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CONTRIBUTION TO THE THEORY OF THE HOLLOW ELLIPTICAL BEAM

by

E. Regenstreif
UNIVERSITE DE RENNES I
Rennes-Beaulieu, France

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E. Regenstreif
UNIVERSITE DE RENNES I
Rennes-Beaulieu, France

ABSTRACT

This paper is an application to the hollow beam of the general theory previously developed for the potential and the field of an elliptic beam coasting coaxially inside a cylindrical vacuum chamber of elliptic cross-section. Examination of the hollow beam reveals some unusual features, e.g. existence of two potential maxima instead of one, inversion of the electric field lines inside the beam, image field in opposition with the field in the hollow.
1. Introduction
2. Inadequacy of symmetry considerations
3. Definition of the hollow beam
4. Arbitrary density distribution
5. Constant density distribution
   5.1. Notations
   5.2. Field components in cartesian coordinates
   5.3. Ellipse $E_y = 0$
   5.4. Potential at the centre of the vacuum chamber and maximum potential
   5.5. Maximum electric field
   5.6. Remarks on the potential
   5.7. Potential in cartesian coordinates
   5.8. Equipotential lines
1. INTRODUCTION

In a previous paper \(^1\) closed expressions have been derived for the electric potential and the field created by a beam of elliptic cross-section and uniform density coasting coaxially inside a cylindrical vacuum chamber of elliptic cross-section; confocality between the ellipse representing the vacuum chamber and the ellipse describing the beam envelope was assumed. An extension of the theory to beams of non-uniform charge density was given subsequently \(^2\).

In this paper an attempt is made to apply the theory to the case of a hollow beam. Although hollow beams have been considered in the past \(^3\), the importance of the hollow elliptical beam in accelerator work seems to be rather limited at the present time. It is however a simple example of a non-uniform beam so that application of the general theory allows a detailed analysis of its physical properties. Moreover it sheds some light on the behaviour of the electric field under unusual conditions, in addition to illustrating rather nicely some general theorems of electrostatic potential theory.

2. INADEQUACY OF SYMMETRY CONSIDERATIONS

In the case of a solid elliptical beam coasting coaxially in an elliptic vacuum chamber one is guided by symmetry considerations to postulate the existence of a maximum for the potential at the centre of the beam. If we consider a hollow beam, we know from general theorems that the potential cannot have its maximum in the hollow for there are no charges there. Now, if the maximum potential is no more at the centre of the vacuum chamber, symmetry with respect to \(x\) on one hand and with respect to \(y\) on the other hand does not provide an argument to decide where the maximum should be. One might actually be tempted to assume that it is azimuthally half-way between the two axes so as not to privilege either of them but this would be wrong.

There is only one piece of information which we can deduce from symmetry considerations: if the centre of the vacuum chamber is excluded, there must be at least two maxima, symmetrically located with respect to the origin.
3. DEFINITION OF THE HOLLOW BEAM

Let (Fig 1) \( G, P \) be the semi-axes of the vacuum chamber, \( g_o, p_o \) the semi-axes of the outer beam envelope, and \( g_i, p_i \) the semi-axes of the inner beam envelope. In elliptic coordinates \( \eta, \psi \) the three ellipses are described respectively by \( \eta_c, \eta_o, \eta_i \), where

\[
theta_c = \frac{P}{G} \quad \eta_o = \frac{P_o}{G_o} \quad \eta_i = \frac{P_i}{G_i}
\]  

(1)

and confocality requires

\[
g^2 - p^2 = g_o^2 - p_o^2 = g_i^2 - p_i^2 = c^2
\]  

(2)

We define the hollow beam by

\[
\begin{align*}
p &= 0, & 0 &\leq \eta &\leq \eta_i \\
p &= \rho(\eta), & \eta_i &\leq \eta &\leq \eta_o \\
p &= 0, & \eta_o &\leq \eta &\leq \eta_c
\end{align*}
\]  

(3)

Some simplification in the formulae is obtained \(^1,2\) by putting

\[
\xi = 2\eta \quad , \quad \phi = 2\psi
\]  

(4)

Eqs (3) take then the form

\[
\begin{align*}
p &= 0, & 0 &\leq \xi &\leq \xi_i \\
p &= \rho(\frac{\xi}{2}), & \xi_i &\leq \xi &\leq \xi_o \\
p &= 0, & \xi_o &\leq \xi &\leq \xi_c
\end{align*}
\]  

(5)

Investigation of the properties of the hollow beam can proceed to some extent without actually specifying the explicit form of the density distribution \( \rho \). We therefore consider first the general case where \( \rho \) is arbitrary, and then the simpler case where \( \rho \) is constant in the beam.
4. ARBITRARY DENSITY DISTRIBUTION

