Analytic Estimates of Magnetic Energy in Superconducting Quadrupoles

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Abstract—In this paper, we propose analytical formulae to determine the magnetic energy stored in superconducting quadrupoles made of sector coils. The study is based on the Fourier transformation of the current density flowing within the coils. The case of real magnets (i.e., magnets which are not made of pure sector coils) is treated and a heuristic corrective coefficient allowing taking into account the energy enhancement due to current grading is derived from a numerical study. We also introduce the effect of an unsaturated iron yoke on the stored energy and we discuss the issue of the yoke saturation.

Index Terms—Superconducting accelerator magnets, stored energy, quadrupole magnets.

I. INTRODUCTION

THE magnetic energy stored in a superconducting accelerator magnet is an important parameter for the quench calculations and for designing its quench protection [1]–[3]. However, an analytical calculation is rather difficult due to the complicated geometry of the coil and the evaluation is usually done with numerical codes [4]–[6]. Indeed, during the conceptual design phase it can be useful to have simple approximated expressions of the magnetic energy as a function of the main features of the quadrupole lay-out (magnet aperture, coil width, grading, iron yoke dimensions). In this paper, we explore the possibility of giving such approximated formula, and their validity limits.

We first derive an equation for an ironless quadrupole with uniform current density made of simple sector coils (Section II). The formula is then extended to include the case of graded coils, i.e., coils with nonuniform current density (Section V), and the impact of an unsaturated ferromagnetic iron around the coils (Section VI). We also investigate the iron saturation effect through a numerical analysis. The study of 31 realistic quadrupole cross-sections shows that this approach allows estimating the magnetic energy of a quadrupole coil within 10% accuracy.

II. MAGNETIC ENERGY IN A SECTOR COIL QUADRUPOLE

A. Magnetic Energy Definition

In the absence of nonlinear magnetic material, the magnetic energy U_t produced by any winding can be obtained by inte-

Fig. 1. Schematic view of a superconducting quadrupole coil.

grating the scalar product between the magnetic induction $\mathbf{B} = \mu\mathbf{H}$ and the magnetic field $\mathbf{H}$ all over the space Ω_∞

$$U_t = \frac{1}{2} \iiint_{\Omega_\infty} \frac{|\mathbf{B}|^2}{\mu} d\Omega. \quad (1)$$

An equivalent approach makes use of the potential vector $\mathbf{A}$ defined as $\mathbf{B} = \text{curl}\mathbf{A}$

$$U_t = \frac{1}{2} \iiint_{\Omega_\infty} \mathbf{A} \cdot \mathbf{j} d\Omega = \frac{1}{2} \iiint_{\Omega} \mathbf{A} \cdot \mathbf{j} d\Omega. \quad (2)$$

Expression (2) is easier to implement since the integration is carried out over the finite window domain Ω where the current density is nonzero. The gauge invariance of the vector potential does not alter the validity of the relation (2). Throughout the paper, we will use the I.S., thus expressing the current density $\mathbf{j}$ is in $[\text{A}/\text{m}^2]$ and the distances in $[\text{m}]$, and the energy in $[\text{J}]$.

In high energy accelerator magnets (see Fig. 1), the transverse dimensions are usually small with respect to the axial one and therefore we can consider the magnet as infinitely long in the z direction. Neglecting the coil ends, the current density and the vector potential are parallel to the z axis, the magnetic field has components only in the transverse plane (x, y) , and the magnetic energy per meter along z is given by

$$U = \frac{1}{2} \iint_S A_z(x, y) j_z(x, y) dx dy \quad (3)$$

where S is the cross-sectional area of the winding. Given the symmetry of the magnet, it is convenient to use in the transverse plane polar coordinates (r, θ) instead of the Cartesian ones (x, y) .

B. Stored Energy in a Quadrupole Made of Sector Coils

The cross-section of a quadrupole can be well approximated by a given number of sectors carrying a constant current density

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Fig. 2. Quadrupole made of one (left) or two (middle) pure sectors blocks and cross-section of the RHIC main quadrupole (right).

