THE $\pi\pi$ INTERACTION

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ABSTRACT

In these lectures the pion-pion interaction is studied from both a theoretical and a practical point of view. On the theoretical side the discussion includes the constraints provided by unitarity and analyticity and the consequences of current algebra. On the practical side a general account is given of the extrapolation necessary to obtain pion-pion cross-sections from production experiments, together with the alternative method starting from Ke⁺ experiments. The problem of phase-shift analysis and the possible removal of ambiguities is discussed in some detail, and the most probable set of phase shifts and resonances is presented.
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\( \pi^+(x), \pi^-(x), \pi^0(x) \)  
pion field operators in charge basis.

\( \pi_i(x), i = 1, 2, 3 \)  
pion field operators in Cartesian basis.

\( F^I_x \)  
\( \pi\pi \)-scattering amplitude in state of isospin = I in the x-channel; \( I = 0, 1, 2; x = s, t, u. \)

\( A, B, C \)  
Chew-Mandelstam amplitudes.

\( M_{\pi\pi} \)  
dipion invariant mass.

\( s, t, u \)  
Mandelstam invariants.

\( \phi_x \)  
c.m. scattering angle in channel \( x, x = s, t, u. \)

\( C_{xy}(I, I') \)  
\( (I, I') \) element of \( \pi\pi \) isospin crossing matrix from channel \( x \) to channel \( y, x, y = s, t, u. \)

\( \mathcal{D}_x \)  
dispersive part of amplitude \( F \) in x-channel, \( x = s, t, u. \)

\( \mathcal{A}_x \)  
absorptive part of amplitude \( F \) in x-channel, \( x = s, t, u. \)

\( \cos \theta_y \)  
boundary of double spectral functions, \( x, y = s, t, u. \)

\( \rho_{xy} \)  
boundaries of double spectral functions, \( x, y = s, t, u. \)

\( M_{lI} \)  
invariant Feynman amplitude.

\( f^I_x \)  
\( \pi\pi \) partial-wave amplitude pertaining to isospin = I and angular momentum = \( \ell. \)

\( P_{\ell}(x) \)  
Legendre polynomial.

\( n^I_x, \delta^I_x \)  
elasticity and phase shift pertaining to \( f^I_x \).

\( a_{\ell} \)  
scattering length pertaining to \( f^I_x \).

\( \sigma_{\text{tot}}, \sigma_{\text{el}}, \sigma_{\text{inel}}^I, \sigma_{\text{inel}}^\ell \)  
total, elastic and inelastic cross-section in state of isospin = I and angular momentum = \( \ell. \)

\( \mathcal{V}_i(x), A_i(x) \)  
vector and axial vector currents at Minkowski point \( x, \) with isospin index \( i = 1, 2, 3 \) and Lorentz index \( \mu = 0, 1, 2, 3. \)

\( Q_i, Q_i^S \)  
vector and axial vector charges.

\( \varepsilon_{ijk} \)  
totally antisymmetric symbol in three dimensions.

\( f_T \)  
pion decay constant.

\( g_A \)  
nucleon axial vector coupling constant.
\( G_{\pi NN} \)
pion-nucleon coupling constant.

\( \mathcal{H} \)
Hamiltonian density.

\( t_{\text{min}}, t_{\text{max}} \)
minus the minimal and maximal value of \(|t|\), where \( t \) is momentum transfer to the final baryon in dipion production experiments.

\( H_{\lambda', \lambda} \)
helicity amplitudes for dipion production, \( \lambda \) and \( \lambda' \) being initial and final baryon helicity.

\( Y_{L}^{M} \)
spherical harmonics.

\( L_{\lambda', \lambda, m} \)
helicity partial-wave amplitudes for dipion production. Dipion angular momentum is 0, 1, 2, ... for \( L = S, P, D, ... \). Dipion helicity = \( m \).

\( D_{\ell \ell'}^{m m'} \)
dipion density matrix elements pertaining to dipion angular momenta = \( \ell \), \( \ell' \), respectively and helicities = \( m \), \( m' \), respectively.

\( L^{(\ell)}_{\lambda', \lambda, m} \)
helicity partial-wave amplitudes for dipion production corresponding to asymptotically definite naturality: natural parity exchange for \( L^{(+)} \) and unnatural parity exchange for \( L^{(-)} \). \( \chi = s \) or \( t \) for \( s \)- or \( t \)-channel helicity states.

\( L_{m} \)
two component helicity partial-wave amplitude corresponding to \( (\lambda', \lambda) = (\frac{1}{2}, \frac{1}{2}) \) for the first component and \( (\lambda', \lambda) = (\frac{1}{2}, -\frac{1}{2}) \) for the second component.

\( \chi^{(\ell)} \)
crossing angle for baryon helicities. \((\ell)\) refer to asymptotic naturality of helicity amplitudes to be crossed from \( t \)-channel to \( s \)-channel.

\( G_{\text{st}}^{(\ell)} \)
baryon-helicity crossing matrix.

\( \chi_{\pi \pi} \)
dipion crossing angle.

\( T_{\pi \pi}^{s} \)
\( \pi \pi \) partial-wave amplitude in Argand diagram normalization.

\( G_{\lambda} (M_{\pi \pi}) \)
short-hand for various \( t \)-independent factors in dipion helicity amplitudes for dipion angular momentum = \( \lambda \).

\( n \)
net helicity flip.

\( L_{g}, L_{h} \)
transversity amplitudes for dipion production.

\( S \)
\( \pi \pi \) S matrix element for definite isospin and angular momentum.

\( q_{\pi}, q_{K} \)
\( \pi \pi \) and \( KK \) c.m. three-momenta, respectively.

\( \omega \)
conformal variable uniformizing the \( KK \) threshold.

\( \mathcal{F} \)
manifestly crossing symmetric part of \( \pi \pi \) amplitude, having purely dispersive partial wave amplitudes for \( \lambda \geq 2 \).
\( \Phi \)  

driving term amplitude in Roy equations \( \equiv F - \hat{F} \).

\( F^{+-}, F^{++} \)  

\( \pi^+ \pi^- \rightarrow \pi^+ \pi^- \) and \( \pi^+ \pi^+ \rightarrow \pi^+ \pi^+ \) scattering amplitudes.

\( \nu \)  

\( \nu \equiv (s - u)/4\mu \).

\( z(\nu) \)  

conformal variable uniformizing the right-hand cut \( \pi \pi \) threshold.

\( \phi(\nu) \)  

"arbitrary" amplitude with "same" analyticity properties as physical amplitude.

\( \chi^2(\phi) \)  

\( \chi^2 \) for \( \phi \) to experimental data.

\( \Phi(\phi) \)  

convergence test function, or penalty functional.

\( a_n \)  

Taylor coefficient for \( z \)-expansion of \( \phi \).

\( \lambda \)  

weight parameter in \( \phi \).

\( F_p, F_{\rho}, F_{\phi}, \mathcal{F}^{+-} \)  

terms parametrizing \( F^{+-} \).

\( P(\nu) \)  

explicit pole part of \( \mathcal{F}^{+-} \).

\( \psi_{++} \)  

\( z \)-polynomial parts of \( \mathcal{F}^{+-} \).
1. GENERAL THEORETICAL IDEAS

1.1 Elementary formalism of $\pi^+ \rightarrow \pi^+$

1.1.1 Isospin

The three pions $\pi^+, \pi^-, \pi^0$ form an isovector. To settle on a definite phase convention, let us describe free pions by the fields $\pi^0(x)$, such that

$$|\pi^0\rangle = (\pi^0)^\dagger |0\rangle$$

(1.1)

and

$$\pi^+ (x) = \pi^- (x) = (\pi^0 (x))^\dagger.$$  

(1.2)

With this convention we cannot use Clebsch-Gordan coefficients with Condon-Shortley phases. Rather, it is the set

$$|I_3 = +1\rangle = -|\pi^+\rangle, \quad |I_3 = 0\rangle = |\pi^0\rangle, \quad |I_3 = -1\rangle = |\pi^-\rangle$$

(1.3)

that transforms correctly under isospin rotation.

Hermitian and Cartesian components are sometimes used to make elegant formulae:

$$\pi_1 = \frac{1}{\sqrt{2}} (\pi^+ + \pi^-), \quad \pi_2 = \frac{i}{\sqrt{2}} (\pi^+ - \pi^-), \quad \pi_3 = \pi^0 = \frac{1}{\sqrt{2}} (\pi_1 + i \pi_2).$$

(1.4)

Correspondingly, we introduce formal states

$$|\pi_1\rangle = \frac{1}{\sqrt{2}} (|\pi^+\rangle + |\pi^-\rangle), \quad |\pi_2\rangle = \frac{i}{\sqrt{2}} (|\pi^+\rangle - |\pi^-\rangle), \quad |\pi_3\rangle = |\pi^0\rangle$$

(1.5)

Two-pion states may be decomposed into states of definite isospin $= 0, 1, 2$. Denoting the transition amplitude for scattering in a state of total isospin $= I$ by $F^{(I)}$, we get for some important special cases

$$F (\pi^+ \pi^+ \rightarrow \pi^+ \pi^+) = F (\pi^- \pi^- \rightarrow \pi^- \pi^-) = F^{(2)}$$

$$F (\pi^+ \pi^0 \rightarrow \pi^+ \pi^0) = F (\pi^- \pi^0 \rightarrow \pi^- \pi^0) = \frac{1}{2} (F^{(2)} + F^{(1)})$$

$$F (\pi^+ \pi^- \rightarrow \pi^0 \pi^0) = \frac{1}{3} (F^{(0)} - F^{(2)})$$

$$F (\pi^+ \pi^- \rightarrow \pi^+ \pi^-) = \frac{1}{3} F^{(0)} + \frac{1}{2} F^{(1)} + \frac{1}{6} F^{(2)}$$

$$F (\pi^0 \pi^0 \rightarrow \pi^0 \pi^0) = \frac{1}{3} (2F^{(0)} + F^{(2)}).$$

(1.6)
The amplitude $F(ab + cd)$ for the general process
\[ \Pi_a + \Pi_b \rightarrow \Pi_c + \Pi_d \]
may be elegantly written, using the Cartesian basis (1.4) and (1.3),
\[ F(ab \rightarrow cd) = A \delta_{ab} \delta_{cd} + B \delta_{ac} \delta_{bd} + C \delta_{ad} \delta_{bc}. \tag{1.7} \]

The amplitudes $A$, $B$, and $C$ introduced by Chew and Mandelstam are related to the $F^{(i)}$'s by
\[
\begin{pmatrix}
F^{(0)} \\
F^{(1)} \\
F^{(2)}
\end{pmatrix} =
\begin{pmatrix}
3 & 1 & 1 \\
0 & 1 & -1 \\
0 & 1 & 1
\end{pmatrix}
\begin{pmatrix}
A \\
B \\
C
\end{pmatrix}
\]

From (1.6) it follows that the amplitude $A$ describes the process
\[ \Pi^+ \Pi^- \rightarrow \Pi^0 \Pi^0 \]
in the s-channel, see below).

1.1.2 Kinematics

Consider the scattering process $\pi_a + \pi_b + \pi_c + \pi_d$ of Fig. 1.1.

![Fig. 1.1](image)

Fig. 1.1 $\pi_a + \pi_b + \pi_c + \pi_d$. For simplicity, $a, b, c, d$ denote charge as well as four-momenta.

The total c.m. energy we denote by $M_\pi$, the c.m. scattering angle between $a$ and $c$ by $\theta$, and the c.m. three-momentum by $q$. The mass of the pion (we ignore all isospin-breaking effects) we denote by $\mu$. 
Then \[ \text{units: } \hbar = c = 1; \text{ metric: } p^2 = p_\mu p^\mu = (p^0)^2 - \vec{p}^2 \]

\[ S \equiv M_{\pi \pi}^2 = (a+c)^2 = (b+d)^2 \]

\[ q^2 = \frac{1}{4} (S-4\mu^2) \]

\[ t = -2q^2 (1-c\cos \theta) = (a-c)^2 = (b-d)^2 \]

\[ u = -2q^2 (1+c\cos \theta) = (a-d)^2 = (b-c)^2 \]

\[ S + t + u = 4\mu^2 \]

where \( s, t, \) and \( u \) are the Mandelstam invariants.