According to our general theory \(^2\) we put

\[ C = C(\xi) = \int_{\xi_1}^{\xi} \rho(\frac{S}{2}) \text{ch}sds, \quad C_0 = \int_{\xi_1}^{\xi_0} \rho(\frac{S}{2}) \text{ch}sds \quad (6) \]

and

\[ S = S(\xi) = \int_{\xi_1}^{\xi} \rho(\frac{S}{2}) \text{sh}sds, \quad S_0 = \int_{\xi_1}^{\xi_0} \rho(\frac{S}{2}) \text{sh}sds \quad (7) \]

and we apply our basic relations [Eqs (104) and (105) of Ref 2] to the three regions specified in Eq (5).

a) Outside the beam we have \(C = C_0, S = S_0\), and the field components are

\[ E_x = \frac{C_0}{Z_0} \left( \text{th} \xi_0 - \text{th} \frac{\xi}{2} \right) x \quad (8) \]

\[ E_y = \frac{C_0}{Z_0} \left( \text{coth} \frac{\xi}{2} - \text{th} \xi_0 \right) y \quad (9) \]

In cartesian coordinates we would write

\[ \text{th} \frac{\xi}{2} = \sqrt{\frac{x^2 - y^2 + z^2 - c^2}{2x^2}} \quad , (10) \]

and

\[ \text{coth} \frac{\xi}{2} = \sqrt{\frac{y^2 - x^2 + z^2 + c^2}{2y^2}} \quad , (11) \]

where

\[ z^4 = (x^2 - y^2 - c^2)^2 + 4x^2y^2 \quad (12) \]

The field described by Eqs (8) and (9) is Laplacian and non-linear.

b) Inside the beam we have

\[ E_x = \frac{1}{2Z_0} \left( C_0 \text{th} \xi_0 - S_0 + S - C \text{th} \frac{\xi}{2} \right) x \quad (13) \]

\[ E_y = \frac{1}{2Z_0} \left( S_0 - C_0 \text{th} \xi_0 - S + C \text{coth} \frac{\xi}{2} \right) y \quad (14) \]

This is a non-linear Poissonian field.

c) In the hollow we have \(C = 0, S = 0\), and the field components
become

\[ E_x = \frac{1}{2\varepsilon_0} (C_0 \text{ th} \xi_C - S_0)x \quad (15) \]

\[ E_y = \frac{1}{2\varepsilon_0} (S_0 - C_0 \text{ th} \xi_C)y \quad (16) \]

The following remarks can be made in connection with the last two equations:

i) The field is Laplacian and linear; the hollow is the only two-dimensional region of a non-uniform beam where the field components are linear.

ii) Although the average field, taken on a circle, is zero, in agreement with Gauss' theorem, the individual components (15) and (16) are not.

iii) The components vanish only in a circular structure for in this case one has \( \text{th} \xi_C = 1 \) and \( C_0 = S_0 \). Any departure from circularity will give rise to an electric field in the hollow.

iii) It can readily be seen that irrespective of the density distribution one always has

\[ C_0 \text{ th} \xi_C - S_0 > 0 \quad (17) \]

From Eqs (15) and (16) we then infer that the field in the hollow is always directed away from the vertical plane of the beam and towards its median plane (Fig 2). This is just the opposite of the field created by the charges in the well of the vacuum chamber, irrespective of whether there is a hollow in the beam or not \(^2\). The ratio of the two components (15) and (16) being the same as \( E_x \text{ vc} / E_y \text{ vc} = -x/y \), these two fields will everywhere be in opposition, irrespective of the distribution.

Eq (16) shows that for \( y > 0 \), the \( y \)-component of electric field is negative in hollow, whereas according to Eq (9), this component is positive outside the beam (for \( y > 0 \)). Consequently \( E_y \) must be zero along some curve situated inside the beam. Putting \( E_y = 0 \) in Eq (14), we get for \( y \neq 0 \)

\[ C \cosh \frac{x}{2} - S = C_0 \text{ th} \xi_C - S_0 \quad (18) \]
Let $\xi = \xi_m$ be the solution of this equation; we have

$$\xi_1 < \xi_m < \xi_0$$  \hspace{1cm} (19)$$

and the confocal ellipse $\xi = \xi_m$ defines the locus of the points in the beam where $E_y = 0$. Along this curve the electric field reduces to $E_x$ (Fig 3) and is given by

$$E_{E_y=0} = \frac{1}{\varepsilon_0(1 + \epsilon h \xi_m)} \left[ C_0 \theta h \xi_c + S(\xi_m) - S_0 \right] x$$ \hspace{1cm} (20)$$

The field is linear along this ellipse.

