. As mentioned at the end of Sec. II, if \( 1 \leq k \leq \lceil \mathcal{N}/2 \rceil \) of the bounds expressed by Eq. (2.28) are saturated, the corresponding extremal BH state is named to be \( \frac{k}{\mathcal{N}} \)-BPS. Thus, the minimal fraction of total supersymmetries (pertaining to the asymptotically flat space-time metric) preserved by the extremal BH background within the considered assumptions is \( \frac{1}{2 \mathcal{N}} \) (for \( k = 1 \)), while the maximal one is \( \frac{k}{\mathcal{N}} \) (for \( k = \frac{\mathcal{N}}{2} \)).

The ADM mass and its symmetries are different, depending on \( k \).

A. \( \mathcal{N} = 8 \)

In \( \mathcal{N} = 8, d = 4 \) supergravity (treated in Sec. III), the \( E_7(7) \)-U-duality symmetry only allows the cases \([3]\) \( k = 1, 2, 4 \). By recalling the review given in Sec. III, the maximal compact symmetries of the supporting charge orbits, respectively, read \([3,4,13,30,32,33]\)
\[ k = 1: SU(2) \times SU(6); \] (6.2)
\[ k = 2: USp(4) \times SU(4); \] (6.3)
\[ k = 4: USp(8), \] (6.4)
and they hold all along the respective scalar flows. While cases \( k = 2 \) and \( 4 \) are small (thus not enjoying the attractor mechanism), case \( k = 1 \) can be either large or small.

In the large \( k = 1 \) case, the attractor mechanism makes the maximal compact symmetry \( SU(2) \times SU(6) \) of the supporting charge orbit \( \tilde{O}_{(1/8)} \)-BPS fully manifest as a symmetry of the central charge matrix \( Z_{AB} \) through the symmetry enhancement (3.17) at the event horizon of the considered extremal BH.

Furthermore, the \( \frac{1}{4} \)-BPS saturation of the \( \mathcal{N} = 8 \) BPS bound (all along the \( \frac{1}{4} \)-BPS scalar flow) has the following peculiar structure [recall Eq. (3.35)] \([3]\)
\[ |Z_1(\phi, \mathcal{P})| = |Z_2(\phi, \mathcal{P})| > |Z_3(\phi, \mathcal{P})| = |Z_4(\phi, \mathcal{P})|, \] (6.5)
where it should be recalled that in Sec. III the notation \( e_i \equiv |Z_i| (i = 1, \ldots, 4) \) was used.
As done in Sec. III, let us denote with $\lambda_i$ ($i = 1, \ldots, 4$) the four real non-negative eigenvalues of the $8 \times 8$ Hermitian matrix $Z_{AB}Z_{CB} = (ZZ^T)_{AB}$. Their relation with the absolute values of the complex skew-eigenvalues $e_i$ of $Z_{AB}$ is given by Eq. (3.29). As mentioned, the ordering $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4$ does not imply any loss of generality. After [9] (see, in particular, Eqs. (4.74), (4.75), (4.86), and (4.87) therein), the explicit expression of $\lambda_i$ in terms of $U(8)$-invariants (namely of $\text{Tr}A$, $\text{Tr}(A^2)$, $\text{Tr}(A^3)$, and $\text{Tr}(A^4)$, and suitable powers) is known, and it can be thus be used in order to compute the ADM mass of the extremal BH states of $N = 8$, $d = 4$ supergravity.

The $\lambda_i$’s are solution of the (square root of) characteristic equation [9]

$$\sqrt{\det(A - \lambda I)} = \prod_{i=1}^{4}(\lambda - \lambda_i) = \lambda^4 + a\lambda^3 + b\lambda^2 + c\lambda + d = 0,$$

where [9]

$$a = -\frac{1}{2}\text{Tr}A = -(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4);$$

$$b = \frac{1}{4} \left[ \frac{1}{2} (\text{Tr}A)^2 - \text{Tr}(A^2) \right] = \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_1\lambda_4 + \lambda_2\lambda_3 + \lambda_2\lambda_4 + \lambda_3\lambda_4;$$