(see Fig. 2). The need of having more than one sector is due to the constraint of generating a pure field component (quadrupole in this case) within tight tolerances. A simple 30° sector coil [see Fig. 2 (left)] sets to zero the first nonzero field harmonic, i.e., the sixth order (12th pole) which is not enough to meet the beam dynamics requirements. Lay-outs with two sectors having appropriate angles can set to zero also the 20th and the 28th poles [see Fig. 2 (middle)], thus providing sufficiently pure fields to satisfy the beam dynamics specifications [7].

A sector coil carrying a constant current density j_0 can be modeled by a thick winding powered by a sum of sinusoidal current densities with different amplitudes and pulsations

$$j(\theta) = j_0 \sum_{n=1}^{\infty} a_n \cos(n\theta) \equiv \sum_{n=1}^{\infty} j_n(\theta) \quad (4)$$

where there is no constant term because for a quadrupole the average current density is null, and the sine terms are absent due to the coil symmetry. The coefficients a_n are given by

$$a_n = \frac{1}{\pi} \int_0^{2\pi} j(\theta) \cos(n\theta) d\theta. \quad (5)$$

For a coil made by a single sector of angular dimension θ_1 the coefficients a_n read

$$a_n = \frac{2}{n\pi} \sin(n\theta_1) \left[1 + \cos(n\pi) - 2 \cos\left(n\frac{\pi}{2}\right) \right]. \quad (6)$$

For any coil made of sectors with quadrupolar symmetry (as for instance the lay-outs shown in Fig. 2), the coefficient a_n is different from zero only for $n = 2, 6, 10, 14, \dots$

The vector potential given by a pure current harmonic of rank n inside the winding is (see Appendix A)

$$A_{z,n}(r, \theta) = C_n(r) j_n(\theta). \quad (7)$$

For $n = 2$

$$C_2(r) = \frac{\mu_0}{4} \left(\frac{1}{4r^2} [r^4 - R_1^4] + r^2 \ln\left(\frac{R_2}{r}\right) \right) \quad (8)$$

and for $n > 2$

$$C_n(r) = \frac{\mu_0}{2n} \left(\frac{1}{n+2} \left[r^2 - \frac{R_1^{n+2}}{r^n} \right] + \frac{1}{n-2} \left[r^2 - \frac{r^n}{R_2^{n-2}} \right] \right) \quad (9)$$

where R_1 and R_2 are the inner and the outer radius of the coil, respectively. Since the harmonic functions are orthogonal, one has that

$$\begin{aligned} U &= \frac{1}{2} \int_0^{2\pi} \int_{R_1}^{R_2} \sum_l A_{z,l} \sum_m j_m ds \\ &= \frac{\pi}{2} j_0^2 \sum_n a_n^2 \int_{R_1}^{R_2} C_n(r) r dr \end{aligned} \quad (10)$$

and, therefore,

$$U = \sum_n U_n \equiv \frac{\pi}{2} j_0^2 \sum_n a_n^2 \int_{R_1}^{R_2} C_n(r) r dr. \quad (11)$$

Substituting (8) and (9) in (11), carrying out the integration and introducing the coil width $w = R_2 - R_1$, one obtains an explicit expression for the magnetic energy coefficients

$$U_n = \frac{\pi \mu_0 j_0^2 a_n^2}{4n} R_1^4 f_n\left(\frac{w}{R_1}\right). \quad (12)$$

For $n = 2$ and $t = w/R_1$, the function f reads

$$f_2(t) = \frac{1}{8} [(1+t)^4 - 1] - \frac{1}{2} \ln(1+t) \quad (13)$$

and for $n > 2$ one has

$$\begin{aligned} f_n(t) &= \frac{1}{(n+2)(n-2)} \\ &\times \left(\frac{2}{(1+t)^{n-2}} - (1+t)^4 - 1 + \frac{n}{2} ((1+t)^4 - 1) \right). \end{aligned} \quad (14)$$

The magnetic energy associated to the first harmonic is U_2

$$U_2 = \frac{\pi \mu_0 j_0^2 a_2^2}{8} R_1^4 f_2\left(\frac{w}{R_1}\right). \quad (15)$$