Nothing similar happens to $E_x$; the equation

$$C_0 \theta h \xi/2 - S = C_0 \theta h \xi_c - S_0$$ \hspace{1cm} (21)$$

has no solution $\xi_m$ such that $\xi_1 < \xi_m < \xi_0$, and we conclude that nowhere can we have $E_x = 0$, except on the y-axis [Eqs (8), (13), and (15)]. Consequently there are two points and only two points inside the beam where we have simultaneously $E_x = 0$, $E_y = 0$, namely the intersections of the y-axis with the ellipse $\xi = \xi_m$: these are the points where the potential is maximum. From $y = c \sin \theta = c \sin \xi_m / 2$ we find for the position of the points of maximum potential

$$y_m = \pm c \sin \left( \frac{\xi_m}{2} \right)$$ \hspace{1cm} (22)$$

Eq (83) of Ref 2 gives for the value of the maximum potential

$$\phi_{\text{Max}} = \frac{c^2}{\varepsilon_0} \left[ C_0 (\xi_c + \theta h \xi_c - \xi_0) + \int_{\xi_m}^{\xi_0} C(v) dv + S(\xi_m) - S_0 \right]$$ \hspace{1cm} (23)$$

and for the value of the potential at the centre of the vacuum chamber

$$\phi_0 = \frac{c^2}{\varepsilon_0} \left[ C_0 (\xi_c + \theta h \xi_c - \xi_0) + \int_{\xi_1}^{\xi_0} C(v) dv - S_0 \right]$$ \hspace{1cm} (24)$$

We can summarize as follows the results of the preceding calculations (Fig 4):

Along the x-axis the potential decreases smoothly from its value $\phi_0$ at the centre of the vacuum chamber to the value it
takes at the wall of the vacuum chamber, i.e. zero. Along the y-axis the potential increases from $\phi_0$ to $\phi_{\text{Max}}$ and decreases then to zero. The centre of the vacuum chamber is a saddle-point of the potential function.

The electric field along the x-axis increases from the centre of the vacuum chamber, where its value is zero, up to the outer envelope where its value is

$$E_{\mid x = e_0 \mid y = 0} = \frac{C_0 e_0}{2e_0} \left( \text{th} \xi_c - \text{th} \frac{\xi_0}{2} \right)$$

(25)

It then decreases from this value to the value it takes at the wall of the vacuum chamber, viz.

$$E_{\mid x = e_0 \mid y = 0} = \frac{C_0 G}{2e_0} \left( \text{th} \xi_c - \text{th} \frac{\xi_c}{2} \right)$$

(26)

Along the y-axis the electric field decreases in the hollow, it starts increasing at the inner edge of the beam, it goes through zero for $y = y_m$, and it then increases up to outer edge of the beam where its value is

$$E_{\mid x = 0 \mid y = p_0} = \frac{C_0 p_0}{2e_0} \left( \cosh \frac{\xi_0}{2} - \text{th} \xi_c \right)$$

(27)

Finally it decreases from this value to the value it takes at the wall of the vacuum chamber, viz.

$$E_{\mid x = 0 \mid y = p} = \frac{C_0 p}{2e_0} \left( \cosh \frac{\xi_c}{2} - \text{th} \xi_c \right)$$

(28)

Eq (27) actually represents the maximum value the electric field can take anywhere inside or outside the beam.

A few equipotential lines of the hollow beam have been sketched in Fig 5. It should be noted that inside the hollow the potential is given by

$$\phi = \phi_0 + \frac{1}{4e_0} \left( S_0 - C_0 \text{ th} \xi_c \right) (x^2 - y^2)$$

(29)
and the equipotential lines are equilateral hyperbolas.

Let us finally remark that the field produced by the vacuum chamber under the action of a hollow beam is given by

\[
E_x \, \nu_c = - \frac{C_0}{2 \varepsilon_0} (1 - \theta \xi_c) x
\]

\[
E_y \, \nu_c = \frac{C_0}{2 \varepsilon_0} (1 - \theta \xi_c) y
\]

These components are the same in the three regions of interest. As in all elliptical beams we have \(E_x \, \nu_c / E_y \, \nu_c = -x/y\) irrespective of the charge distribution ²).

5. CONSTANT DENSITY IN THE BEAM

5.1. Notations

In the preceding analysis we have made no assumption concerning the density distribution. If we now put \(\rho = \text{const.}\) inside the beam, we have

\[
C = \rho (sh \xi - sh \xi_1) , \quad C_0 = \rho (sh \xi_0 - sh \xi_1)
\]

\[
S = \rho (ch \xi - ch \xi_1) , \quad S_0 = \rho (ch \xi_0 - ch \xi_1)
\]

In terms of the beam dimensions, we can write

\[
sh \xi_0 = \frac{2g_0^2 \rho_0}{c^2} , \quad ch \xi_0 = \frac{g_0^2 + p_0^2}{c^2} , \quad \xi_0 = \ln \frac{g_0 + p_0}{g_0 - p_0}
\]

and

\[
sh \xi_1 = \frac{2g_1^2 \rho_1}{c^2} , \quad ch \xi_1 = \frac{g_1^2 + p_1^2}{c^2} , \quad \xi_1 = \ln \frac{g_1 + p_1}{g_1 - p_1}
\]