$$c = -\frac{1}{6} \left[ \frac{1}{8} (\text{Tr}A)^3 + \text{Tr}(A^3) - \frac{3}{4} \text{Tr}(A^2) \text{Tr}A \right] = -(\lambda_1\lambda_2\lambda_3 + \lambda_1\lambda_2\lambda_4 + \lambda_1\lambda_3\lambda_4 + \lambda_2\lambda_3\lambda_4);$$

$$d = \frac{1}{4} \left[ \frac{1}{96} (\text{Tr}A)^4 + \frac{1}{8} \text{Tr}^2(A^2) + \frac{1}{4} \text{Tr}(A^3) \text{Tr}A + -\frac{1}{2} \text{Tr}(A^4) - \frac{1}{8} \text{Tr}(A^2) \text{Tr}^2A \right] = \sqrt{\det A} = \lambda_1\lambda_2\lambda_3\lambda_4.$$  

The system (6.7), (6.8), (6.9), and (6.10) can be inverted, yielding

$$\lambda_{1,2} = -\frac{a + s}{4} \pm \frac{1}{2} \sqrt{\frac{a^2 - 4b}{2} - \frac{2}{3} (\frac{a^2 - 2ab + 4c}{4s}) - \frac{u}{3w} - \frac{w}{3}};$$

$$\lambda_{3,4} = -\frac{a - s}{4} \pm \frac{1}{2} \sqrt{\frac{a^2 - 4b}{2} + \frac{2}{3} (\frac{a^2 - 2ab + 4c}{4s}) - \frac{u}{3w} - \frac{w}{3}}.$$  

where

$$u \equiv b^2 + 12d - 3ac;$$

$$v \equiv 2b^3 + 27c^2 - 72bd - 9abc + 27a^2d;$$

$$w \equiv \frac{(v + \sqrt{v^2 - 4u^3})^1/3}{2};$$

$$s \equiv \frac{a^2 - 2b + u}{3w} + \frac{w}{3}.$$  

Notice that the positivity of quantities under square root in Eqs. (6.11), (6.12), (6.15), and (6.16) always holds. Furthermore, Eq. (6.6) is at most of fourth order (for $k = 1$), of second-order for $k = 2$, and of first order for $k = 1$.

(1) $k = 1$ ($\frac{1}{8}$ - BPS, either large or small). The $\frac{1}{8}$ - BPS extremal BH square ADM mass is

$$M_{\text{ADM}(1/8)-\text{BPS}}^2(\phi, P) = \lambda_1(\phi, P).$$

As mentioned above, the maximal (compact) symmetry is manifest when $\lambda_2$ (in the renaming of Eq. (6.18)) vanishes (see treatment in Sec. III). Equation (3.35) implies [9]

$$c = \frac{1}{2} d \left( 1 - \frac{1}{4} a^2 \right);$$

$$d = \frac{1}{4} \left( 1 - \frac{1}{4} a^2 \right)^2.$$  

In [9] Eqs. (6.19) and (6.20) were shown to be consequences of the criticality constraints (3.34). Thus, the $\frac{1}{4}$-BPS extremal BH square ADM mass is

$$M_{\text{ADM}(1/4)-\text{BPS}}^2(\phi, P) = \lambda_1(\phi, P),$$

where $\lambda_1$ ($\lambda_2$) is given by Eq. (6.18):
putting $\lambda_1 = \lambda_2$ in Eq. (6.18). Thus, all eigenvalues of the Hermitian $8 \times 8$ matrix $A$ are equal:

$$A_A^C = \frac{1}{8} (\text{Tr} A) \delta_A^C. \quad (6.23)$$

which implies

$$\text{Tr} (A^2) = \frac{1}{8} (\text{Tr} A)^2. \quad (6.24)$$

Therefore, $\frac{1}{2} - \text{BPS}$ extremal BH square ADM mass is given by

$$M_{\text{ADM},(1/2)-\text{BPS}}^2(\phi_\infty, \mathcal{P}) = \frac{1}{8} \text{Tr} A(\phi_\infty, \mathcal{P})$$

$$= \frac{1}{16} Z_{AB}(\phi_\infty, \mathcal{P}) Z^{AB}(\phi_\infty, \mathcal{P}). \quad (6.25)$$