1.1.3 Crossing

From the \( s \)-channel process of Fig. 1.1 we define \( t \)-channel and \( u \)-channel processes, respectively:

- **s-channel**: \( \pi_a + \pi_b \rightarrow \pi_c + \pi_d \); \( \Theta_s = \chi (\vec{a}, \vec{c}) \)

- **t-channel**: \( \pi_a + \pi_b \rightarrow \pi_c + \pi_d \); \( \Theta_t = \chi (\vec{d}, \vec{c}) \)

- **u-channel**: \( \pi_a + \pi_b \rightarrow \pi_c + \pi_d \); \( \Theta_u = \chi (\vec{d}, \vec{a}) \)

Note that the order is important for the definition of quadrants for the scattering angles.

The relation between amplitudes describing scattering in states of definite isospin in the various channels is in an obvious notation

\[ F_{st}^{(I)} = \sum_{I' = 0,1,2} C_{sI}^{(-1)} (I,I') F_{s}^{(I')} \]

\[ F_{tu}^{(I)} = \sum_{I' = 0,1,2} C_{su}^{(-1)} (I,I') F_{s}^{(I')} \]

where

\[ C_{st} = C_{st}^{(-1)} = C_{sI}^{(-1)} \]

\[ C_{su} = C_{su}^{(-1)} = C_{sI}^{(-1)} \]

and

\[ C_{st} = \frac{1}{6} \begin{pmatrix} 2 & 6 & 10 \\ 2 & 3 & -5 \\ 2 & -3 & 1 \end{pmatrix} \]

\[ C_{su} = \frac{1}{6} \begin{pmatrix} 2 & -6 & 10 \\ -2 & 3 & 5 \\ 2 & 3 & 1 \end{pmatrix} \]

The distribution of minus signs depends on our particular crossing convention.

Again these relations are elegantly summarized by the Chew-Mandelstam amplitudes:

\[ A(s, t, u) = A(s, u, t) = B(t, u, s) = C(u, t, s) \]
1.1.4 Analyticity

Figure 1.2 shows the Mandelstam plane with the physical regions for $s^-$, $t^-$, and $u$-channels. Also shown are the physical thresholds $s$, $t$, $u = 4\mu^2$ as well as double spectral function boundaries.

![Mandelstam Plane Diagram](image)

**Fig. 1.2** The Mandelstam $(s, t, u)$-plane for $\pi\pi$ scattering. Physical regions are bounded by shades. The physical thresholds $s$, $t$, $u = 4\mu^2$ are shown by dashed lines. The boundaries of the double spectral function regions are shown by solid lines. To give an idea of the mass scale, the lines $s$, $t$, $u = m^2$ as well as the lines $s$, $t$, $u = 4m^2$ are shown. The fixed-$s$ line $s = s_0$ with points A, B, C, D are shown to indicate regions of convergence of partial wave expansions (see Section 1.1.4).

For a fixed value of $s$ say, with $s > 4\mu^2$, the amplitude is a function of $\cos \theta_s = x_s$ only. Writing

$$F(s, x_s) = \Re F(s, x_s) + i \Im F(s, x_s)$$

and

$$\mathcal{O}_s F(s, x_s) = \Re F(s, x_s) \quad \text{for} \quad |x_s| \leq 1$$

$$\mathcal{A}_s F(s, x_s) = \Im F(s, x_s)$$

(1.13)
\( \mathcal{F}(s, x_s) \) is an analytic function in the \( x_s \)-plane cut along \((-\infty, B)\), \((A, \infty)\), whereas \( \mathcal{A}_s \mathcal{F}(s, x_s) \) is supposed to be an analytic function in the \( x_s \)-plane cut along \((-\infty, D)\) and \((C, \infty)\) (see Fig. 1.2). We shall not discuss how much of this can be proven from axiomatic field theory (most of it can\(^{1-5}\)).

The spectral function boundaries (points C and D) are given by

\[
\mathcal{O}^T_{st} : \quad (s - 4\mu^2)(t - 16\mu^2) = 0
\]

\[
\mathcal{O}^T_{st} : \quad (s - 16\mu^2)(t - 4\mu^2) = 0,
\]

and similarly for the \( t \)- and \( u \)-channels. These analyticity domains determine where partial wave expansions converge outside physical regions: the largest ellipse in the \( x \)-plane, having foci at \( x = \pm 1 \) and not intersecting any cut. It follows that imaginary parts at low energies converge in a much larger domain than real parts. This plays an important role in dispersion relation studies (Sections 3, 5, and 6 below).

Crossing and analyticity are closely linked. Thus the relation

\[
\mathcal{F}_t^{(0)} = \frac{4}{3} \mathcal{F}_s^{(0)} + \frac{2}{3} \mathcal{F}_s^{(1)} + \frac{5}{3} \mathcal{F}_s^{(2)}
\]

obtained from Eq. (1.11) means that when the left-hand side, defined in the physical region of the \( t \)-channel, is analytically continued to the physical region of the \( s \)-channel, the formula holds with the \( \mathcal{F}_s^{(1)} \)'s defined in the \( s \)-channel.

### 1.1.5 Unitarity

The S-matrix element for the transition \( |i\rangle \rightarrow |f\rangle \), where \( |i\rangle \) and \( |f\rangle \) are two-pion states, is connected to the invariant Feynman amplitude \( M_{fi} \) by

\[
\langle f | S' | i \rangle = \langle f | i \rangle - i (2\pi)^4 \delta^4(p_f - p_i) M_{fi} \tag{1.15}
\]

[normalization: \( \langle p | p' \rangle = 2p^0(2\pi)^3\delta^3(p - p') \)].

We normalize amplitudes as follows:

\[
\mathcal{F} = -\frac{M}{32\pi} \tag{1.16}
\]

so that

\[
\frac{d\sigma}{d\Omega} = \frac{16\pi}{3} |\mathcal{F}|^2 \quad j \frac{d\sigma}{dt} = \frac{16\pi}{s q^2} |\mathcal{F}|^2 \tag{1.17}
\]

The optical theorem is
\[ \text{Im } F(s, \{ i \}) = \frac{q N_s}{16 \pi} \sigma_{\text{tot}}^I(s). \] (1.18)

The partial wave expansion is
\[ F(s, \cos \theta) = \sum_I (2I+1) f_I(s) P_I(\cos \theta). \] (1.19)

Then unitarity gives for scattering in a state of definite isospin \( I \),
\[ f_I(s) = \frac{\sqrt{N_I}}{2} \left\{ \begin{array}{l} 
\frac{\eta_I^I e^{i \delta_I^I} - 1}{2 i q} \\
\frac{\eta_I^I e^{2i \delta_I^I} - 1}{2 i q} 
\end{array} \right\} \] (1.20)
\( \delta_I^I \) is the phase shift and \( \eta_I^I \) the elasticity: \( 0 \leq \eta_I^I \leq 1 \). In the normalization (1.16),
\[ F(s=4\mu^2, t=0, \nu=4\mu^2 = a_0 \cdot \mu = \text{s-wave scattering length in pion mass units}. \] (1.21)

For general angular momentum we define scattering lengths by
\[ a_{\ell}^I = \lim_{s \to 4\mu^2} \frac{\delta_{\ell}^I}{q^2 s + 1} \] (1.22)

In deriving relations like (1.18) and (1.20) it is important to remember that two pions in a state of definite isospin effectively behave as identical particles\(^5\).

The same remark applies to the following expressions:
\[ \frac{4\pi (2\ell+1)}{q^2} \left( 1 - \eta_I^I \cos 2 \delta_I^I \right) \]
\[ \frac{2 \pi (2\ell+1)}{q^2} \left| \eta_I^I e^{2i \delta_I^I} - 1 \right|^2 \] (1.23)

which all differ by a factor of 2 from the corresponding formulae pertaining to unequal particle reactions.

1.2 Current algebra and breaking of chiral symmetry\(^7\)\(^9\)

Pion-pion scattering is a very good place to test ideas on current algebra and chiral symmetry. Below, we briefly review these ideas.
We use weak and electromagnetic interactions to define vector currents $V^\mu_i(x)$ and axial currents $A^\mu_i(x)$. Here $\mu$ is a Lorenz index and $i$ is an isospin index, $i = 1, 2, 3$ (we need only consider $\Delta S = 0$ currents). To these we define formal charges $Q^i(t)$ and $Q^5_i(t)$ at time $t$:

$$Q^i(t) = \int d^3x \, V^0_i(\vec{x}, t)$$

$$Q^5_i(t) = \int d^3x \, A^0_i(\vec{x}, t)$$

They are believed to satisfy the commutation relations proposed by Gell-Mann:

$$[Q^i(t), Q^j(t)] = i \epsilon_{ijk} Q_k(t)$$

$$[Q^i(t), Q^5_j(t)] = i \epsilon_{ijk} Q^5_k(t)$$

$$[Q^5_i(t), Q^5_j(t)] = i \epsilon_{ijk} Q^5_k(t)$$

or, equivalently, the chiral $\text{SU}(2) \times \text{SU}(2)$ algebra:

$$[Q^i_+(t), Q^j_+(t)] = i \epsilon_{ijk} Q^5_k(t)$$

$$[Q^i_+(t), Q^j_-(t)] = 0$$

with

$$Q^\pm_i(t) \equiv \frac{1}{2} \left[ Q^i(t) \pm Q^5_i(t) \right].$$

These equal time commutation rules may hold independently of whether the charges commute with the Hamiltonian. Actually the vector charges are believed to indeed commute with the strong Hamiltonian. They are the generators of isospin and they are time-independent. Further, it seems that

$$Q^i |0\rangle = 0$$

so that isospin symmetry is realized in a standard Wigner-Weyl fashion: hadrons group themselves into isomultiplets.

In contrast, the axial charge $Q^5_i$ cannot commute with the Hamiltonian. This would imply the conservation of the axial current:

$$\partial_\mu A^\mu_i = 0$$

However, describing $\pi^0 - \mu$ decay by the matrix element

$$\langle 0 | A^\mu_i (0) | \Pi^0_q (q) \rangle = i f_{\pi^-} q^\mu \delta_{ij}$$

where experimentally\textsuperscript{10}
\[ |f_\pi| = 93.3 \text{ MeV}, \]  

Eq. (1.28) is seen to imply either \( f_\pi = 0 \) or \( \mu = 0 \).

Nonetheless, there are indications that the idea of a *partially conserved axial current* (PCAC) is attractive. One consequence is the Goldberger-Treiman relation,

\[ M \cdot g_A = G_{\pi NN} \cdot f_\pi \]  \hspace{1cm} (1.31)

where \( g_A = 1.26 \) is the nucleon axial vector coupling constant, \( G_{\pi NN}^2 / 4\pi = 14.5 \) is the \( \pi N \) coupling constant, and where \( M \) is the nucleon mass. Relation (1.31) is satisfied to 6-8%.

In the limit (1.28) the Hamiltonian is symmetric not only under isospin but under the full chiral \( SU(2) \times SU(2) \) group. The axial part, however, cannot be realized in the Wigner-Weyl fashion: particles do not appear in near-degenerate pairs with opposite parity and baryons are not approximately massless.

Instead, a Nambu-Goldstone realization might be possible. If so,

\[ Q_i \phi^0 \neq 0 \]  \hspace{1cm} (i.e. \( f_\pi \neq 0 \))

and instead the symmetry manifests itself in the existence of a Goldstone boson, the pion (i.e. \( \mu = 0 \)). The smallness of the pion mass compared to other hadronic masses again is evidence that breakings of the symmetry are at the few percent level.

To test those ideas, we would like to know the size and the transformation properties of the chiral-symmetry breaking part of the Hamiltonian. One way of studying this is by looking at \( \sigma \)-terms. This we shall now briefly outline in the case of \( \pi \pi \) scattering.