Similarly, we recall that in terms of the dimensions of the vacuum chamber one has

\[
sh \xi_C = \frac{2G \rho}{c^2} , \quad ch \xi_C = \frac{G^2 + P^2}{c^2} , \quad \xi_C = \ln \frac{G + P}{G - P}
\]
5.2. Field components in cartesian coordinates

Using the relation \( \text{ch} \xi = \frac{1}{c^2} (x^2 + y^2 + z^2) \),

\[
\text{ch} \xi = \frac{1}{c^2} (x^2 + y^2 + z^2) ,
\]

with \( z \) given by Eq (12), we can express everywhere the field components in cartesian coordinates. We find

a) outside the beam

\[
|E_x| = \frac{\rho}{\varepsilon_0 c^2} \left( g_0 p_o - g_1 p_1 \right) \left( \frac{2Gp}{G^2 + p^2} |x| - \sqrt{\frac{x^2 - y^2 - z^2 - c^2}{2}} \right)
\]

\[
|E_y| = \frac{\rho}{\varepsilon_0 c^2} \left( g_0 p_o - g_1 p_1 \right) \left( \sqrt{\frac{y^2 - x^2 - z^2 - c^2}{2}} - \frac{2Gp}{G^2 + p^2} |y| \right)
\]

b) inside the beam

\[
|E_x| = \frac{\rho}{\varepsilon_0 c^2} \left( g_1 p_1 \sqrt{\frac{x^2 - y^2 - z^2 - c^2}{2}} - \left[ p_0^2 - 2 \left( g_0 p_o - g_1 p_1 \right) \frac{Gp}{G^2 + p^2} \right] |x| \right)
\]

\[
|E_y| = \frac{\rho}{\varepsilon_0 c^2} \left( \left[ \frac{G^2 - 2 \left( g_0 p_o - g_1 p_1 \right) \frac{Gp}{G^2 + p^2} }{G^2 + p^2} \right] |y| - g_1 p_1 \sqrt{\frac{x^2 - y^2 - z^2 - c^2}{2}} \right)
\]

c) inside the hollow

\[
E_x = \frac{\rho}{\varepsilon_0 c^2} \left[ 2 \left( g_0 p_o - g_1 p_1 \right) \frac{Gp}{G^2 + p^2} - \left( g_0^2 - g_1^2 \right) \right] x
\]

\[
E_y = \frac{\rho}{\varepsilon_0 c^2} \left[ \left( g_0^2 - g_1^2 \right) - 2 \left( g_0 p_o - g_1 p_1 \right) \frac{Gp}{G^2 + p^2} \right] y
\]

5.3. Ellipse \( E_y = 0 \)

The solution of Eq (18) becomes now

\[
\frac{\xi}{2} = \frac{\text{ch} \xi_o + 1 - (\text{sh} \xi_o - \text{sh} \xi_1) \text{th} \xi_1}{\text{sh} \xi_1}
\]
whereas the solution of Eq (21) can be written

\[
\frac{\xi_m'}{2} = \frac{\text{ch}\xi_0 - 1 - (\text{sh}\xi_0 - \text{sh}\xi_1)\text{th}\xi_0}{\text{sh}\xi_1}
\]  

(45)

It is readily seen that

\[
\xi_m' < \xi_1 < \xi_m < \xi_0
\]  

(46)

Therefore the solution (44) will always exist whereas the solution (45) must be rejected. In terms of the geometric parameters of the problem, Eq (44) can be written

\[
\xi_m = \ln \frac{g_0^2 + g_1p_1 - 2(g_0p_0 - g_1p_1) \frac{GP}{G^2 + p^2}}{g_0^2 - g_1p_1 - 2(g_0p_0 - g_1p_1) \frac{GP}{G^2 + p^2}}
\]  

(47)

As mentioned in the general theory, the ellipse \(\xi = \xi_m\) defines the locus of the points in the beam where \(E_y = 0\). The semi-axes of this ellipse (Fig. 3) are \(a = c \text{ sh} \frac{\xi_m}{2}\) and \(b = c \text{ sh} \frac{\xi_m}{2}\). Replacing \(\xi_m\) by its value we find for the semi-axes of this ellipse

\[
a = \frac{g_0^2 - 2(g_0p_0 - g_1p_1) \frac{GP}{G^2 + p^2}}{\sqrt{\left[g_0^2 - 2(g_0p_0 - g_1p_1) \frac{GP}{G^2 + p^2}\right]^2 - g_1^2p_1^2}}
\]  

(48)

\[
b = \frac{g_1p_1}{c} \sqrt{\left[g_0^2 - 2(g_0p_0 - g_1p_1) \frac{GP}{G^2 + p^2}\right]^2 - g_1^2p_1^2}
\]  

(49)