We now estimate the quantity of the energy carried by U_2 with respect to the total energy U as a function of w/R_1 . In Fig. 3, we study the function U_2/U for the $[0^\circ-30^\circ]$ sector coil [see Fig. 2 (left)] which allows cancelling the first nonzero 12th pole and for four cases allowing cancelling both the 12th pole and the 20th pole, made of two blocks separated by a copper wedge [see Fig. 2 (center)], and limited by the angles $[0^\circ-12^\circ]$, $[18^\circ-30^\circ]$, $[0^\circ-18^\circ, 22^\circ-32^\circ]$, $[0^\circ-24^\circ, 30^\circ-36^\circ]$, and $[0^\circ-26^\circ, 36^\circ-44^\circ]$. For w/R_1 larger than 0.2 (as in most superconducting

Fig. 3. Ratio between the magnetic energy produced by the main harmonic ($n = 2$) and the total energy versus coil width w over aperture radius R_1 for five different sector coils.

Fig. 4. Function h defined in (18).

magnets), the first harmonic U_2 accounts for 98%–99% of the total energy.

C. Stored Energy Versus Gradient

Here we link the gradient with the magnetic energy produced by the main harmonic of current given by (15). The field gradient of any sector coil is

$$G = \frac{\mu_0 j_0 a_2}{2} \ln \left(1 + \frac{w}{R_1} \right) \quad (16)$$

and, therefore, the magnetic energy is proportional to the square of the gradient and the fourth power of the aperture radius

$$U_2 = \frac{R_1^4 \pi G^2}{2 \mu_0} h \left(\frac{w}{R_1} \right) \quad (17)$$

where the geometric function h , depending on the ratio t of coil width to aperture radius, is

$$h(t) \equiv \frac{f_2(t)}{[\ln(1+t)]^2}. \quad (18)$$

The function h versus t is plotted Fig. 4.

III. COMPARISON OF ANALYTIC VERSUS NUMERICAL ESTIMATE FOR REALISTIC LAY-OUTS

We now compare the numerical computation of the magnetic energy done using the numerical code [6] to the analytical estimate (15) for several realistic coil lay-outs, assuming that there is no iron yoke and that the current density j in the coils is uniform (no grading). The input parameters for the estimate (15)

Fig. 5. Agreement between numerical estimate of magnetic energy and analytical estimate for a 30° sector coil based on (15).

are the coil inner radius R_1 and its cross-sectional area S . The area provides an equivalent width, i.e., the width of a 30° sector coil having the same coil surface S

$$w_{eq} = \left(\sqrt{1 + \frac{3S}{2\pi R_1^2}} - 1 \right) R_1. \quad (19)$$

This equivalent width is used to compute the magnetic energy of a sector coil through (15).

We consider 31 quadrupoles, all based on the $\cos 2\theta$ lay-out. Some of them are used in particles accelerators such as the Intersection Storage Ring [8], the Tevatron [9], the Large Electron Positron [10], HERA [11], the Relativistic Heavy Ion Collider [12], and the Large Hadron Collider [13]. Others have been designed in the framework of the LHC luminosity upgrade: MQY 90/100/110 [14] and MQXC [15] and have been designed for the phase one of the LHC luminosity upgrade (Nb–Ti); HQ1, HQ2 [16], 90 mm 2/4 Layers [17] and IRQ 90/100/110 [18], have been designed for the phase two of the LHC luminosity upgrade (Nb₃Sn). The TQ magnet [19] is a Nb₃Sn short model quadrupole built in the framework of the U.S. LHC Accelerator Research Program (LARP). The quadrupoles lay-outs named Ap. 50/100/150/200 mm have been sketched for a field quality study in superconducting quadrupoles [20]. LHC MQ+1 and LHC MQ+2 denote an LHC MQ coil with one or two additional layers, sketched to analyze coils with very large w/R_1 . Please note that the current grading is removed from graded magnets.