For a scattering process such as \( \pi \pi \rightarrow \pi \pi \), current algebra and chiral symmetry give no immediate predictions on-shell. Instead, one must go to off-shell points where one or two pion four-momenta vanish. The off-shell continuation must be carried out using the axial current in order to bring in current algebra. Thus one uses on-shell and off-shell the pion field

\[ \Pi_i^r (\alpha) = \frac{1}{N_\Sigma f_\pi} \partial^r A_i^r (\alpha) \]  \hspace{1cm} (1.32)

where the "PCAC" constant \( (\sqrt{2} f_\pi)^{-1} \) ensures that on-shell the pion wave function is correctly normalized [Eq. (1.29)]. We can then employ the LSZ technique\(^{11} \) to derive low-energy theorems relating values and derivatives at off-shell points.
to $\sigma$-terms and current algebra commutators. This information is only enough to
determine the on-shell amplitude in a linear approximation.

Figure 1.3 summarizes the information obtained this way in the case of the
$\pi\pi \to \pi\pi$ amplitude $A(s,t,u)$, Eq. (1.7), in the linear approximation

$$A(s,\hat{t},\hat{u},\hat{a}^2,\hat{b}^2,\hat{c}^2,\hat{d}^2) = \alpha + \beta(t+u) + \delta^s$$  \hspace{1cm} (1.33)

where

$$s + t + u = \hat{a}^2 + \hat{b}^2 + \hat{c}^2 + \hat{d}^2.$$

The tilted plane $s + t + u = 4\mu^2$ is the on-shell Mandelstam plane of Fig. 1.2.

The point $A$, $s = t = u = \mu^2$ is the Adler point corresponding to one pion
being soft (i.e. having vanishing four-momentum). At that point the amplitude
must vanish by (1.32):

$$A(\mu^2,\mu^2,\mu^2) = 0$$  \hspace{1cm} (1.34)

![Diagram showing on- and off-shell $\pi\pi \to \pi\pi$. Off-shell points relevant to current algebra are indicated.](image-url)
This is the Adler condition, which in itself gives no information on on-shell zeros.

At the three current algebra points CA and CA' corresponding to pairs of soft pions, the Gell-Mann algebra (1.25) imposes the conditions (in our normalization)

\[
\frac{\partial}{\partial x} A(\mu^2 + \kappa, 0, \mu^2 - \kappa) = \frac{1}{32\pi \mu f_\pi^2} \tag{CA}
\]

\[
\frac{\partial}{\partial x} A(\mu^2, \mu^2, 0) = \frac{1}{32\pi \mu f_\pi^2} \tag{CA}'
\]

\[
\frac{\partial}{\partial x} A(0, \mu^2 + \kappa, \mu^2 - \kappa) = 0 \tag{1.35}
\]

the last condition being a trivial consequence of crossing symmetry.

The values of the amplitude at these same points are the \(\sigma\)-terms. They are not specified by the Gell-Mann algebra. They may be shown to be integrals over matrix elements of commutators of the form

\[
[Q_i^S(t), \partial_\mu A_d^A(\vec{x}, t)] = -i \left[ [Q_i^S(t), [Q_d^S(t), \mathcal{H}(\vec{x}, t)]] \right]
\]

\[
= -i \left[ [Q_i^S(t), [Q_d^S(t), \varepsilon \mathcal{H}'(\vec{x}, t)]] \right] \tag{1.36}
\]

Here \(\mathcal{H}\) is the Hamiltonian density which we may decompose into a piece symmetric under chiral SU(2) \(\times\) SU(2) and a piece breaking that symmetry:

\[
\mathcal{H} = \mathcal{H}_{SU(2) \times SU(2)} + \varepsilon \mathcal{H}'.
\]

We see that the \(\sigma\)-terms precisely test on the size and the transformation properties of the breaking part. Symmetry under \(i \leftrightarrow j\) implies that \(\Delta I\) for the \(\sigma\)-terms is 0 or 2. At CA' \((s = 0, t = u = \mu^2)\) the value is given by the \(\Delta I = 0\) part \(\sigma^{(0)}\) and at CA \((s = \mu^2, t = 0, u = \mu^2\) or \(s = \mu^2, t = \mu^2, u = 0\) it is given by the \(\Delta I = 2\) part of \(\sigma^{(2)}\).

The breaking \(\varepsilon \mathcal{H}'\) may be classified according to its transformation properties under the more general SU(3) \(\times\) SU(3) group. The most attractive scheme is probably that of Gell-Mann, Oakes and Renner\(^8\) \(^{9,12}\) in which \(\mathcal{H}'\) transforms purely according to \((3, \bar{3}) + (3, 3)\). In that scheme \(\sigma^{(2)} \equiv 0\), as assumed also originally by Weinberg\(^7\).

If \(\sigma^{(2)} = 0\) we can immediately write down the final form for \(A:\)

\[
A(s, t, u) = \frac{s - \mu^2}{32\pi f_\pi^2 \mu^2} \tag{1.37}
\]
bringing out the vanishing of $A$ on the plane $s = \mu^2$, including the on-shell line, $s = \mu^2$. This is often called the on-shell appearance of the Adler zero (1.34), although it should be clear that the exact position is determined by $\sigma^{(2)}$.

It can easily be shown that the linearity assumption itself implies the rule

$$2a_0^2 - 5a_0^2 = 18\mu^2 \alpha_4^1.$$  \hspace{1cm} (1.38)

Independent of assumptions on the $\sigma$-terms, current algebra predicts a value for that (we often use $\mu = 1$ units for scattering lengths):

$$2a_0^2 - 5a_0^2 = 18\mu^2 \alpha_4^1 \equiv 6L = \frac{3\mu}{4\pi f_\pi^2} = 0.54$$  \hspace{1cm} (1.39)

or

$$\alpha_4^1 = 0.03$$

Using the Goldberger-Treiman relation for $f_\pi$ increases this number by $\approx 10\%$.

![Graph](image)

Fig. 1.4 Summary of current algebra predictions in the plane of $s$-wave scattering lengths.
Finally, the non-exoticity condition for the $\sigma$-terms imply

\[
\frac{\alpha_o^2}{\alpha_o^2} = -\frac{7}{2}
\]

or (1.39):

\[
\begin{align*}
\alpha_o^2 &\approx 0.16 \\
\alpha_o^2 &\approx -0.05
\end{align*}
\]

(1.40)

Figure 1.4 summarizes the situation for various possible transformation possibilities for $\epsilon \mathcal{J} C'$ (Ref. 8).

Compared to $\pi N$ scattering$^8,9^9$, $\pi \pi$ scattering offers a much nicer testing ground for ideas on chiral symmetry breaking: i) certain breaking assumptions lead to unique predictions; ii) it is possible to test for the presence or absence of $\Delta I = 2$ components (which are trivially projected out in $\pi N \rightarrow \pi N$).

In Section 3 we shall discuss how predictions compare with experimental information.

1.3 Spectroscopy

The richest problem of present-day $\pi \pi$ phenomenology is that of sorting out the resonance structure. Whereas this is reasonably straightforward for leading, peripheral ("parent") states, it requires very detailed and accurate phase-shift analysis to clarify the picture at the "daughter" level.

Although it is dangerous to use one particular theoretical framework, we find it convenient to make a comparison with the predictions of the Lovelace-Shapiro-Veneziano model for $\pi \pi$ scattering$^{13}$). As we shall see, it is clearly contradicted by experiment in detail; however, it very neatly summarizes a number of basic ideas such as duality, absence of exotics, chiral symmetry, etc.

In Fig. 1.5 we show the Chew-Frautschi plot predicted by the model. In Table 1.1 we compare the prediction with experiment$^{14}$) at the parent level. The theory is very successful in describing the masses and two-pion partial widths. In particular, the recent discovery of the $h(2000)^{15,16}$ was a triumph for the notion of straight line, exchange degenerate, $\rho$-f trajectories (see Figs. 1.6 to 1.9).

At the level of first daughters, however, the model appears to fail almost completely (see Table 1.2). None of the phase-shift solutions to be presented in Sections 4, 5, and 6, see any trace of a $\rho ^*(1300)$ which is predicted to couple very strongly. The same applies to the predicted $d$-wave state under the $g$ -- although, being near the upper end of present analysis, the conclusion is less firm. The status of $\epsilon (760)$ is controversial. A strong $s$-wave effect is certainly present below 1 GeV. However, as discussed in Section 3.2, it may well be just
the tail of an $\varepsilon$ with a mass $> 1100$ MeV, or no resonance at all. The clear-cut presence of a strong s-wave in the $f^0(1270)$ region is another serious defect of the LSV model (Table 1.2).

![Diagram](image)

Fig. 1.5 Chew-Frautschi plot pertaining to the LSV-model (Ref. 13). Solid lines are the exchange-degenerate $\rho$-$f$ trajectory with daughters. Dashed lines are the $\phi$-$f'$ trajectory with daughters. These are predicted not to couple to the $\pi\pi$ system. Any state predicted to have zero $\pi\pi$ coupling is shown by an open circle.

Table 1.1

"Parent states" as predicted by the LSV model (Ref. 13) and as given by PDG (Ref. 14).

$\Gamma_{\pi\pi}(p) = 150$ MeV was used for the LSV prediction.

<table>
<thead>
<tr>
<th>Particle</th>
<th>LSV</th>
<th>Experiment, PDG</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\text{Mass}$</td>
<td>$\Gamma_{\pi\pi}$</td>
</tr>
<tr>
<td>$\rho$</td>
<td>764 (150)</td>
<td>773 $\pm$ 3</td>
</tr>
<tr>
<td>$f$</td>
<td>1300</td>
<td>1271 $\pm$ 5</td>
</tr>
<tr>
<td>$g$</td>
<td>1670</td>
<td>1690 $\pm$ 20</td>
</tr>
<tr>
<td>$h$</td>
<td>1980</td>
<td>2040 $\pm$ 20</td>
</tr>
<tr>
<td>$f'$</td>
<td>1455</td>
<td>1516 $\pm$ 3</td>
</tr>
</tbody>
</table>
Fig. 1.6 The $h$ meson as seen in the CERN–THEP–Karlsruhe–Pisa–Vienna experiment (Ref. 15) on $\pi^- p \rightarrow \pi^0 \pi^0 n$ at 40 GeV/c. Figures (a) and (b) show the three-shower and four-shower invariant mass spectra.
Fig. 1.7  Same experiment as in Fig. 1.6. Eighth unnormalized moment of the $\pi^0\eta^0$ system as a function of $\pi\pi$ mass.

Fig. 1.8  The $h$ meson as seen in the CERN-Munich experiment (Ref. 16) on $\pi^-p \to K^+K^-n$ at 18.4 GeV/c. $K^+K^-$ invariant mass spectrum for $0.025 \text{ GeV}^2 < |t| < 0.25 \text{ GeV}^2$. 

Fig. 1.9 Same experiment as in Fig. 1.8. Mass dependence of the unnormalized moments. Same t-interval as in Fig. 1.9.

Table 1.2
Daughters predicted by the LSV model (Ref. 13). Subscripts indicate the parent particle.

<table>
<thead>
<tr>
<th>Particle</th>
<th>Mass (MeV)</th>
<th>$\Gamma_{\pi\pi}$ (MeV)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$e_0$</td>
<td>764</td>
<td>760</td>
</tr>
<tr>
<td>$s^*$</td>
<td>1009</td>
<td>0</td>
</tr>
<tr>
<td>$\rho_f^*$</td>
<td>1300</td>
<td>150</td>
</tr>
<tr>
<td>$e_f$</td>
<td>1300</td>
<td>0</td>
</tr>
<tr>
<td>$f_n^*$</td>
<td>1670</td>
<td>110</td>
</tr>
<tr>
<td>$g^*$</td>
<td>1670</td>
<td>19</td>
</tr>
</tbody>
</table>
The model predicts a $p'(1670)$ with $\Gamma_{\pi\pi} \approx 19$ MeV. A similar object appears in some phase-shift solutions and not in others. A discussion on this will be a major subject in Sections 4–6.

General duality arguments lead us to expect a pair of exchange-degenerate trajectories associated with the $\phi(1019)$. The next expected recurrence (with opposite signature) is the $f'(1455)$ (Table 1.1), expected to have vanishing coupling to $2\pi$ if its quark structure is $(\bar{s}s)$ as for the $\phi$. Indeed the experimental $f'(1516)$ has only been seen in the $K\bar{K}$ channel. Recent upper limits on $\Gamma_{\pi\pi}$ are very impressive:

$$\frac{\Gamma (f' \to 2\pi)}{\Gamma (f' \to all)} < 1\%.$$  

The situation seems very different in the case of the $S^*$, which theory would like to be degenerate in mass with the $\phi$ and to have the same quark content. Although experimentally the mass of the $S^*$ (assuming it is a resonance, see Section 3.2) is close to that of the $\phi$, a clear $2\pi$ decay mode is seen. The general analysis of the $s$-wave in the region from, say, 600 MeV to 1400 MeV, is a very interesting subject to which we return in Section 3.2.