These values can also be derived from Eq (41) by putting \(E_y = 0\). Everywhere on this ellipse the electric field reduces to \(E_x\) and is given by

\[
E_x (E_y = 0) = \frac{\xi_0}{\xi_m} \frac{(\text{sh}\xi_0 - \text{sh}\xi_1)\text{th}\xi_0 - (\text{ch}\xi_0 - \text{ch}\xi_m)}{1 + \text{ch}\xi_m} \cdot x
\]  

(50)
By means of Eqs (34), (35), (36) and (44) the coefficient of $x$ in this equation can be expressed in terms of the geometric parameters $G, P, G_0, P_0,$ and $g_1, p_1$. On finds

$$E(E_{y=0}) = \frac{1}{\varepsilon_0 c^2} \left[ \frac{(g_1 p_1)^2}{G^2 - 2(g_0 p_0 - g_1 p_1) \frac{GP}{G^2 + p^2}} - p_0^2 \right. $$

$$\left. + 2(g_0 p_0 - g_1 p_1) \frac{GP}{G^2 + p^2} \right] \times (51)$$

This can also be deduced from Eqs (40) and (41).

5.4. Potential at the centre of the vacuum chamber and maximum potential

The position of the two maxima of the potential is given by

$$x = 0 \quad , \quad y = \pm b (52)$$

To calculate its value, we apply Eqs (23) and (24) to the case we consider, i.e. $\rho = \text{const}$. We find

$$\phi_{\text{Max}} = \frac{\rho c^2}{\varepsilon_0} \left[ (\xi_0 + \text{th} \xi_0)(\text{sh} \xi_0 - \text{sh} \xi_1) + \xi_m \text{sh} \xi_1 - \xi_0 \text{sh} \xi_0 \right] (53)$$

and

$$\phi_0 = \frac{\rho c^2}{\varepsilon_0} \left[ (\xi_0 + \text{th} \xi_0)(\text{sh} \xi_0 - \text{sh} \xi_1) + \xi_1 \text{sh} \xi_1 - \xi_0 \text{sh} \xi_0 \right] (54)$$

Using Eqs (34), (35), and (36) the expression for the potential at the centre of the vacuum chamber takes the form

$$\phi_0 = \frac{\rho}{2\varepsilon_0} \left[ g_0 p_0 \left( \frac{GP}{G^2 + p^2} + \ln \frac{G + P}{G_0 + P_0} \right) - g_1 p_1 \left( \frac{GP}{G^2 + p^2} + \ln \frac{G + P}{g_1 + p_1} \right) \right] (55)$$

On the other hand, from Eqs (53) and (54) we have

$$\phi_{\text{Max}} - \phi_0 = \frac{\rho c^2}{\varepsilon_0} (\xi_m - \xi_1) \text{sh} \xi_1 (56)$$

Substituting $\xi_m$ and $\xi_1$ from Eqs (47) and (35) we obtain
\[ \phi_{\text{Max}} = \phi_0 + \frac{\rho \delta \phi_0^2}{4\epsilon_0} \ln \frac{g_o^2 g_1^2 P_1^2 - 2(g_o P_0 - g_1 P_1)}{g_o^2 - g_1^2 P_1^2 - 2(g_o P_0 - g_1 P_1)} \frac{G^2 P^2}{G^2 + P^2} \cdot \frac{g_1 - P_1}{g_1 + P_1} \] \tag{57}

5.5. Maximum electric field

For the maximum electric field (located at \(x = 0, y = \pm p_o\)), we find from Eq (27)

\[ E_{\text{max}} = \frac{\rho \delta \phi_0^2}{\epsilon_0^2} \left( g_o^2 - g_1 P_1 \right) \left( \frac{g_o}{p_o} - 2 \frac{G^2}{G^2 + P^2} \right) \] \tag{58}

5.6. Remarks on the potential

Let us consider in somewhat more detail the potential in the three regions of interest. We calculate it from Eqs (82) and (83) of Ref 2 and write it in the form

\[ \phi = \frac{\rho \delta \phi_0^2}{\epsilon_0} \left[ \frac{\xi_c - \xi - \frac{\text{sh}(\xi_c - \xi)}{\text{ch} \xi_c}}{\text{sh} \xi_c} \right] \] \tag{59}

\[ \phi = \frac{\rho \delta \phi_0^2}{\epsilon_0} \left[ \frac{\xi_c - \xi - \frac{\text{sh}(\xi_c - \xi)}{\text{ch} \xi_c}}{\text{sh} \xi_c} \right] \] \tag{60}

for \( \xi_o \leq \xi \leq \xi_c \)

\[ \phi = \frac{\rho \delta \phi_0^2}{\epsilon_0} \left\{ \left[ \frac{\text{ch}(\xi_c - \xi)}{\text{ch} \xi_c} \right] \left[ \text{ch} \xi - 1 \right] \cos \phi + \text{ch} \xi_o - \text{ch} \xi + (\xi_c - \xi_o) \text{sh} \xi_o \right\} \]

