Information, Physics, and Computation

Marc Mézard

*Laboratoire de Physique Théorique et Modèles Statistiques,
CNRS, and Université Paris Sud*

Andrea Montanari

*Department of Electrical Engineering and Department of Statistics,
Stanford University,
and Laboratoire de Physique Théorique de l'ENS, Paris*

OXFORD
UNIVERSITY PRESS

Contents

PART I BACKGROUND

1	Introduction to information theory	3
1.1	Random variables	3
1.2	Entropy	5
1.3	Sequences of random variables and their entropy rate	8
1.4	Correlated variables and mutual information	10
1.5	Data compression	12
1.6	Data transmission	16
	Notes	21
2	Statistical physics and probability theory	23
2.1	The Boltzmann distribution	24
2.2	Thermodynamic potentials	28
2.3	The fluctuation–dissipation relations	32
2.4	The thermodynamic limit	33
2.5	Ferromagnets and Ising models	35
2.6	The Ising spin glass	44
	Notes	46
3	Introduction to combinatorial optimization	47
3.1	A first example: The minimum spanning tree	48
3.2	General definitions	51
3.3	More examples	51
3.4	Elements of the theory of computational complexity	54
3.5	Optimization and statistical physics	60
3.6	Optimization and coding	61
	Notes	62
4	A probabilistic toolbox	65
4.1	Many random variables: A qualitative preview	65
4.2	Large deviations for independent variables	66
4.3	Correlated variables	72
4.4	The Gibbs free energy	77
4.5	The Monte Carlo method	80
4.6	Simulated annealing	86
4.7	Appendix: A physicist’s approach to Sanov’s theorem	87
	Notes	89

PART II INDEPENDENCE

5	The random energy model	93
5.1	Definition of the model	93
5.2	Thermodynamics of the REM	94
5.3	The condensation phenomenon	100
5.4	A comment on quenched and annealed averages	101
5.5	The random subcube model	103
	Notes	105
6	The random code ensemble	107
6.1	Code ensembles	107
6.2	The geometry of the random code ensemble	110
6.3	Communicating over a binary symmetric channel	112
6.4	Error-free communication with random codes	120
6.5	Geometry again: Sphere packing	123
6.6	Other random codes	126
6.7	A remark on coding theory and disordered systems	127
6.8	Appendix: Proof of Lemma 6.2	128
	Notes	128
7	Number partitioning	131
7.1	A fair distribution into two groups?	131
7.2	Algorithmic issues	132
7.3	Partition of a random list: Experiments	133
7.4	The random cost model	136
7.5	Partition of a random list: Rigorous results	140
	Notes	143
8	Introduction to replica theory	145
8.1	Replica solution of the random energy model	145
8.2	The fully connected p -spin glass model	155
8.3	Extreme value statistics and the REM	163
8.4	Appendix: Stability of the RS saddle point	166
	Notes	169

PART III MODELS ON GRAPHS

9	Factor graphs and graph ensembles	173
9.1	Factor graphs	173
9.2	Ensembles of factor graphs: Definitions	180
9.3	Random factor graphs: Basic properties	182
9.4	Random factor graphs: The giant component	187
9.5	The locally tree-like structure of random graphs	191
	Notes	194
10	Satisfiability	197
10.1	The satisfiability problem	197

10.2 Algorithms	199
10.3 Random K -satisfiability ensembles	206
10.4 Random 2-SAT	209
10.5 The phase transition in random $K (\geq 3)$ -SAT	209
Notes	217
11 Low-density parity-check codes	219
11.1 Definitions	220
11.2 The geometry of the codebook	222
11.3 LDPC codes for the binary symmetric channel	231
11.4 A simple decoder: Bit flipping	236
Notes	239
12 Spin glasses	241
12.1 Spin glasses and factor graphs	241
12.2 Spin glasses: Constraints and frustration	245
12.3 What is a glass phase?	250
12.4 An example: The phase diagram of the SK model	262
Notes	265
13 Bridges: Inference and the Monte Carlo method	267
13.1 Statistical inference	268
13.2 The Monte Carlo method: Inference via sampling	272
13.3 Free-energy barriers	281
Notes	287
 PART IV SHORT-RANGE CORRELATIONS	
14 Belief propagation	291
14.1 Two examples	292
14.2 Belief propagation on tree graphs	296
14.3 Optimization: Max-product and min-sum	305
14.4 Loopy BP	310
14.5 General message-passing algorithms	316
14.6 Probabilistic analysis	317
Notes	325
15 Decoding with belief propagation	327
15.1 BP decoding: The algorithm	327
15.2 Analysis: Density evolution	329
15.3 BP decoding for an erasure channel	342
15.4 The Bethe free energy and MAP decoding	347
Notes	352
16 The assignment problem	355
16.1 The assignment problem and random assignment ensembles	356
16.2 Message passing and its probabilistic analysis	357
16.3 A polynomial message-passing algorithm	366

16.4	Combinatorial results	371
16.5	An exercise: Multi-index assignment	376
	Notes	378
17	Ising models on random graphs	381
17.1	The BP equations for Ising spins	381
17.2	RS cavity analysis	384
17.3	Ferromagnetic model	386
17.4	Spin glass models	391
	Notes	399
 PART V LONG-RANGE CORRELATIONS		
18	Linear equations with Boolean variables	403
18.1	Definitions and general remarks	404
18.2	Belief propagation	409
18.3	Core percolation and BP	412
18.4	The SAT-UNSAT threshold in random XORSAT	415
18.5	The Hard-SAT phase: Clusters of solutions	421
18.6	An alternative approach: The cavity method	422
	Notes	427
19	The 1RSB cavity method	429
19.1	Beyond BP: Many states	430
19.2	The 1RSB cavity equations	434
19.3	A first application: XORSAT	444
19.4	The special value $x = 1$	449
19.5	Survey propagation	453
19.6	The nature of 1RSB phases	459
19.7	Appendix: The $SP(y)$ equations for XORSAT	463
	Notes	465
20	Random K-satisfiability	467
20.1	Belief propagation and the replica-symmetric analysis	468
20.2	Survey propagation and the 1RSB phase	474
20.3	Some ideas about the full phase diagram	485
20.4	An exercise: Colouring random graphs	488
	Notes	491
21	Glassy states in coding theory	493
21.1	Local search algorithms and metastable states	493
21.2	The binary erasure channel	500
21.3	General binary memoryless symmetric channels	506
21.4	Metastable states and near-codewords	513
	Notes	515
22	An ongoing story	517
22.1	Gibbs measures and long-range correlations	518

22.2 Higher levels of replica symmetry breaking	524
22.3 Phase structure and the behaviour of algorithms	535
Notes	538
Appendix A Symbols and notation	541
A.1 Equivalence relations	541
A.2 Orders of growth	542
A.3 Combinatorics and probability	543
A.4 Summary of mathematical notation	544
A.5 Information theory	545
A.6 Factor graphs	545
A.7 Cavity and message-passing methods	545
References	547
Index	565