What we want to emphasize here is the importance of sorting out the general structure of daughter states, demanded both by duality arguments and most quark schemes. This should have a profound effect on our thinking about $\pi\pi$ dynamics.

1.4 Concluding remarks

In this introductory section we have emphasized the importance of $\pi\pi$ scattering in throwing light on such diverse ideas as current algebra, chiral symmetry, and spectroscopy in general. The kind of questions which we have posed and to which we would like to have answers are of such a detailed nature that precision analyses of high quality experiment are required. In recent years the experimental situation has considerably improved. In the rest of this course we shall concentrate on the efforts made to obtain from experimental information, amplitudes of sufficient quality that one can hope to clarify some of the issues discussed above.

2. ANALYSIS OF DIPION PRODUCTION EXPERIMENTS

The overwhelming majority of our knowledge on $\pi\pi$ scattering has been obtained from studies of the processes

$$\pi\pi N \rightarrow \pi\pi N$$
$$\pi N \rightarrow \pi\pi \Delta.$$  

(2.1)
In this section we shall review the state of the art of extracting the $\pi\pi$ cross-section from such reactions. We shall mainly concentrate on the high statistics experiments performed over the last years by the CERN-Munich group\textsuperscript{19-22}. This will mean ignoring crucial pioneering work performed by a large number of other experimental groups. For an account of these we refer to the several existing review articles\textsuperscript{6,10,23-33}.

A recent, very comprehensive account of all details pertaining to dipion amplitude analysis (as well as $\pi\pi$ scattering in general) can be found in Ref. 32. Here we shall be content with emphasizing general features.

2.1 Kinematics

We consider the process

$$\pi N \rightarrow (\pi \pi) + N_f$$  \hspace{1cm} (2.2)

where we allow for the possibility that the final baryon may differ from the initial one. Much of the kinematical notation is summarized in Fig. 2.1.

![Kinematic diagram](image)

**Fig. 2.1** Kinematics of the process $\pi N \rightarrow 2\pi N_f$.
Symbols in parenthesis after the names of the particles are masses. The final $2\pi$-system has polar decay angles $(\theta, \phi)$ in some appropriate system (see text). The total c.m. energy squared is $s$, and $t$ is the square of the momentum transfer from the initial to the final baryon.

Initial and final c.m. momenta (for the over-all reaction) $q_i$ and $q_f$ are given by

$$4s q_i^2 = \left[ s - (M + \mu)^2 \right] \left[ s - (M - \mu)^2 \right]$$

$$4s q_f^2 = \left[ s - (M_f + M_{\pi\pi})^2 \right] \left[ s - (M_f - M_{\pi\pi})^2 \right].$$  \hspace{1cm} (2.3)
These are related to $t$ and $s$ via the c.m. scattering angle $\theta_s$ (note that $t$ and $s$ in this section have a different meaning from the one in Section 1, where they referred to the $\pi\pi$ scattering process itself):

$$t = -q_i^2 - q_f^2 + 2q_i q_f \cos \theta_s + \frac{1}{4s} \left(M_i^2 - \mu_i^2 - M_f^2 + M_{\pi\pi}^2\right)^2.$$  

(2.4)

The boundary $-t_{\min} \leq t \leq -t_{\max}$ of the physical region for fixed incident energy is given by Eq. (2.3), putting $\cos \theta_s = 1$ for $t_{\min}$ and $\cos \theta_s = -1$ for $t_{\max}$.

Clearly

$$\sin \frac{\theta_s}{2} = \left\{ \frac{t_{\min} - t}{t_{\min} - t_{\max}} \right\}^{1/2},$$

(2.5)

$$\cos \frac{\theta_s}{2} = \left\{ \frac{t - t_{\max}}{t_{\min} - t_{\max}} \right\}^{1/2}.$$  

For large $s$ or lab. momentum $p_L$ we have approximately

$$t_{\min} \approx \begin{cases} -\frac{\left(M_{\pi\pi}^2 - \mu_i^2\right)^2}{4p_L^2} + O(p_L^{-3}) & \text{for } M_f = M \\ -\frac{\left(M_i^2 - M^2\right)\left(M_f^2 - \mu_i^2\right)}{2Mp_L} + O(p_L^{-2}) & \text{for } M_f \neq M \end{cases}$$

(2.6)

For the CERN–Munich experiment\(^{19-22}\) on $\pi^- p \rightarrow \pi^+ \pi^- n$ at $p_L = 17.2$ GeV/c (CM) and for the Berkeley experiment on $\pi^+ p \rightarrow \pi^+ \pi^- \Delta^{++}$ at $p_L = 7.1$ GeV/c (BKLY)\(^{36}\) we find

$$M_{\pi\pi} = 1.6\, \text{GeV} \Rightarrow -t_{\min} / \mu_i^2 = \begin{cases} 0.04 \ (\text{CM}) \\ 3.2 \ (\text{BKLY}) \end{cases}$$

$$M_{\pi\pi} = 2.6\, \text{GeV} \Rightarrow -t_{\min} / \mu_i^2 = \begin{cases} 0.7 \ (\text{CM}) \\ 2.43 \ (\text{BKLY}) \end{cases}$$

(2.7)

This illustrates one drawback of using reactions with $\Delta$ in the final state: the forward direction is comparatively far removed from the one-pion exchange (OPE) pole at $t = \mu_i^2$. Also the spread in $\Delta$ introduces a lack of definition of $t_{\min}$.

2.2 Chew-Low-Goble extrapolation\(^{35}\)

The objective in studies of dipion production experiments is to somehow isolate the OPE contribution (Fig. 2.2). Independently of dynamical complication, the amplitude, analytically continued outside the physical region, has an OPE pole at $t = \mu_i^2$. The residue of that is given exactly by the on-shell $\pi\pi$ scattering amplitude and the coupling $\pi NN_f$.  

If OPE was the only reaction mechanism, we would have for the intensity

\[
I(s,t,M_{\pi\pi},\Theta,\varphi) = \frac{\partial \sigma}{\partial M_{\pi\pi} \partial t \partial S^2_{\pi\pi}} = \frac{2}{4\pi s q_\pi^2} \frac{G^2}{4\pi} \frac{-t}{(t-m^2)^2} \left( \frac{d\sigma}{dS^2}_{\pi\pi} \right)
\]

(2.8)

for the reaction $\pi N \rightarrow 2\pi N$, and

\[
I(s,t,M_f,M_{\pi\pi},\Theta,\varphi) = \frac{\partial \sigma}{\partial M_{\pi\pi} \partial t \partial M_f \partial S^2_{\pi\pi}} = \frac{2}{4\pi^3 s q_{\pi N}^2} \frac{q_{\pi N}^2}{q_{\pi N}^2} \frac{M_f^2 \sigma(M_f)}{(t-m^2)^2} \left( \frac{d\sigma}{dS^2}_{\pi\pi} \right)
\]

(2.9)

for $\pi N \rightarrow 2\pi \Delta$.

Here $q_{\pi\pi}$ is the break-up momentum of the dipion system in its own c.m. Similarly, $q_{\pi N}$ is the break-up momentum of the $\Delta$ in the $\Delta$ rest system; $\sigma(M_f)$ is the $\pi N$ cross-section at $\pi N$ energy $= M_f$ (in the $\Delta$ region).

Equations (2.8) and (2.9) represent exact relations for the intensity analytically continued in $t$ to $t = \mu^2$. Thus we may attempt to analytically continue the quantity $(t-m^2)^2$. Or, equivalently, multiply Eqs. (2.8) and (2.9) by form factors and fit for $t$-physical, using, however, the result for $t = \mu^2$.

This process of analytic continuation of experimental information is a delicate one and can be expected to work reasonably only when the OPE contribution shows up clearly in the data. As we shall see, this is indeed the case. However, background effects are expected to be complicated enough that a more detailed type of analysis is desirable. But, it seems worth emphasizing that if the $t$-dependence can be determined with adequate accuracy, the procedure would yield unambiguous determinations of the $\pi N$ cross-section.
The following comments are in order:

- For the \( \pi N \rightarrow \pi \pi N \) reaction, the OPE signal vanishes at \( t = 0 \), a point between the physical region and \( t = \mu^2 \). This results in background effects playing a crucial role for very small \( t \). It is therefore to be feared that the \( t \)-behaviour is not sufficiently smooth for simple extrapolations to work accurately.

- A great advantage of the "\( \Delta \) reactions" \( \pi N \rightarrow \pi \pi \Delta \) is that no such suppression of the OPE signal takes place. The standard procedure for those reactions, then, is to use some kind of extrapolation \( [\text{for a discussion of different extrapolation strategies and further references, see Refs. 6, 36, and 37}] \). The main worry is that the Chew-Low boundary lacks definition and does not come close to the OPE pole \( [\text{Eqs. (2.7)}] \).

The CERN-Munich experiment on \( \pi^- p \rightarrow \pi^+ \pi^- n \) at 17.2 GeV offers sufficient information for a potentially much richer type of analysis to be attempted. The different spin-structure of OPE and background can be exploited to obtain information on the background contributions themselves. Thus the OPE-signal gets "purified" and its very characteristic \( t \)-dependence hopefully allows a much safer extrapolation. This procedure is equivalent to an amplitude analysis of the production process. Such amplitude analyses have serious ambiguity problems of their own, as we shall see. As emphasized, however, provided the \( t \)-dependence is properly exploited, the data do in fact determine the \( \pi \pi \) cross-section in principle.

In practice it will prove useful to use models for the reaction mechanisms as a guide to how unobserved contributions should be parametrized. Having outlined this, we shall try to point to the most important biases which are hereby introduced, and to see how some of them could be removed by further experimental studies.

2.3 Amplitudes for dipion production

It is customary to describe spin correlations by helicity amplitudes \( H_{\lambda, \lambda'} \), where \( \lambda' \) and \( \lambda \) are final and initial nucleon helicities. Restricting ourselves to reactions of the type \( \pi N \rightarrow 2 \pi N \), the intensity may be written

\[
I(s, t, M_{\pi \pi}, \Theta, \phi) = \frac{1}{2} \sum_{\lambda' \lambda} \left| H_{\lambda', \lambda} \right|^2,
\]

which defines the normalization of the helicity amplitudes. Equation (2.10) applies to experiments on an unpolarized target and with undetected final nucleon polarization. In Section 2.6 we shall briefly comment on the important case of experiments on a polarized target now being analysed.

*) For a complete account, see Ref. 32, the conventions of which we have largely adopted.
The dipion polar angle dependence is made explicit by further expanding on \( \pi \pi \) partial waves. For \( \pi \pi \) angular momentum \( \ell \) and helicity \( m \) we denote the production amplitude by \( L^{\lambda', \lambda, m}_{\lambda, \ell, \ell} \) (i.e., \( S_{\lambda', \lambda, 0} \) for \( \ell = 0 \), \( P_{\lambda', \lambda, m} \) for \( \ell = 1 \), etc.). Then

\[
\hat{H}^{\lambda', \lambda, m}_{\lambda, \ell, \ell}(\theta, \varphi) = \sum_{\ell = 0}^{\infty} \sum_{m = -\ell}^{\ell} (2\ell + 1)^{1/2} L^{\lambda', \lambda, m}_{\lambda, \ell, \ell}(\theta) e^{im\varphi}
\tag{2.11}
\]

\[
I(\theta, \varphi) = N \sum_{\ell, \ell', m, m'} (2\ell + 1)^{1/2} (2\ell'+1)^{1/2} \int_{m, m'} L^{\ell, \ell'}_{m, m'}(\theta) \varphi_{m, m'}(\varphi) e^{im\varphi} e^{im'\varphi}
\tag{2.12}
\]

with the (normalized) density matrix elements defined by

\[
\rho^{\ell, \ell'}_{m, m'} = \frac{1}{2N} \sum_{\lambda, \lambda'} L^{\ast}_{\lambda, \lambda', \ell}(\theta) \cdot L^{\lambda', \lambda, \ell'}_{m, m'}(\theta)
\tag{2.13}
\]

and

\[
N = \frac{\partial^{2} \sigma}{\partial M_{\pi \pi} \partial t}
\]

The density matrix elements are not all observable. A complete set of observables is provided by the moments \( \langle \Re Y^{m}_{\ell} \rangle \) or \( \langle Y^{m}_{\ell} \rangle \) for short (we use the so-called unnormalized moments):

\[
I(\theta, \varphi) = \sum_{\ell} \left\{ \langle Y^{0}_{\ell} \rangle \Re Y^{0}_{\ell}(\theta, \varphi) \\
+ 2 \sum_{m = 1}^{\ell} \langle Y^{m}_{\ell} \rangle \Re Y^{m}_{\ell}(\theta, \varphi) \right\}.
\tag{2.14}
\]

The moments may be expressed as sums of density matrix elements using standard addition theorems for spherical harmonics. Examples will be given below.