for \( \xi_1 \leq \xi \leq \xi_o \)

and
\[ \phi = \frac{\partial^2 \phi}{\partial \xi^2} \left\{ \left[ \frac{\text{ch}(\xi - \xi_0)}{\text{ch} \xi_0} \right] \text{ch} \xi - 1 \right\} \cos \phi + \text{ch} \xi_0 - \text{ch} \xi + (\xi_0 - \xi) \text{sh} \xi_0 \right\} \]

\[ - \frac{\partial^2 \phi}{\partial \xi^2} \left\{ \left[ \frac{\text{ch}(\xi - \xi_1)}{\text{ch} \xi_1} \right] \text{ch} \xi - 1 \right\} \cos \phi + \text{ch} \xi_1 - \text{ch} \xi + (\xi_0 - \xi_1) \text{sh} \xi_1 \right\} \]

for

\[ 0 \leq \xi \leq \xi_1 \]

More compact forms for the potential are obviously possible but the preceding equations have the virtue to show that

a) outside the beam the potential is the difference between two Laplacian potentials, one corresponding to a solid beam of envelope \( \xi_0 \), and the other to a solid beam of envelope \( \xi_1 \).

b) inside the beam the potential is the difference between a Poissonian potential corresponding to a solid beam of envelope \( \xi_0 \) and a Laplacian potential corresponding to a solid beam of envelope \( \xi_1 \).

c) in the hollow the potential is the difference between two Poissonian potentials, one corresponding to a solid beam of envelope \( \xi_0 \), and the other corresponding to a solid beam of envelope \( \xi_1 \).

Although we have determined the maximum potential by searching for the points where the field is zero, it is of some interest to apply the equations

\[ \frac{\partial \phi}{\partial \xi} = 0, \quad \frac{\partial \phi}{\partial \xi} = 0 \] (62)

to the potential inside the beam [Eq (50)]. These equations yield respectively

\[ \left[ \left( \text{sh} \xi_0 - \text{sh} \xi_1 \right) \frac{\text{sh}(\xi - \xi_0)}{\text{ch} \xi_0} + 1 - \text{ch}(\xi - \xi_0) \right] \sin \phi = 0 \] (63)

and

\[ \frac{\cos \phi}{\text{ch} \xi_0} \left[ \text{ch}(\xi - \xi_0) \text{sh} \xi - \text{ch}(\xi - \xi_1) \text{sh} \xi_1 \right] + \text{sh} \xi_1 - \text{sh} \xi = 0 \] (64)

The quantity in brackets in Eq (63) is always positive for \( \xi < \xi_0 \). The
... component of the electric field can therefore be zero only for \( \phi = 0 \) (which corresponds to \( \psi = 0 \)) and \( \phi = \pm \pi \) (which corresponds to \( \psi = \pm \frac{\pi}{2} \)). From Eq (64) we find however that the \( \xi \)-component of the electric field is zero (let us recall that \( \xi = 0 \) does not belong to the domain we now consider) for

\[
\cos \phi = -\frac{\sh \xi - \sh \xi_i}{\sh \xi \left[ 1 - \sh \xi_i \left( \coth \frac{\xi_m}{2} - \coth \xi \right) \right]}
\]

(65)

where we have made use of Eq (44) giving the value of \( \coth \frac{\xi_m}{2} \). The curve represented by Eq (55) goes through the points \( \xi = \xi_m, \phi = \pm \pi \), i.e. the points defined by Eq (52), and intersects the inner envelope of the beam for \( \xi = \xi_i \) where \( \phi = \pm \frac{\pi}{2} \) (corresponding to \( \psi = \pm \frac{\pi}{4} \) or \( \psi = \mp \frac{3\pi}{4} \)). These are in fact the intersections of the inner envelope with the equilateral hyperbola

\[
x^2 - y^2 = \frac{c^2}{2}
\]

(66)

For \( |\psi| < \frac{\pi}{4} \) or \( \frac{3\pi}{4} < |\psi| < \pi \) the \( \xi \)-component of the electric field cannot be zero (Fig 6). We have seen that in cartesian coordinates there is an ellipse inside the beam \( \xi = \xi_m \) along which the \( y \)-component of the electric field vanishes throughout. In elliptic coordinates there is a curve \( \Gamma \) inside the beam along which the \( \xi \)-(or \( \eta \))-component of the electric field is zero. This curve, given by Eq (65) is not an ellipse and it does not extend throughout the beam; however, in those parts of the beam where \( \Gamma \) exists (Fig 6), the electric field is everywhere tangential to it.