For these magnets, we compared the stored energy numerically computed and analytically determined with (15) (see Fig. 5). All values agree within 10%, and 2/3 of them agree within 5%. It was also found that the 30° sector coil provides a better agreement than coils with two sectors.

IV. MAGNETIC ENERGY AT CRITICAL CURRENT DENSITY

In superconducting magnets, the current density in the coils cannot exceed the critical current density of superconductor j_c at peak field in the coil. Here we will compute the magnetic energy when the current density is set at its critical value.

In order to estimate the critical current density of a generic superconducting quadrupole, we define the following basic parameters, according to the approach of [7]:

Fig. 6. Analytical estimate of the stored energy (MJ/m) in quadrupoles with Nb-Ti at 1.9 K, filling factor of 0.3, powered at 80% of their short sample current and with aperture radii varying from 15 to 120 mm. On each curve, the ratio w/R_1 varies from 0.16 to 1.2.

Fig. 7. Same as Fig. 6 for Nb₃Sn at 1.9 K.

$$\beta = \frac{B_p}{j} [\text{Tm}^2/\text{A}] \quad (20)$$

$$\gamma = \frac{G}{j} [\text{Tm}/\text{A}] \quad (21)$$

where j [A/m²] is the current density in the insulated coil, G [T/m] is the gradient, and B_p [T] is the peak field in the coil. For a 30° sector coil, γ is determined analytically

$$\gamma = \gamma_0 \log \left(1 + \frac{w}{R_1} \right) \quad (22)$$

with $\gamma_0 = 6.93 \times 10^{-7}$ [A/(Tm²)] and β is well fit by an empirical expression

$$\beta = R_1 \gamma_0 \left(a_{-1} \frac{R_1}{w} + 1 + a_1 \frac{w}{R_1} \right) \quad (23)$$

with $a_{-1} = 0.04$ and $a_1 = 0.11$ (for a 30° sector coil).

The Nb-Ti critical surface is well fit by a linear relation

$$j_{c,s} = c(b - B) \quad (24)$$

where $j_{c,s}$ [A/m²] is the critical current in the Nb-Ti material, c [A/(Tm²)] is the slope of the line in the $(j_{c,s}, B)$ plane, and b [T] is the critical field according to the linear fit. We assume

that $b \sim 10$ T at 4.2 K and ~ 13 T at 1.9 K, and $c \sim 6 \times 10^8$ A/(Tm²) is independent of the temperature. We introduce the filling factor k which takes into account the presence of copper, void, and insulation in the coil. Therefore, the critical surface for the engineering current density becomes

$$j_c = kc(b - B) \quad B < b. \quad (25)$$

The expression for the critical current of a sector winding made from Nb-Ti material is [7]

$$j_c = \frac{kcb}{1 + kcR_1 \left(a_{-1} \frac{R_1}{w} + 1 + a_1 \frac{w}{R_1} \right) \gamma_0 \ln \left(1 + \frac{w}{R_1} \right)}. \quad (26)$$

A good fit for the critical current of the Nb₃Sn is given by the following relation:

$$j_{c,s} = c \left(\frac{b}{B} - 1 \right), \quad B < b. \quad (27)$$

Typical values for a cable having 3000 A/mm² at 4.2 K and 12 T are $c = 4.0 \times 10^9$ A/m² and $b = 21$ T (4.2 K) or $b = 23$ T (1.9 K). The critical current expression is then

$$j_c = \frac{kc}{2} \left(\sqrt{\frac{4b}{\beta kc} + 1} - 1 \right). \quad (28)$$

Substituting the current density j_0 in (15) either by the critical current density in a Nb-Ti coil (26) or in a Nb₃Sn coil (28), we obtain the energy as a function of the coil lay-out at the critical current (sometimes called short sample condition). As superconducting magnets are usually powered with about 20% operational margin from the short sample current, we computed the magnetic energy at the nominal current defined as 80% of the critical current. Plots of the magnetic energy versus the operational gradient for different magnet apertures in a semilogarithmic scale are given in Figs. 6 and 7. For each aperture radius, the coil width is varied from $R_1/6$ to $1.2R_1$. One observes that for a fixed aperture, the energy is close to be the exponential of the gradient, thus giving straight lines in a semilogarithmic plot.