To give full meaning to the above formulae, we must specify the coordinate frame in which helicities are defined. Two frames are of special interest: the s-channel or helicity frame, and the t-channel or Gottfried-Jackson frame. The polar angles \( (\theta, \varphi) \) refer to a coordinate system in the rest frame of the dipion system. The y-axis is always taken normal to the over-all reaction plane. In the s-channel frame then, \( \hat{z} \) is the direction of "\( 2\pi \)" as seen in the s-channel c.m. In the \( 2\pi \) rest system this is the direction opposite to the outgoing baryon.

In the t-channel frame, \( \hat{z} \) is likewise opposite to the direction of the incoming \( \pi \).

The t-channel frame is the natural one for describing \( \pi \pi \) scattering and for describing definite t-channel exchange mechanisms.

The s-channel system, on the other hand, is thought to be the system in which important corrections arising from unitarity (absorption) are most naturally described.
It is convenient for the description of t-channel exchanges to introduce the following notation (be it in the s-channel or in the t-channel):

\[
\mathcal{L}^{(\pm)}_{\lambda', \lambda, m} \equiv \frac{1}{\sqrt{2}} \left[ \mathcal{L}^{(\lambda', \lambda, m)}_{m} \mp (-)^m \mathcal{L}^{(\lambda', -\lambda, -m)}_{-m} \right] \quad \text{for } m \neq 0
\]  \hspace{1cm} (2.15a)

and

\[
\mathcal{L}^{(-)}_{\lambda', \lambda, 0} \equiv \mathcal{L}^{(\lambda', -\lambda, 0)}_{-\lambda, 0} \quad \mathcal{L}^{(+)}_{\lambda', \lambda, 0} \equiv 0.
\]  \hspace{1cm} (2.15b)

For \( t \) fixed these can be shown\(^{38}\) asymptotically (i.e. for \( s \rightarrow \infty \)) to be dominated by specific naturality in the t-channel: natural parity exchange for \( \pi^+ \) and unnatural parity exchange for \( \pi^- \). These properties are valid equally in the s-channel and in the t-channel.

The nucleon helicities can assume four sets of values: \( ++, +- , -+, -- \) (short for \( +\frac{1}{2}, +\frac{1}{2} \), etc.). However, parity invariance imposes the constraint

\[
\mathcal{L}^{(\lambda', \lambda, m)} = (-)^{\delta_{m,0}} \mathcal{L}^{(-\lambda', -\lambda, -m)}
\]  \hspace{1cm} (2.16)

So we need only consider \( \lambda' = \frac{1}{2} \). This allows us to work only with two-component complex "vectors" in nucleon spin-space:

\[
\mathcal{L}^{(\lambda, m)} = \left( \frac{\mathcal{L}^{(+, +, m)}_{++, m}}{\mathcal{L}^{(+, -, m)}_{++, m}} \right).
\]  \hspace{1cm} (2.17)

We define the four-dimensional Euclidean metric by

\[
|| \mathcal{L}^{(m)}_m \cdot \mathcal{L}^{(m')}{m'} || = \Re \left( \mathcal{L}^{(K, +, +, m)}_{++, m} \cdot \mathcal{L}^{(K', +, +, m')}_{++, m'} + \mathcal{L}^{(K, +, -, m)}_{++, m} \cdot \mathcal{L}^{(K', +, -, m')}_{++, m'} \right).
\]  \hspace{1cm} (2.18)

It is important to be able to transform helicity amplitudes from the t-channel frame to the s-channel frame and vice versa. This is achieved by the crossing transformation, which we do not give in full generality (see Ref. 32). It is a direct product of operations acting on individual helicities, a statement true even for the definite naturality amplitudes. We confine ourselves to the simple case of dipion s- and p-waves, an acceptable first approximation for \( M_{\pi \pi} \lesssim 1 \text{ GeV} \), referring to Ref. 32 for full details in the general case.

Thus for dipion s-waves (for which obviously only unnatural parity exchange is allowed) we get (superscripts referring to s- and t-channel frames)

\[
\mathcal{S}^{(s)} = \mathcal{S}^{(1)} = \left[ \begin{array}{c} 
\mathcal{S}^{(1)(+)}_{++, 0} \\
\mathcal{S}^{(1)(-)}_{++, 0}
\end{array} \right] = -i \left[ \begin{array}{cc}
\cos \chi^{(-)} & \sin \chi^{(-)} \\
-\sin \chi^{(-)} & \cos \chi^{(-)}
\end{array} \right] \left[ \begin{array}{c}
\mathcal{S}^{(1)(+)}_{++, 0} \\
\mathcal{S}^{(1)(-)}_{++, 0}
\end{array} \right]
\]  \hspace{1cm} (2.19)
and for p-waves

\[ \mathcal{P}^{(s)(-)} = -i \mathcal{C}_{st} \otimes \begin{bmatrix} \cos \chi_{\pi \pi} & -\sin \chi_{\pi \pi} \\ \sin \chi_{\pi \pi} & \cos \chi_{\pi \pi} \end{bmatrix} \mathcal{P}^{(t)(-)} \]  

(2.20)

with \( \mathcal{C}_{st} \) as before acting on the nucleon helicities and the second matrix acting on the two \( \pi \pi \) helicity indices 0 and 1. Finally,

\[ \mathcal{P}^{(s)(+)} = -i \mathcal{C}_{st} \mathcal{P}_{1}^{(t)(+)} \]  

(2.21)

where only \( m = 1 \) occurs [see Eqs. (2.15)].

With

\[ \mathcal{C}_{st}^{(\pm)} \equiv \begin{bmatrix} \cos \chi^{(\pm)} & \sin \chi^{(\pm)} \\ -\sin \chi^{(\pm)} & \cos \chi^{(\pm)} \end{bmatrix} \]

the crossing angles are given by \(^{32}\)

\[ \cos \chi^{-} = \left( \frac{t_{\text{min}}}{t} \right)^{\frac{1}{2}} \cos \frac{\Theta_{s}}{2} \]

\[ \sin \chi^{-} = \left( \frac{t_{\text{max}}}{t} \right)^{\frac{1}{2}} \sin \frac{\Theta_{s}}{2} \]

\[ \cos \chi^{(\pm)} = \left( \frac{4 M^2 - t_{\text{min}}}{4 M^2 - t} \right)^{\frac{1}{2}} \cos \frac{\Theta_{s}}{2} \]

\[ \sin \chi^{(\pm)} = -\left( \frac{4 M^2 - t_{\text{max}}}{4 M^2 - t} \right)^{\frac{1}{2}} \sin \frac{\Theta_{s}}{2} \]  

(2.22)

For \( \chi_{\pi \pi} \) we give only the asymptotic result for \( s \to \infty \):

\[ \cos \chi_{\pi \pi} \approx \frac{t + M_{\pi \pi}^2 - \mu^2}{\sqrt{t}} \]

\[ \sin \chi_{\pi \pi} \approx \frac{-2 M_{\pi \pi} \sqrt{t}}{\sqrt{r}} \]  

\[ \sqrt{r} \equiv (t - (M_{\pi \pi} + \mu)^2)(t - (M_{\pi \pi} - \mu)^2). \]  

(2.23)

The crossing transformation on the nucleon helicities has the following "anomalous" property: In the forward direction \( (\Theta_{s} = 0, t = t_{\text{min}}) \)

\[ \cos \chi^{(+)} = 1 \quad \sin \chi^{(+)} = 0 \]  

(2.24)

i.e. flip \( \leftrightarrow \) flip; non-flip \( \leftrightarrow \) non-flip. However, from Eqs. (2.5) and (2.18) we get at fixed \( t \) for \( s \to \infty \) [or equivalently for \( |t| \gg |t_{\text{min}}| \), which is true even for \( |t|/\mu^2 << 1 \) at large \( s \), see Eqs. (2.7)]:

\[ \cos \chi^{-} \approx \left( \frac{t_{\text{min}}}{t} \right)^{\frac{1}{2}} \approx 0 \quad \sin \chi^{-} \approx 1; \]  

\[ \cos \chi^{(+)\approx} \frac{2 M}{(4 M^2 - t)^{\frac{1}{2}} \approx 1; \sin \chi^{(+)} \approx \frac{-\sqrt{t}}{(4 M^2 - t)^{\frac{1}{2}} \approx 0}. \]  

(2.25)
Thus, except very close to the forward direction, the unnatural parity exchange amplitudes transform in an antidiagonal fashion for the nucleon spin:
flip ↔ non-flip. For the natural parity exchange amplitudes, on the other hand, the transformation continues to be approximately diagonal. For accurate numerical work, corrections to the above simplified picture must be retained.

Near the forward direction, helicity amplitudes have a characteristic behaviour described by the "half-angle factors" and expressing conservation of angular momentum along the direction of motion. Thus let \( n = \lambda - \lambda' + m \), then

\[
\angle(x', \lambda, m) \propto \left( \frac{t_{min} - t}{s} \right)^{\frac{1}{2}n}\tag{2.26}
\]

The analogous rules for the t-channel amplitudes are

\[
\angle(t, x', \lambda, m) \propto \left[ \cos \frac{\theta_{\pi}}{2} \right]^{|m + \lambda - \lambda'|} \left[ \sin \frac{\theta_{\pi}}{2} \right]^{|m - \lambda' + \lambda'|}\tag{2.27}
\]

with

\[
\left( \begin{array}{c}
\cos \frac{\theta_{\pi}}{2} \\
\sin \frac{\theta_{\pi}}{2}
\end{array} \right) \approx \left\{ \frac{1}{x} \left[ 1 \pm \left( \frac{t}{t_{min}} \right)^{\frac{1}{2}} \right] \right\}^{\frac{1}{2}}.
\]

This shows that t-channel helicity amplitudes have characteristic kinematical singularities for small \( t \). Provided they are handled consistently, they should introduce no essential complication. They cancel in observable quantities.