Simultaneous satisfaction of Eqs (63) and (64) requires \( \xi = \xi_m \) and \( \phi = \pm \pi \) (corresponding to \( \psi = \pm \frac{\pi}{2} \)). These are the two points where the potential is maximum; their position is given by Eq (52). Replacement of \( \xi \) and \( \phi \) by \( \xi_m \) and \( \pi \) in Eq (60) leads to the value of the maximum given by Eq (53).

5.7 Potential in cartesian coordinates

The form in which the potential was cast in elliptic coordinates [Eqs (59), (60), and (61)] is appropriate for translation
into cartesian coordinates. Using the results we have obtained in Ref 1, we can write without any calculation

\[
\phi = \frac{\phi_0}{2\varepsilon_0} \left( g_0 p_0 - g_1 p_1 \right) \left[ \frac{G}{G^2 + P^2} \left( 1 - 2 \frac{x^2 - y^2}{c^2} \right) + \frac{G + P}{\sqrt{x^2 + y^2 + z^2 + c^2}} \right]
+ \frac{1}{c^2} \left( |x| \sqrt{\frac{x^2 - y^2 + z^2 - c^2}{2}} - |y| \sqrt{\frac{y^2 - x^2 + z^2 + c^2}{2}} \right) \tag{67}
\]

for the potential between the outer envelope and the vacuum chamber, i.e. for

\[
\left( \frac{x}{g_0} \right)^2 + \left( \frac{y}{p_0} \right)^2 < 1, \quad \left( \frac{x}{g_1} \right)^2 + \left( \frac{y}{p_1} \right)^2 \geq 1
\]

\[
\phi = \frac{\phi_0}{2\varepsilon_0} \left[ \frac{G}{G^2 + P^2} \left( 1 - 2 \frac{x^2 - y^2}{c^2} \right) + \frac{G + P}{\sqrt{x^2 + y^2 + z^2 + c^2}} \right]
+ \frac{1}{c^2} \left( |x| \sqrt{\frac{x^2 - y^2 + z^2 - c^2}{2}} - |y| \sqrt{\frac{y^2 - x^2 + z^2 + c^2}{2}} \right) \tag{68}
\]

for the potential inside the beam, i.e. for

\[
\left( \frac{x}{g_0} \right)^2 + \left( \frac{y}{p_0} \right)^2 < 1, \quad \left( \frac{x}{g_1} \right)^2 + \left( \frac{y}{p_1} \right)^2 \geq 1
\]

and

\[
\phi = \phi_0 + \frac{\phi_0}{2\varepsilon_0 c^2} \left[ \frac{g_0^2 - g_1^2}{G^2 + P^2} - 2(g_0 p_0 - g_1 p_1) \frac{G}{G^2 + P^2} \right] (x^2 - y^2) \tag{69}
\]

for the potential in the hollow, i.e. for

\[
\left( \frac{x}{g_1} \right)^2 + \left( \frac{y}{p_1} \right)^2 \leq 1
\]

The value of \( \phi_0 \) appearing in Eq (69) is given by Eq (55).
5.8 Equipotential lines

As already mentioned the equipotential lines in the hollow are equilateral hyperbolas and this property holds for an arbitrary distribution \( \rho \) in the beam [Eq (29)]. The case \( \phi = \phi_0 \) translates the degenerescence \( y = \pm x \) and corresponds to straight lines passing through the origin [Fig 5]. Inside the beam and in the region between the outer beam envelope and the vacuum chamber, the equipotential lines are complicated curves as can be seen by inspection of Eqs (67) and (68). These curves can however easily be determined by reverting to elliptic coordinates, i.e. by calculating from Eqs (59) and (60) \( \cos \phi \) as a function of \( \xi \) for a given value of the potential \( \Phi \). Translation into cartesian coordinates \( x = c \, \text{ch} \frac{\xi}{2} \cos \frac{\Phi}{2}, \quad y = c \, \text{sh} \frac{\xi}{2} \sin \frac{\Phi}{2} \) allows then an easy plotting of the considered equipotential line.

We limit ourselves here to the separatrix \( \Phi = \Phi_0 \) passing through the origin. Its equation does not depend on \( \rho \) but only on the geometric dimensions. In the hollow it is simply \( y = \pm x \). Inside the beam the equation of the separatrix is obtained by equating the r.h.s. of Eqs (54) and (60). We find

\[
\cos \phi = \frac{\text{cosh} \frac{\xi_m}{2} + \xi - \xi_1}{\text{cosh} \frac{\xi_m}{2} \text{ - sh} \xi \text{sh} \xi_1 - (1 + \text{ch} \xi)} \quad (70)
\]

where we have made use of Eq (44) defining \( \xi_m \). Eq (70) is applicable in the range