**B. $\mathcal{N} = 4$**

In $\mathcal{N} = 4$, $d = 4$ supergravity (treated in Sec. IV), the $SL(2, \mathbb{R}) \times SO(6, M)$ U-duality symmetry only allows the cases $[3] k = 1, 2$. By recalling the treatment of Sec. IV, the respective maximal compact symmetries read $[3, 4, 13, 39]$

$$k = 1: (SU(2))^2 \times SO(M) \times SO(2); \quad (6.26)$$

$$k = 2: USp(4) \times SO(M), \quad (6.27)$$

and they hold all along the respective scalar flows. While case $k = 1$ is large, case $k = 2$ is small (thus not enjoying the attractor mechanism).

In the large $k = 1$ case, the attractor mechanism makes the maximal compact symmetry $(SU(2))^2 \times SO(M) \times SO(2)$ of the supporting charge orbit $O_{(1/4)-\text{BPS,large}}$ fully manifest as a symmetry of the central charge matrix $Z_{AB}$ through the symmetry enhancement (recall Eq. (4.25))

$$(SU(2))^2 \times SO(M - 2) \times SO(2) \xrightarrow{\text{eigenvalues}} (SU(2))^2 \times SO(2) \times SO(2) \quad (6.28)$$

at the event horizon of the considered extremal BH.

As done in Sec. IV and in the treatment of case $\mathcal{N} = 8$, $d = 4$ above, let us denote with $\lambda_1$ and $\lambda_2$ the two real non-negative eigenvalues of the $4 \times 4$ Hermitian matrix $Z_{AB} Z^{CB} = (Z Z^\dagger)^C_A = A_A^C$. Their relation with the absolute values of the complex skew-eigenvalues $\epsilon_j$ of $Z_{AB}$ is given by Eq. (3.29). As mentioned, the ordering $\lambda_1 \geq \lambda_2$ does not imply any loss of generality. After [9], the explicit expression of $\lambda_1$ and $\lambda_2$ in terms of $(U(4) \times SO(M))$-invariants (namely of $\text{Tr} A$, $\text{Tr} (A^2)$ and $(\text{Tr} A)^2$) is known, and it can be thus be used in order to compute the ADM mass of the $\frac{1}{2} - \text{BPS}$ extremal BH states of $\mathcal{N} = 4$, $d = 4$ supergravity.

Indeed, $\lambda_1$ and $\lambda_2$ are solutions of the (square root of) characteristic equation [9]

$$\sqrt{\text{det}(A - \lambda I)} = \prod_{j=1}^{2} (\lambda - \lambda_j)$$

$$= \lambda^2 - \frac{1}{2} (\text{Tr} A) \lambda + (\text{det} A)^{1/2} = 0, \quad (6.29)$$

whose solution reads

$$\lambda_{1,2} = \frac{1}{2} \left[ \frac{1}{2} \left( \text{Tr} A \pm \sqrt{\text{Tr}(A^2) - \frac{1}{4} (\text{Tr} A)^2} \right) \right]. \quad (6.30)$$

Notice that the positivity of quantities under square root in Eq. (6.30) always holds. Furthermore, Eq. (6.29) is at most of second-order (for $k = 1$) and of first order for $k = 2$.

1. $k = 1$ ($\frac{1}{2} - \text{BPS}$ large). The $\frac{1}{2}$-BPS extremal BH square ADM mass is

$$M_{\text{ADM},(1/2)-\text{BPS}}^2(\phi_\infty, \mathcal{P})$$

$$= \lambda_1(\phi_\infty, \mathcal{P})$$

$$= \frac{1}{2} \left[ \frac{1}{2} \left( \text{Tr} A \pm \sqrt{\text{Tr}(A^2) - \frac{1}{4} (\text{Tr} A)^2} \right) \right], \quad (6.31)$$

where $\lambda_1 > \lambda_2$. Notice that $\lambda_2 = 0$ at the event horizon of the extremal BH, as given by Eq. (4.23).