2.4 OPE and absorption

The one-pion exchange contribution gives rise to t-channel helicity amplitudes of the following form [see Eqs. (2.8), (1.16) and (1.17)]:

\[
\angle(t) = \frac{N^2}{\pi q_i \sqrt{q_{\pi}^2 (on)}} \frac{M_{\pi \pi}^2}{q_{\pi}^{(off)}} \left( 2\ell + 1 \right)^{\frac{1}{2}} \left( \frac{q_{\pi}^{(off)}}{q_{\pi}^{(on)}} \right)^{\ell}
\]

\[
\times \left( \frac{\sqrt{\ell}}{\ell - \mu^2} \right) \int_{\pi \pi} \tag{2.28}
\]

where

\[
4M_{\pi \pi}^2 q_{\pi}^2 \frac{q_{\pi}^{(off)}}{q_{\pi}^{(on)}} = \left[ M_{\pi \pi}^2 - (\sqrt{\ell} \mu)^2 \right] \left[ M_{\pi \pi}^2 - (\sqrt{\ell} \mu)^2 \right] = \ell^2 - (M_{\pi \pi}^2 \mu)^2 = \ell^2.
\]

[see Eq. (2.23)].
\[ T_{\pi\pi}^l \] is the on-shell \( \pi\pi \) scattering partial wave amplitude in Argand-diagram normalization. The factor \( [q_{\pi(\text{off})}/q_{\pi(\text{on})}]^2 \) is chosen to give the correct centrifugal barrier factor near \( q_{\pi(\text{off/on})} = 0 \). It behaves very badly for large \( t \) where it is customary to replace it by a Dürf-Pilkuhn form factor, for example^{39}. It is convenient to simplify the notation by writing

\[
L_{++0} = i \, G_{e}(M_{\pi\pi}) \left[ 2q_{\pi(\text{off})}^2 \right]^{\ell} \frac{\sqrt{-t}}{t - \mu^2} \frac{M_{\pi\pi}}{N q_{\pi(\text{on})}^2} \left[ 2q_{\pi(\text{on})}^2 \right]^{-\ell} G_{\piNN}^{(2\ell+1)} \sum_{\ell}^{(2.29)} \frac{1}{\ell!} M_{\ell}(t) \frac{d}{dt} \frac{1}{\sqrt{t}} \frac{\beta_{\pi\pi}^{(2\ell+1)}}{\beta_{\pi\pi}^{(2\ell+1)}} \]

The distinctive features of OPE are:

a) the pole \((t-\mu^2)^{-1}\);

b) the pure non-flip structure in the t-channel frame (this follows by considering the quantum numbers of the different NN helicity states);

c) the vanishing as \( \sqrt{-t} \) at \( t = 0 \), combining with the pion pole to give a dramatic and characteristic t-dependence;

d) only \( m = 0 \) \( \pi\pi \) helicities are produced in the t-channel, implying absence of any \( \phi \)-dependence in the Gottfried-Jackson frame (Treiman-Yang test), i.e. \( \langle Y_{M}^{2} \rangle = 0 \) for \( M \neq 0 \) in the t-channel.

![Graph](image)

**Fig. 2.3** Differential cross-section for the reaction \( \pi^- p \rightarrow \pi^+ \pi^- n \) at 17.2 GeV/c (Ref. 19-22) in the \( p \)-region, \( 0.71 \text{ GeV} < M_{\pi\pi} < 0.83 \text{ GeV} \). The dashed line is the function \( -t/(t-\mu^2)^2 \) in arbitrary normalization. The predicted maximum near \( \sqrt{-t} = \mu^* \) as well as the forward dip is clearly seen. The combination \( [\rho_{\pi\pi}^{(0)} + (1/3)\rho_{\pi\pi}^{(2)}] \frac{d\sigma}{dt} \) is directly obtainable from data and is supposed to contain less background (see text).
In Figs. 2.3 and 2.4 these properties are confronted with the results of the high statistics CERN-Munich 17.2 GeV/c experiment on $\pi^- p \rightarrow \pi^+ \pi^- n$ in the $\rho$ region of $M_{\pi\pi}$ (Refs. 19-22).

In $d\sigma/dt$ a very clear component with $t$-dependence $t/(t-\mu^2)^2$ at small $|t|$ is seen. Deviations from this $t$-dependence could be partly described by form factors. However, form factors would not suffice to account for the non-vanishing of $M \neq 0$ $t$-channel moments (Fig. 2.4).

![Diagram](image)

**Fig. 2.4** Unnormalized $t$-channel and $s$-channel moments of the CERN-Munich experiment at 17.2 GeV/c. Same $M_{\pi\pi}$ interval as in Fig. 2.3.
Thus, clear corrections to OPE are indicated experimentally. In fact, Eq. (2.28) is also quite unacceptable theoretically, in particular owing to an abnormally large s-channel s-wave (see below). This fact is exploited in absorption models where unitarity is restored by brute force. Most of the effect is achieved by the so-called Williams' prescription\(^\text{10}\) or poor man's absorption model \((\text{PMA})^\text{11}\). To study unitarity we must transform the OPE signal to the s-channel helicity frame. Also, absorption is commonly thought to be approximately diagonal in s-channel helicities.

The relevant formulae are given by Eqs. (2.21)-(2.25). At \(t \approx t_{\text{min}}\), OPE is pure non-flip in nucleon helicities in the s-channel just as in the t-channel. However, this is true only in a tiny region of size \(|t_{\text{min}}|\), which is almost negligible at 17 GeV/c \([\text{see Eq. (2.7)}]\). For just slightly larger \(-t\)-values, the nucleon helicity matrix is antidiagonal \([\text{Eq. (2.25)}]\) and the OPE signal is pure flip in the s-channel. (For accurate numerical work, corrections must be kept.) Remembering this and suppressing the nucleon helicities, we get for OPE contributions to \(\Pi\) s- and p-waves:

\[
\begin{align*}
S_0^{(s-y^-)} &= -G_0 \frac{\sqrt{-t}}{t - \mu^2} \\
P_0^{(s-y^-)} &= G_1 \frac{\sqrt{-t}}{t - \mu^2} \frac{t + M_{\pi\pi}^2}{M_{\pi\pi}} \\
P_1^{(s-y^-)} &= G_1 \frac{2(-t)}{t - \mu^2} \\
P_{-1} &= 0
\end{align*}
\]

(2.30)

for unnatural parity exchange, and

for natural parity exchange.

Equivalently,

\[
\begin{align*}
S_0^{(s)} &= -G_0 \frac{\sqrt{-t}}{t - \mu^2} \quad (n = 1, \sqrt{-t}) \\
P_0^{(s)} &= G_1 \frac{\sqrt{-t}}{t - \mu^2} \frac{t + M_{\pi\pi}^2}{M_{\pi\pi}} \quad (n = 1, \sqrt{-t}) \\
P_1^{(s)} &= -G_1 \frac{\sqrt{t}}{t - \mu^2} \quad (n = 0, 1) \\
P_{-1}^{(s)} &= +G_1 \frac{\sqrt{t}}{t - \mu^2} \quad (n = 2, -t)
\end{align*}
\]

(2.31)

The number \(n\) in parentheses is the net helicity flip. Also given is the expected kinematical behaviour near \(t = 0\) \([\text{Eq. (2.26)}]\). The vanishing as \(t\) of the \(n = 0\) amplitude is peculiar to OPE. Kinematically this behaviour is required only for
n = 2. However, it is forced upon $P_1(s)$ as well if there is no natural parity exchange [Eqs. (2.30) and (2.15)], $P_1(s)(\pm) = 1/\sqrt{2} [P_1(s) + P_1(s)] = 0$. The resulting form for $P_1(s)$ contains an s-wave violating the unitarity bound by an arbitrary amount as $s \to \infty$. This results from the t-independent part of $P_1(s)$, giving rise to an s-wave cross-section constant in energy, in contradiction to the unitarity bound, $\sigma_s \propto \lambda_s^2 \propto s^{-1}$.

The Williams' or the PMA prescription consists in removing that constant by substitution $t + \mu^2$ in $P_1(s)$ (and making the approximation elsewhere). This does not affect the $\pi\pi$ s-wave (!). For the $\pi\pi$ p-wave we obtain (nucleon flip throughout)

$$
\begin{align*}
P_0(s) &= G_1 \frac{\sqrt{-t}}{t-\mu^2} M_{\pi\pi} \\
P_1(s) &= -G_1 \frac{t+\mu^2}{t-\mu^2} + G_1 (C-1) \\
P_4(s) &= CG_4,
\end{align*}
$$

(2.32)

where PMA has $C \equiv 1$.

The specific predictions of this model are:

i) **Phase coherence:** The phase of all helicity amplitudes equal that of $G$, the $\pi\pi$ phase. Phase coherence would be destroyed, for example, if the absorptive "$\pi$-cut" contribution $C$ (so called for its possible relation to a $\pi \otimes F$ cut in the t-channel angular momentum plane) was complex.

ii) **Spin coherence:** All amplitudes, including the predicted non-OPE contributions, are pure flip in the s-channel.

iii) Definite predictions for the $M$-dependence of observed moments. As usual let us restrict ourselves to s- and p-wave $\pi\pi$ production. The angular $(\theta,\phi)$ distribution is then given by the following expressions, generally valid in the s-channel or in the t-channel frames:

$$
\frac{d^2 \sigma}{d \theta d \phi M_{\pi\pi}} = \left\| P_{\pi\pi}^{(s)} \right\|^2 + \left\| P_{\pi\pi}^{(p)} \right\|^2 + \left\| P_{\pi\pi}^{(t)} \right\|^2 = N
$$

$$
(\theta_{11}^{11} - \eta_{11}^{11}) N = \left\| P_{\pi\pi}^{(t)} \right\|^2 - \frac{1}{2} \left( \left\| P_{\pi\pi}^{(s)} \right\|^2 + \left\| P_{\pi\pi}^{(p)} \right\|^2 \right) = \sqrt{5\pi} < Y_2 >
$$

$$
\text{Re} \theta_{10}^{11} N = \frac{1}{\sqrt{2}} \left\| P_{\pi\pi}^{(t)} \cdot P_{\pi\pi}^{(t)} \right\|^2 = \sqrt{5\pi} < Y_2 >
$$

$$
\text{Re} \theta_{00}^{10} N = \left\| P_{\pi\pi}^{(t)} \cdot S_{\pi\pi}^{(t)} \right\|^2 = \sqrt{5\pi} < Y_1 >
$$

$$
\text{Re} \theta_{10}^{01} N = \frac{1}{\sqrt{2}} \left\| P_{\pi\pi}^{(t)} \cdot S_{\pi\pi}^{(t)} \right\|^2 = \sqrt{5\pi} < Y_1 > .
$$

(2.33)
The $\rho$'s are the density matrix elements, Eq. (2.13), normalized as
\[ \sum_{\ell, m} \rho_{\ell m}^{\ell m} = 1 \]  
(2.34)
or (using parity invariance)
\[ \rho_{00}^{00} + \rho_{00}^{11} + 2 \rho_{11}^{11} = 1. \]
(2.35)
The $(Y_L^M)$'s are the unnormalized moments, Eq. (2.14). The metric $\| \|$ was defined by Eq. (2.18).

Equations (2.33) involve no other assumptions than dominance of $s$- and $p$-wave production.

From Eqs. (2.32) with $C = 1$ follows characteristic relations between the different M-components $(Y_L^M)$ for every $L^{21,22}$. As shown by Fig. 2.5, data are described with quite remarkable success in the $\rho$ region. The $t$-dependence of

![Graph showing ratios of s-channel moments.](image)

Fig. 2.5 Ratios of s-channel moments as predicted by the Williams model (curves) and as measured in $\pi^- p + \pi^- p$ in the $\rho$-region by the CERN-Munich 17.2 GeV/c experiment (Refs. 21 and 43) (data points).
individual moments is not accounted for by OPE even after the PMA modification. A reasonable fit, however, is obtained by multiplying all helicity amplitudes by a common "collimation factor" such as

$$\Phi(t) = e^{b(t-\mu^2)}$$

(2.36)
or a more sophisticated one.

iv) An important prediction of the scheme concerns the OPE to "background" ratio. From Eqs. (2.32) we see that the "OPE amplitude" $P_0(s)(-)$ has a factor $M_\Pi^\Pi$ on it compared to the "non-OPE" amplitude $P_1(s)(-)$.

The effect of this is to enhance the OPE term at the higher values of $M_\Pi^\Pi$, thus making $\Pi\Pi$ analyses easier here. At low $M_\Pi^\Pi$, on the other hand, we may expect additional difficulties, especially for the $p$-wave. We shall come back to these points in Sections 3 and 4.

2.5 Amplitude analysis

We continue to base the arguments on the $s+p$ approximation [Eqs. (2.33)], referring to Ref. 32 for formulae for more general $x$-values.

For the unpolarized target experiments which we shall discuss in this subsection, we then have the six observable quantities $\langle Y^M_{LM} \rangle$ $0 \leq M \leq L \leq 2$. There are, however, 15 real parameters, plus one overall phase in the problem (!) corresponding to the eight complex amplitudes $(s^{(-)}_{++}, s^{(-)}_{+-}, s^{(-)}_{-+}, s^{(-)}_{--}, P^{(-)}_{++}, P^{(-)}_{+-}, P^{(-)}_{-+}, P^{(-)}_{--})$, for unnatural parity exchange, and $(P^{(+)}_{++}, P^{(+)}_{+-}, P^{(+)}_{-+}, P^{(+)}_{--})$ for natural parity exchange.