V. GRADED COILS

The grading method consists of setting a higher current density in the outer layers, where the peak field is lower. This technique is widely used in superconducting accelerator magnets made of two or more layers. For graded magnets the previous approach cannot be directly used. Here we propose a heuristic corrective coefficient allowing taking into account the grading effect in the usual case where one has only two different current densities j_1 and j_2 . The grading parameter is defined by

$$g = \frac{j_2}{j_1}, \quad g > 1. \quad (29)$$

The average current density value j flowing through the overall coil cross-sectional area $S = S_1 + S_2$ is

$$j = \frac{j_1 S_1 + j_2 S_2}{S} \quad (30)$$

where S_1 and S_2 are defined as the area of the inner and of the outer layer, respectively. We express j as a function of g

Fig. 8. Increase of magnetic energy due to grading as a function of x as defined in (32).

$$j = j_1 \left[1 + \frac{S_2}{S}(g-1) \right]. \quad (31)$$

Since the magnetic energy is proportional to j^2 , we expect that the variation caused by current grading is proportional to

$$x \equiv \left[1 + \frac{S_2}{S}(g-1) \right]^2. \quad (32)$$

Indeed, this is the case: in Fig. 8, we show the numerical evaluation based on the numerical code [6] of the increase of the magnetic energy due to the grading in 11 quadrupoles presented in Section III. The increase is linear in x

$$\frac{U_{\text{grad}}}{U_{\text{not-grad}}} = d(x-1) + 1 \quad (33)$$

where the fit gives $d = 1.167$. One can, therefore, obtain the following heuristic law allowing estimating the stored energy in graded quadrupoles:

$$U_{\text{grad}} = U_2[d(x-1) + 1]. \quad (34)$$

Comparison of (34) to numerical results including grading gives an agreement within 8% (see Fig. 9).

VI. IRON YOKE CONTRIBUTION

An iron yoke has the function of closing the magnetic field circuit and increasing the magnetic field produced by the coil for a given current density. It can also be used to withstand the forces. The iron yoke can be partially saturated and in this case an analytical formulation is not possible. In this section, analytical calculations are performed considering an infinitely long cylindrical iron yoke of inner radius R_f and of infinite permeability. The impact of the saturation on the magnetic energy is then evaluated through a numerical analysis. Analytical expressions of the vector potential within the winding is obtained by solving equations $\Delta \mathbf{A} = 0$ outside the coil, $\Delta \mathbf{A} = -\mu_0 \mathbf{j}$ inside the coil, and matching the boundary conditions between coil and air, and between iron and air [21]. The expression of $\mathbf{A}$ inside the winding can be split in two parts $\mathbf{A} = \mathbf{A}_c + \mathbf{A}_i$, where $\mathbf{A}_c$ is the vector potential created by the coil alone [given in (7)] i.e.,

Fig. 9. Energy agreement between numerical computations of the magnetic energy in graded coils and estimates based on (34).

there is no iron, and $\mathbf{A}_i$ is the supplementary vector potential due to the iron yoke

$$A_{i,n}(r, \theta) = \frac{\mu_0 a_2 j_0}{2n(n+1)} \frac{r^n}{R_f^{2n}} R_1^{n+2} \left(\left(1 + \frac{w}{R_1} \right)^{n+2} - 1 \right) \times \cos(n\theta) \quad (35)$$

where $w = R_2 - R_1$. The total stored energy U_{tot} is obtained integrating the scalar product between the total vector potential $\mathbf{A} = \mathbf{A}_c + \mathbf{A}_i$, and the current density in the winding. Therefore, the total energy can be written as a sum of two terms

$$U_{\text{tot}} = U_c + U_i \quad (36)$$

where U_c is the stored energy created by the coil alone, and U_i is the supplementary energy due to the iron yoke contribution to the vector potential.