\( \xi_1 < \xi < \xi_0 \)

and it is obviously simpler to manipulate than its cartesian counterpart

\[
\begin{align*}
\frac{g_0p_0}{g_1p_1} & \left[ (2 \frac{Gp}{G^2+p^2} - \frac{p_0}{g_0}) \frac{x}{c} \right] \left( \frac{y}{c} \right)^2 + \left( \frac{g_0}{p_0} - 2 \frac{Gp}{G^2+p^2} \right) \left( \frac{y}{c} \right)^2 \\
-2 \frac{Gp}{G^2+p^2} \frac{x^2-y^2}{c^2} & + \ln \frac{\sqrt{g_1 + p_1}}{\sqrt{x^2+y^2+z^2-c^2}} \\
& + \frac{1}{\sqrt{2c^2}} (|x| \sqrt{x^2-y^2+z^2-c^2} - |y| \sqrt{y^2-x^2+z^2-c^2}) = 0
\end{align*}
\]
which can be obtained either from Eq (70) or by equating the r.h.s. of Eqs (55) and (88). For the intersections of the separatrix with the inner envelope we find, putting $\xi = \xi_1$ in Eq (70)

$$\cos \phi \cosh \xi_1 + 1 = 0$$  \hspace{1cm} (72)

or in cartesian terms

$$|x| = |y| = \frac{\xi_1 p_1 \xi_1}{\sqrt{\xi_1^2 + p_1^2}}$$  \hspace{1cm} (73)

as one would expect. The intersections of the separatrix with the outer envelope would be obtained either by simultaneous solution of Eq (71) and

$$\frac{x^2}{\xi_0^2} + \frac{y^2}{p_0^2} = 1$$  \hspace{1cm} (74)

or by putting $\xi = \xi_0$ in Eq (70); the latter procedure gives

$$\cos \phi = -\frac{(\sinh \xi_0 - \sinh \xi_1) \tanh \xi_0 - (\xi_0 - \xi_1) \sinh \xi_1}{(\sinh \xi_0 - \sinh \xi_1) (\cosh \xi_0 \tanh \xi_0 - \sinh \xi_0)}$$  \hspace{1cm} (75)

The condition $|\cos \phi| < 1$ requires

$$\tanh \xi_0 > \frac{1}{\cosh \xi_0 - 1} \left[ \sinh \xi_0 - \frac{(\xi_0 - \xi_1) \sinh \xi_1}{\sinh \xi_0 - \sinh \xi_1} \right]$$  \hspace{1cm} (76)

and this in turn would demand

$$(\sinh \xi_0 - \sinh \xi_1) (\sinh \xi_0 - \cosh \xi_0 + 1) < (\xi_0 - \xi_1) \sinh \xi_1,$$  \hspace{1cm} (77)

an inequality which it is impossible to satisfy. We conclude that the separatrix cannot intersect the outer envelope, but makes a closed loop inside the beam (Fig 5).

The intersections of the separatrix with the $y$-axis can be obtained either by putting $x = 0$ in Eq (71) or by taking $\cos \phi = -1$ in Eq (70). The latter procedure leads to the transcendental equation in $\xi$
\[ \xi = \xi_1 + \sh(\frac{\xi}{2^2} - 1) \]  
(78)

If \( \xi = \xi_s \) is the solution of this equation, the intersections of the separatrix with the y-axis are given by

\[ y_s = \pm c \ch \xi_s \]  
(79)

It can be seen that \( \xi_m < \xi_s < \xi_o \). If we put \( x = 0 \) in Eq (71) we obtain for the cartesian counterpart of Eq (78)

\[ \left[ \frac{g_0}{g_0} + \frac{2Gp}{G^2 + p^2} \left( 1 - \frac{g_0p_0}{g_0p_0} \right) \right] \left( \frac{y_s}{c} \right)^2 \]

\[ = \ln \left[ \frac{|y_s| + \sqrt{y_s^2 + c^2}}{g_1 + p_1} \right] + \frac{|y_s|}{c} \sqrt{\left( \frac{y_s}{c} \right)^2 + 1} \]

This can also obtained directly from Eqs (78) and (79) but its numerical solution appears more complicated than that of Eq (78).

The analysis of the other equipotential lines could be carried out on a basis similar to the treatment of the separatrix which we have sketched here.


DEFINITION OF THE HOLLOW ELLIPTICAL BEAM

Fig. 1
DIRECTION OF THE ELECTRIC FIELD IN A HOLLOW ELLIPTICAL BEAM

Fig. 2
ON THE ELLIPSE \( \xi = \xi_m \) THE \( y \)-COMPONENT OF THE ELECTRIC FIELD IS ZERO

\[ 2a \]

\[ 2b \]
POTENTIAL AND FIELD IN A HOLLOW ELLIPTICAL BEAM

Fig. 4
EQUIPOTENTIAL LINES IN A HOLLOW ELLIPTICAL BEAM

FIG. 5
ON THE CURVE \( T \) THE \( \xi \)-COMPONENT OF THE ELECTRIC FIELD IS ZERO.

\[
\frac{x^2 - y^2}{2} = c^2
\]