For the study of $\pi\pi$ scattering, we are predominantly interested in two of these (four real parameters), $s^{(-)}_{++}, s^{(-)}_{+-}$ in the $t$-channel. Even so, the problem is clearly severely underdetermined.

We have already emphasized that the $t$-dependence of the observables in principle contains enough information to allow an unambiguous determination of the $\pi\pi$ cross-section. Let us, however, proceed for the moment without exploiting this, and consider the $t$-dependence a posteriori.

Estabrooks and Martin suggested performing a model-dependent amplitude analysis, inserting ideas on the possible reaction mechanisms mainly as derived from the Williams' model but also from elsewhere in hadron phenomenology. Possible reaction mechanisms are:

i) Reggeized $\pi$-exchange with absorptive cuts;

ii) Reggeized $A_2$-exchange with absorptive cuts;

iii) unnatural parity exchanges other than OPE — especially to the $s$-channel non-flip amplitude; these are normally designated $A_1$-exchanges;

iv) $N^*$ formation.
Not all of these can be allowed for in unpolarized observables. The scheme of Estabrooks and Martin is as follows:

a) No assumption is made about the natural parity exchange amplitudes $P_1^{(+)}$, thus allowing Reggeized $A_2$-exchange.

b) The unnatural parity exchange amplitudes are assumed spin-coherent, in fact to be pure flip in the $s$-channel as expected from absorbed OPE. Violations by absorbed $A_2$-exchange are expected to be small. The assumption would be violated, however, by $A_1$-exchange.

c) $S_0^{(-)}$ and $P_0^{(-)}$ (the principal amplitudes of interest for $\pi\pi$ physics) are assumed to have the same production phase (in the $s$-channel). This would be violated by $N^*$ formation, a mechanism which is expected to be quite unimportant in the kinematically relevant regions. Phase coherence can be relaxed, however, for the relative production phase between $P_0^{(-)}$ and $P_1^{(-)}$. This corresponds to allowing a complex "cut" [the $C$ in Eqs. (2.32)]. Also an arbitrary value for $C$ can be allowed. If $C$ is complex it seems preferable to work in the $s$-channel. In fact, transforming Eqs. (2.32) (with $C \neq 0$) back to the $t$-channel we find (all amplitudes, except $P_1^{(t)(+)}$, are pure non-flip):

\[
\begin{align*}
P_0^{(t)(-)} &= P_0^{(t)\text{ (OPE)}} + i \frac{3\sqrt{-t}}{M_{\pi\pi}} \frac{G_{A^*}}{m_{\pi\pi}} C \\
P_1^{(t)(-)} &= -i G_{A^*} C \\
P_1^{(t)(+)} &= -i G_{A^*} C.
\end{align*}
\]

If $C$ is real, the extra term in $P_0^{(t)(-)}$ could easily be transferred to an effective multiplicative form factor. If $C$ is complex, on the other hand, it could interfere with OPE in $\| P_0^{(t)(-)} \|$ in an undesirable way. In the end, however, we shall see that $C$ real is in fact preferred.

In passing, we notice the specific prediction

\[
\| P_1^{(t)(-)} \| = \| P_1^{(t)(+)} \|
\]

of Eqs. (2.37). This would be violated by additional $A_2$-exchange contributions in $P_1^{(t)(+)}$. It predicts $\langle Y_2 \rangle = 0$ [see Eqs. (2.33)] in the $t$-channel, which in fact is very well satisfied at not too large $t$-values (see Fig. 2.4), thus indicating the unimportance of $A_2$-exchange for small $t$. This property will be exploited in Section 4.
Returning to the Estabrooks and Martin scheme, the assumptions imply
\[
\begin{align}
\|S_0^{-}\|^2 &= |S_{+,-,0}^{-}\|^2, \quad \|P_0^{-}\|^2 &= |P_{+,-,0}^{-}\|^2 \\
\|P_1^{-}\|^2 &= |P_{+,-,1}^{-}\|^2 \\
\|P_1^{-}\cdot S_0^{-}\|^2 &= \|P_1^{-}\|^2 \cdot \|S_0^{-}\|^2 \cos \phi \\
\|P_0^{-}\cdot S_0^{-}\|^2 &= \|P_0^{-}\|^2 \cdot \|S_0^{-}\|^2 \cos \Delta \\
\|P_1^{-}\cdot S_0^{-}\|^2 &= \|P_1^{-}\|^2 \cdot \|S_0^{-}\|^2 \cos (\phi - \Delta).
\end{align}
\] (2.38)

The four moduli and the two angles may then be determined from the six observables. Equations (2.33) in general have three discrete solutions, of which one is unphysical ($\|P_0^{-}\|^2 < 0$). Of the remaining two, "solution 1" is an approximately phase-coherent solution with $|\cos \phi| = 1$ and is eventually preferred as the physical solution (Section 3). The other, "solution 2", has a complicated $\cos \phi$ dependence on $t$. The results for the $\rho$ region are shown in Fig. 2.6.

Definite successes of the analysis, and therefore to some extent of the assumptions are:

i) The $t$-dependence of $\|P_0^{-}\|$ has the very characteristic $\sqrt{-t/(t-\mu^2)}$ form (collimated at larger $-t$) expected for an OPE amplitude. This in itself is weak evidence against $A_1$-exchange.

ii) The ratio $\gamma_s \equiv \|S_0^{-}\|/\|P_0^{-}\|$ is independent of $t$, i.e. $\|S_0^{-}\|$ is OPE dominated precisely as $\|P_0^{-}\|$. This makes an extrapolation to the pion pole quite convincing. Thus $\gamma_s$ is directly the ratio of physical $\pi\pi$ $s$- and $p$-waves.

Fig. 2.6 Estabrooks-Martin amplitude analysis in the $\rho$- region (Ref. 43).
iii) The t-dependences of $\| P_{1}^{(+)} \|$ and $\| P_{1}^{(-)} \|$ are very different but roughly as expected. $\| P_{1}^{(+)} \|$ is featureless but dominating the cross-section at large t. Similar analyses at other beam energies (Refs. 47, 48) indicate that the energy dependence is consistent with $A_{2}$-exchange. The feature in $\| P_{1}^{(-)} \|$ near $t = -\mu^{2}$ is reminiscent of the zero predicted by PMA Eqs. (2.32).

iv) The relative phase $\Delta$ between $S_{0}^{-}$ and $P_{0}^{-}$ gives, in principle, an independent check on the s- and p-waves found from $P_{0}$ and $\gamma_{8}$ (and therefore also on the assumptions of the scheme), provided $\pi\pi$ scattering is elastic in the $\rho$ region (which we think it certainly is, see Section 3). In practice the constraint is not too stringent.

2.6 Discussion: Polarized target experiments

What are the systematic biases introduced by the model assumptions? Interesting clarifying remarks have been provided by Donohue and by Morgan $^{3,5,6,7,8,9}$. Suppose for a moment one more observable was available. Then one could obtain a completely model-independent determination of the Euclidean moduli $\| S_{0}^{(-)} \|$, $\| P_{0}^{(-)} \|$, $\| P_{1}^{(-)} \|$, $\| P_{1}^{(+)} \|$ as well as of the angles between the unnatural parity exchange amplitudes $S_{0}^{(-)}$, $P_{0}^{(-)}$, and $P_{1}^{(-)}$, always in the metric (2.18). In Eqs. (2.38) this would correspond to retaining the first three equations and changing the last equation into

$$\| P_{1}^{(-)*} \cdot S_{0}^{(-)} \| = \| P_{1}^{(-)} \| \cdot \| S_{0}^{(-)} \| \cos X$$  \hspace{1cm} (2.39)

The specific Estabrooks and Martin assumption is

$$\cos X = \cos (\varphi - \Delta)$$  \hspace{1cm} (2.40)

which implies that three amplitudes $S_{0}^{-}$, $P_{0}^{-}$, and $P_{1}^{-}$ lie in a two-dimensional plane. In general, of course, $\cos X$ is an independent parameter subject only to certain "triangular-type" inequalities, and $S_{0}^{-}$, $P_{0}^{-}$, and $P_{1}^{-}$ span a full three-dimensional subspace. It can be shown that the degeneracy introduced by the assumption (2.40) implies that the two solutions, 1 and 2, of Fig. 2.6 (and of Fig. 3.3) give extremal values for $\| S_{0}^{(-)} \|$, maximal for the phase-coherent solution 1 and minimal for solution 2.

It is not clear that all those intermediate possibilities would have satisfactory t-dependences. In fact, if data were accurate enough, we know that only one solution is allowed. However, since both solutions 1 and 2 do have reasonable t-dependences at the present level of accuracy, it is doubtful that intermediate ones could be ruled out.
Further, having obtained the amplitude modulus of $\| S_o^{(-)} \|$, say, what is the effect of the assumed absence of $A_1$-exchange? Since in the scheme we ascribe the full amount to the pure flip OPE amplitude, again the presence of an $A_1$-exchange no-flip amplitude would bias the result towards too high values of the $\pi\pi$ s-wave.

It is at least interesting to be able to realize that all biases of the model assumptions (restricting ourselves now to the phase-coherent case of solution 1) go in the same direction: towards too large phase shifts.

Before the advent of amplitude analysis, the full set of observables (2.33), "polluted" as they were by non-OPE terms, were not used. Instead one restricted oneself to the "clean" quantity

$$\langle Y^{0}_1 \rangle = \frac{1}{N_{\pi}} \| P^{(-)*}_o \cdot S^{(\cdots)}_o \|$$

involving OPE amplitudes only. Assuming a reasonable form (Breit-Wigner or such) for the p-wave, this gave rise to the famous up-down ambiguity. The principal objective of amplitude analysis was to resolve this (at the level of $\pi^+\pi^-$ physics; for other ways of solving it, see Section 3) by separating "depolarized" $\varphi$-effects ($P^{(-)}_1$, $P^{(\ast)}_1$) from OPE. The disappointing fact discovered by Estabrooks and Martin is that the separation can be done in several (at least two) ways, effectively making an up-down ambiguity for the s-wave reappear, albeit of a very different origin. In Section 3 we shall discuss physical ways of getting rid of this.

A potentially very important breakthrough in our understanding of production mechanisms, and therefore in our understanding of biases in $\pi\pi$ phase shift analysis, could be introduced by the advent of polarized target experiments, now in advanced stages of analysis$^{52}$. We shall restrict ourselves to the case of a transversely polarized proton target for the reaction $\pi^- p \rightarrow \pi^+ \pi^- n$ at 17.2 GeV/c and with no polarization detection of the final neutron. This is the case studied by the CERN-Munich group.

A very neat way of describing the potential of the experiment is in terms of transversity amplitudes$^{32,52,53}$. These refer to transitions between states in which the initial and final nucleon are polarized along the normal of the over-all reaction plane. Transversity amplitudes have very simple properties under crossing and parity.

There are two possible nucleon spin configurations (analogous to flip and no-flip) which we designate by subscripts $g$ and $h$ (Ref. 52). The transversity amplitudes are then given by the helicity amplitudes as

$$L^{(\pm)}_{g,m} = \frac{1}{\sqrt{2}} (L^{(+)}_{++,m} \pm i L^{(+)}_{+-,m})$$

(2.41)
At the s+p level there are now 15 (!) observables instead of 6 as in the unpolarized target case. Only 14 of these, however, are independent. We refer to Refs. 32 and 52 for expressions for the observables. It can be shown that the experiment determines the set of $L^g$ amplitudes uniquely up to one over-all phase, i.e. all moduli and all relative phases. Likewise, the set of $L^h$ amplitudes is determined to the same extent. In addition, there are discrete ambiguities corresponding to multiplying any set by $-1$.

We can see from Eqs. (2.41) that the assumption of pure flip implies that the sets $L^g$ and $L^h$ are identical (up to the ambiguity described). This in turn implies no dependence on the orientation of the proton polarization. More generally, spin coherence means that the $L^h$ set is a similarity transformation of the $L^g$ set. All these assumptions may now be tested. Preliminary analysis suggests they are violated, and a clear $A_1$-exchange signal is seen at small $t$ (Ref. 52).

In the end we would like to work with the helicity amplitudes rather than with the transversity amplitudes, since the former are related more directly to our ideas about reaction mechanisms. Unfortunately, the helicity amplitudes still have a rather complicated relation to the observables. We therefore have to resort to semi-model-dependent parametrizations again.