Considering only the main harmonic of the current, i.e., $n = 2$, one has $U_c = U_2$ [see (15)]; the expression of U_i is given by integrating the scalar product of the potential vector $A_{i,n}$ (expression (35) and $n = 2$) and the current density in the winding over the winding cross-section S

$$U_i = \frac{1}{2} \iint_S A_{i,2} j_2 dS \quad (37)$$

where j_2 is the main harmonic of the current density

$$j_2 = j_0 a_2 \cos(2\theta) \quad (38)$$

and, therefore, U_i is

$$U_i = \frac{\pi}{128} \mu_0 j_0^2 a_2^2 R_1^4 \left(\frac{R_1}{R_f} \right)^4 [(1+t)^4 - 1]^2 \quad (39)$$

where $t = w/R_1$. We now express the ratio between the magnetic energy without iron yoke and with iron yoke as

$$\frac{U_{\text{tot}}}{U} = 1 + \frac{U_i}{U_c}. \quad (40)$$

The ratio U_{tot}/U depends only on $t = w/R_1$ and on the forth power of R_1/R_f

TABLE I
COMPARISON OF THE U_{tot}/U RATIO BETWEEN ANALYTICAL FORMULA (42) AND NUMERICAL COMPUTATION PERFORMED BOTH WITH AN UNSATURATED YOKE (NUM UNSAT.) AND WITH A REAL YOKE (NUM SAT.)

	U_{tot}/U			relative error (%)		
	Num unsat.	Num sat.	Ana	Ana/Num unsat.	Ana/Num sat.	Num unsat/sat
RHIC MQY	1.43	1.43	1.43	-0.2	-0.2	0.0
RHIC MQ	1.52	1.52	1.50	-1.3	-1.3	0.0
ISR MQ	1.42	1.34	1.38	-2.5	3.4	6.0
Tevatron MQ	1.10	1.10	1.09	-0.5	-0.1	0.4
HERA MQ	1.18	1.17	1.16	-1.6	-1.4	0.2
LHC MQM	1.12	1.12	1.11	-0.9	-0.8	0.1
ssc MQ	1.12	1.12	1.10	-1.8	-1.9	0.0
LHC MQ	1.10	1.10	1.09	-1.4	-1.4	0.0
MQXC	1.29	1.21	1.27	-1.0	4.9	5.9
LHC MQXB	1.16	1.14	1.12	-3.0	-1.2	1.8
LHC MQY	1.18	1.18	1.14	-3.8	-4.0	0.0
HQ2	1.15	1.10	1.12	-2.7	2.0	4.8
LHC MQXA	1.32	1.19	1.22	-7.6	2.6	11.1

$$\frac{U_{\text{tot}}}{U} = 1 + \frac{1}{8} \left(\frac{R_1}{R_f} \right)^4 \frac{[(1+t)^4 - 1]^2}{\frac{1}{4}[(1+t)^4 - 1] - \ln(1+t)}. \quad (41)$$

In range of t , used on accelerator magnets, one can fit the complicated expression with a second-order polynomial

$$\frac{U_{\text{tot}}}{U} = 1 + \left(\frac{R_1}{R_f} \right)^4 (at^2 + bt + c) \quad (42)$$

with $a = 8.52$, $b = -0.80$, and $c = 1.56$. For cross-checking the validity of this equation, we compare the semianalytical formula (42) with the numerical computation [6] of the U_{tot}/U ratio carried out for eight not graded magnets and five graded magnets (see Table I) surrounded either by an unsaturated yoke of relative permeability $\mu \gg 1$ (Num unsat.) or a real iron yoke (Num sat.). Also in this case, the $t = w/R_1$ ratio used in (42) is replaced by the w_{eq}/R_1 ratio according to definition (19).

Let us consider the unsaturated case (Num unsat.), for both the not graded (Table I, upper part) and the graded magnets (Table I, lower part); the effect of the iron yoke on the stored energy is foreseen within 3% using the analytical formula (42) (excepted for the LHC MQXA). Note that the formula always underestimates the ratio U_{tot}/U and that the iron contribution to the magnetic energy can be large (for example, $\sim 50\%$ in the RHIC MQ). For a quadrupole made with a current sheet and surrounded by iron, the contribution of the iron to the energy reaches its maximum value of 100%.