2. $k = 2$ ($\frac{1}{2} - \text{BPS, small}$). This case can be obtained from the $\frac{1}{4} - \text{BPS}$ considered at point 1 by further putting $\lambda_1 = \lambda_2$ in Eq. (6.30). Thus, all eigenvalues of the Hermitian $4 \times 4$ matrix $A$ are equal:

$$A_A^C = \frac{1}{4} (\text{Tr} A) \delta_A^C. \quad (6.32)$$

which implies

$$\text{Tr} (A^2) = \frac{1}{4} (\text{Tr} A)^2. \quad (6.33)$$

Thus, the $\frac{1}{2} - \text{BPS}$ extremal BH square ADM mass is

$$M_{\text{ADM},(1/2)-\text{BPS}}^2(\phi_\infty, \mathcal{P}) = \lambda_1(\phi_\infty, \mathcal{P})$$

$$= \lambda_2(\phi_\infty, \mathcal{P})$$

$$= \frac{1}{4} \text{Tr} A(\phi_\infty, \mathcal{P}). \quad (6.34)$$

It should be here remarked that the $R$-symmetry of the $\frac{k}{\mathcal{N}} - \text{BPS}$ extremal BH states, i.e. the compact symmetry of the solution in the normal frame (determining the automorphism group of the supersymmetry algebra in the rest frame) gets broken as follows:
This is precisely the symmetry of the $\mathbb{K}$ – BPS saturated massive multiplets of the $\mathcal{N}$-extended, $d = 4$ Poincaré supersymmetry algebra [54].

We end this section by finally commenting about the ADM mass for non-BPS extremal BH states.

In non-BPS cases, ADM mass of extremal BH states is not directly related to the skew-eigenvalues of the central charge matrix $Z_{AB}$. For some non-BPS extremal BHs a fake supergravity (first order) formalism [55] can be consistently formulated in terms of a fake superpotential $W(\phi, P)$ [56–58] such that (also recall Eq. (1.5))

$$W^{2}_{\text{non-BPS}}(\phi, P)(\phi V_{BPS}) = 0$$

$$= W^{2}_{\text{non-BPS}}(\phi_{H,\text{non-BPS}}(P), P)$$

$$= V_{BH}(\phi_{H,\text{non-BPS}}(P), P) = \frac{S_{BH,\text{non-BPS}}(P)}{\pi},$$

with $W_{\text{non-BPS}}$ varying, dependently on whether $Z_{AB} = 0$ or not. In such frameworks, the general expression of the non-BPS ADM mass reads as follows [56–58]

$$M_{\text{ADM, non-BPS}}(\phi_{\infty}, P) = W_{\text{non-BPS}}(\phi_{\infty}, P).$$

Acknowledgments

This work is supported in part by the ERC Advanced Grant no. 226455, “Supersymmetry, Quantum Gravity and Gauge Fields” (SUPERFIELDS). We would like to thank M. Trigiane for enlightening discussions. A. M. would like to thank the CTP of the University of California, Berkeley, CA USA, the Department of Physics, University of Cincinnati, OH USA, and the Department of Physics, Theory Unit Group at CERN, Geneva CH, where part of this work was done, for kind hospitality and stimulating environment. The work of B. L. C. and B. Z. has been supported in part by the Director, Office of Science, Office of High Energy and Nuclear Physics, Division of High Energy Physics of the U.S. Department of Energy under Contract No. DE-AC02-05CH11231, and in part by NSF grant 10996-13607-44 PHXHM. A substantial part of S. F.’s investigation was performed at the Center for Theoretical Physics (CTP), University of California, Berkeley, CA USA, with S. F. sponsored by a “Miller Visiting Professorship” from the Miller Institute for Basic Research on Science. The work of S. F. has been supported also in part by INFN-Frascati National Laboratories, and by D.O.E. grant DE-FG03-91ER40662, Task C. The work of A. M. has been supported by INFN at SITP, Stanford University, Stanford, CA, USA.

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