As expected, the saturation of the iron yoke (Num sat.) leads to a lower stored energy compared to what we have for an unsaturated yoke (see Table I, last column). The decrease of the energy due to the iron saturation does not exceed $\sim 5\%$, except for the MQXA where it is of 11%. In Fig. 10, we compare the analytical estimate of the stored energy in superconducting quadrupoles graded (filled markers) or not (empty markers), and surrounded by an iron yoke using (15), (34), and (42), with a numerical computation [6] in case of a real iron yoke. The relative error is within 5% for 70% one the cases, and within 13% for the others.

Fig. 10. Energy agreement between numerical computations of the stored energy in real quadrupoles (with real saturated iron) and analytical estimates based on (42).

VII. CONCLUSION

We used the Fourier decomposition for evaluating the magnetic stored energy of a quadrupole made of sector coils as a simple function of the coil width and of the aperture (12). We showed that the ratio between the total stored energy and the energy produced by the first harmonic of current depends only on the ratio of coil width to the aperture radius. We also showed that the stored energy in a quadrupole made of sector coil is well approximated by the first harmonic of the energy (15). To link the analytical formula (15) to a real quadrupole cross-section, we defined the 30° sector coil equivalent through (19). Using this approach, the lay-out details as the number of blocks, and their angular position of a real coil, are ignored. The analytical equation allows predicting the magnetic energy within 10% (Fig. 5). The approach was generalized (33) to take into account of grading, which can contribute to a large amount of the stored energy (for instance, 42% for the MQY). Also in this case, the energy is foreseen within 10% (Fig. 9). A third coefficient allowing taking into account the contribution of an unsaturated iron yoke of infinite permeability is given (42). A comparison of the stored energy in real quadrupoles computed numerically

(i.e., real cross-section, real iron yoke and grading) and the analytical estimates of the energy through (15), (34), and (42) shows agreement within 10% (Fig. 10).

APPENDIX A

VECTOR POTENTIAL IN A COS THETA WINDING

Let us consider a thick $\cos(n\theta)$ winding with an inner radius R_1 and an outer radius R_2 and powered by a current density pointed toward z and expressed in $[A/m^2]$. Inside the winding, for $n = 1$ (pure dipolar field) one has

$$A_{z,1}(r, \theta) = \frac{\mu_0 j_0}{2} \cos(\theta) \left(r R_2 - \frac{R_1^3}{3r} - \frac{2}{3} r^2 \right) \quad (A1)$$

and for $n = 2$ (pure quadrupole field)

$$A_{z,2}(r, \theta) = \frac{\mu_0 j_0}{4} \cos(2\theta) \times \left(\frac{1}{4r^2} [r^4 - R_1^4] + r^2 \ln \left(\frac{R_2}{r} \right) \right). \quad (A2)$$

For $n > 2$ one has

$$A_{z,n}(r, \theta) = \frac{\mu_0 j_0}{2n} q_n(r, R_1, R_2) \cos(n\theta) \\ q_n \equiv \left(\frac{1}{n+2} \left[r^2 - \frac{R_1^{n+2}}{r^n} \right] + \frac{1}{n-2} \left[r^2 - \frac{r^n}{R_2^{n-2}} \right] \right). \quad (A3)$$

For $n = 1$, the vector potential within the aperture is

$$A_{z,1}(r, \theta) = \frac{\mu_0 j_0}{2} r (R_2 - R_1) \cos \theta. \quad (A4)$$

For $n = 2$, the vector potential within the aperture is

$$A_{z,2}(r, \theta) = \frac{\mu_0 j_0}{4} r^2 \ln \left(\frac{R_2}{R_1} \right) \cos(2\theta). \quad (A5)$$

For $n > 2$, the vector potential within the aperture is

$$A_{z,n} = \frac{\mu_0 j_0}{2} \frac{r^n}{n(n-2)} \left(\frac{1}{R_1^{n-2}} - \frac{1}{R_2^{n-2}} \right) \cos(n\theta). \quad (A6)$$

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