A very convenient starting point in this respect seems to be the scheme proposed some time ago by Froggatt and Morgan. Apart from the assumption that non-OPE amplitudes have a much smoother $t$-dependence than OPE ones, the scheme is quite general. With data from polarized targets, the full potential of the scheme could be exploited.

3. **The Elastic Region and the $K\bar{R}$ Threshold Region**

3.1 **Survey of results below 1 GeV**

The most detailed studies of $\pi\pi$ scattering have been concerned with non-exotic channels, in particular $\pi^+\pi^-$ scattering. Experimentally the $I = 2$ channel is much less interesting. As we shall see repeatedly, however, reasonable $I = 2$ amplitudes are crucial for being able to make quantitative assessments.

Figure 3.1 shows results of a few recent studies. The largest single one appears to be that of the CERN-Munich Group [Hoogland et al.\(^{55}\)] on $\pi^+p \rightarrow \pi^+\pi^-n$ at 12.5 GeV/c. Using $\approx 17,000$ events, results were obtained using a simple extrapolation to the $\pi$-pole, and incorporating absorption effects by the Williams' prescription (Section 2). Similar techniques were employed for the other studies indicated. With the possible exception of the results of Prukop et al.\(^{57}\) at very low energies, the over-all agreement is reasonable. In general analyses, assume $I = 2$ $\pi\pi$ scattering is described by elastic $s$- and $d$-waves. Above
Fig. 3.1 Measurements of $\delta_0^2$ and $\delta_2^2$, the exotic s- and d-wave phase shifts. Curves are predictions of $\delta_2^2$ based on the BFP study of the Roy equations (see Section 3.4).

$M_{\pi\pi} = 1$ GeV, say, inelasticities could well set in. Little is known about them, and they cannot be obtained from a $\pi\pi$ study alone. We return to the important phenomenon of phase-shift ambiguities in inelastic regions in Sections 4 and 5.

Figure 3.2 shows results for the p-wave $\delta_1^1$. Plotted are the Berkeley findings\(^{1b}\) as well as two early analyses of the CERN-Munich experiment\(^{29,44}\). This should give a rather conservative idea of the current uncertainties. Four parameters suffice to describe our knowledge of $\delta_1^1$ below 1 GeV: the scattering length $a_1^1$ (to which we come back in detail in Section 3.4), $m_\rho$, $\Gamma_\rho$, and $\delta_1^1$ at, say, 900 MeV. The newest PDG parameters for the $\rho$ are \(^{1b}\)

$$m_\rho = 773 \pm 3 \text{ MeV}, \quad \Gamma_\rho = 152 \pm 3 \text{ MeV}.$$

Individual determinations are not, in fact, consistent to that level of accuracy (for determination of $\rho$ parameters in other reactions, see PDG, Ref. 14). At 900 MeV, $\delta_1^1$ is somewhat smaller than expected from a simple Breit-Wigner

$$\delta_1^1 (900\text{ MeV}) \approx 147^0 - 151^0.$$

The main interest in $\pi\pi$ phase-shift analysis below 1 GeV has been in clarifying the $I = 0$ s-wave and the question of scalar resonances.

Figure 3.3 shows results of several analyses of the CERN-Munich experiment (Refs. 19-22, 44 and 46). The most recent findings\(^{46}\) are indicated by the band.
Fig. 3.2 The p-wave phase shift from Ref. 34 (squares), Ref. 44 (dots), and Ref. 20 (crosses).

Fig. 3.3 Various determinations of $\delta^p_1$ based on the CERN-Munich experiment (Refs. 19-22): Grayer et al. (Ref. 19), Estabrooks et al. (Ref. 44), with two solutions (EM1 and EM2), and finally Estabrooks and Martin (Ref. 46), the band indicating the average of their results from analysing the s-channel and the t-channel moments.
This agrees quite well with the very first analysis of the data by Grayer et al.\textsuperscript{19} based on a comparatively simple pole-extrapolation of the $t$-channel $M = 0$ moments (see Section 2). It also agrees very well with the results of the Berkeley group\textsuperscript{34} using a similar extrapolation technique, which should be much safer in their data on $\pi^+p \rightarrow \pi^+\pi^-\Delta^{++}$. (See the discussion in Section 2.) It thus appears at long last that a satisfactory consensus regarding $\delta_8$ has been reached.

Ironically, however, the first attempts at a more sophisticated amplitude analysis by Estabrooks and Martin produced great confusion, as described in Section 2. Their original solutions 1 and 2 are shown in Fig. 3.3 as well. Solution 2 was subsequently discarded owing to inconsistency with the $\pi^0\pi^0$ data of Apel et al.\textsuperscript{59} (Fig. 3.4). We shall come back to the final corrections of solution 1 below.

![Mass Distribution Diagram](image)

Fig. 3.4 The histogram is the $\pi^0\pi^0$ mass spectrum for $2\mu^2 < -t < 8\mu^2$ from $\pi^-p \rightarrow \pi^0\pi^0n$ at 8 GeV (Ref. 59). The circles (triangles) are the shape of the spectrum calculated from the $\pi\pi$ phases of EM1 (EM2), respectively, of Fig. 3.3 (Ref. 44). The scale is arbitrary.

3.2 The $S^*$ effect: How about $\epsilon$'s?

An alternative way of ruling out solution 2 would be to make use of the $S^*$ effect. A resolution of the similar old up-down ambiguity was given some time ago by the Berkeley group\textsuperscript{34} in what must be considered an important breakthrough in our understanding of $\pi\pi$ scattering. We briefly review their argument.
Data were used on the reactions

\[ \pi^+ p \rightarrow \pi^+ \pi^- \Delta^{++} \]
\[ \rightarrow K^+ K^- \Delta^{++} \]
\[ \rightarrow \pi^+ \pi^- \pi^0 \pi^- \Delta^{++} \]
\[ \rightarrow \pi^+ \pi^- (\Delta \Delta) \Delta^{++} \]

at 7.1 GeV/c.

First, Fig. 3.5 shows the important absence of any large \(4\pi\) inelasticity below 1 GeV. The data with \(\Delta^{++}\) selected and \(|t'_p| < 0.1\) GeV\(^2\) are shown. Apart

![Graph](image)

**Fig. 3.5** Mass distribution for \(\pi^+ \pi^+ \pi^- M(\Delta^{-})\) (missing mass \(\geq 2\pi^0\)) in 20 MeV bins from Ref. 34. Cuts corresponding to \(\Delta^{++} + \pi^+ p\) and to \(|t'_p| = |t_p - t_{p(\text{miss})}| < 0.1\) GeV\(^2\) were performed.

from a small bump above the \(\omega \pi\) threshold, no inelasticity is seen below 1 GeV. The authors estimate \(|1 - \frac{T}{T_L}|\) to be less than 2%. For the rest we shall neglect \(4\pi\) inelasticity below 1 GeV.

Figure 3.6 shows the \(\pi^+ \pi^-\) cross-section with a drop before the \(K\bar{K}\) threshold roughly the size of the \(s\)-wave unitarity limit. Also shown is the sharp rise of \(K\bar{K}\) production above threshold. Finally, Fig. 3.7 shows the normalized \(\langle \gamma \rangle\), essentially \(s\)-\(p\) interference, exhibiting a spectacular drop to zero at the \(K\bar{K}\) threshold.

All of this is fully confirmed (and with greater precision) in the CERN-Munich experiments on \(\bar{p} p \rightarrow \pi^+ \pi^- n\) at 17.2 GeV/c and on \(\bar{p} p \rightarrow K^+ K^- n\) at 9.8 GeV/c (Ref. 60). The result of a coupled channel \(\pi\pi \rightarrow \pi\pi\) and \(\pi\pi \rightarrow K\bar{K}\) K-matrix analysis by the CERN-Munich Group is shown for the \(I = \frac{3}{2} = 0\) \(\pi\pi\) Argand diagram in Fig. 3.8.
Fig. 3.6 Cross-sections from Ref. 34 from the reactions $\pi^+ p \rightarrow \pi^+ \pi^- \Delta^{++}$ and $\pi^+ p \rightarrow K^+ K^- \bar{\Delta}^{++}$. Curves are from Ref. 34.

Fig. 3.7 Normalized $\langle Y_1^0 \rangle$ from Ref. 34.

The steps in the qualitative arguments\textsuperscript{34}) leading to the behaviour shown in Fig. 3.8 are as follows:

i) The sudden rise of $K\bar{K}$ production (Fig. 3.9) indicates a sudden drop in $I = \lambda = 0$ $\pi\pi$ elasticity $\eta_0^2$ [Eq. (1.23)]:

$$
\sigma_0^{(0)}(\pi\pi \rightarrow K\bar{K}) = \frac{2\pi}{\sqrt{t}} \left( I - (\eta_0^2)^2 \right). 
$$

(3.1)
Fig. 3.8 Argand diagram for the $I = 0$ s-wave obtained from the coupled channel $\pi\pi + \pi\pi$ and $\pi\pi + KK$ analysis of Ref. 60.

Fig. 3.9 The cross-section $\pi^+\pi^- + K^+K^-$ between threshold and 1300 MeV from Ref. 60. Error bars: results of (two different) pole extrapolations from $\pi^-p + K^+K^-n$ at 9.8 GeV. Solid curve: result of energy-dependent fit of Ref. 60. Also shown is the unitarity limit for the s-wave cross-section.
The threshold behaviour ($\alpha q_K \equiv \text{c.m. } K \text{ momentum}$) indicates that the effect is in the $s$-wave. The $S$-matrix element
\[ S^0_\infty (M_{\pi\pi}) = \eta_0^0 \xi^0_0 \left( 2i \delta^0_0 - \frac{i}{2} M_{\pi\pi} \eta_0^0 \right) \] is an analytic function of $M_{\pi\pi}$ with a square root singularity ($\alpha q_K$) at $M_{\pi\pi} = 2m_K$. Hence a sharp drop in $\eta^0_0$ above $M_{\pi\pi} = 2m_K$ must be accompanied by a sharp rise in $\delta^0_0$ just below $M_{\pi\pi} = 2m_K$.

ii) The fact that $\sigma_{\pi\pi^-\pi^-}$ drops between 900 MeV and 990 MeV ($\simeq 2m_K$) implies that at 900 MeV the $\pi\pi$ $I = 0$ $s$-wave must be near the top of the Argand diagram, i.e. $\delta^0_0 \simeq 90^\circ$. This already rules out solution 2 of Fig. 3.3 [as indeed it ruled out the old up-solution $^{34}$ for $\delta^0_0$].

iii) The behaviour of ($Y^0_1$) is consistent with the above interpretation for a slowly varying $p$-wave with $\eta_1^1 \simeq 1$ and $\delta^1_1 \simeq 160^\circ$. The vanishing of ($Y^0_1$) at the $KK$ threshold implies that $\delta^0_0$ must be close to $180^\circ$ at $M_{\pi\pi} = 2m_K$.

Now let us consider the dynamical interpretation of the $S^*$ effect in some detail. We shall in particular be interested in the question of the occurrence of poles, their positions, and their sheets. We shall find an analysis in terms of $S$-matrix elements very attractive. As we shall see, this allows us to argue directly from features in the data to singularities of the physical amplitude. In contrast, the more standard analysis in terms of a $K$ matrix is such that the parameters only have a very complicated relation to experimental effects. Nevertheless, in the end we shall comment on some $K$-matrix studies.

Let us start by considering a purely elastic Breit-Wigner resonance. The amplitude (neglecting left-hand cuts) is an analytic function in the $s = M_{\pi\pi}^2$ plane cut along $(4\mu^2, \infty)$. Time reversal ($T$) invariance implies
\[ S(s*) = (S(s))^* \] on the physical sheet ($I$). Also, for $\text{Re} \ s > 4\mu^2$, $\text{Im} \ \varepsilon = \pm i\varepsilon$, unitarity gives
\[ S(s) = e^{\pm 2i\varepsilon}. \]

It is convenient to translate all of this to the $q = \frac{1}{2} (s-4\mu^2)^{1/2}$ plane, "uniformizing" the branch point $s = 4\mu^2$ ($q = 0$) and unfolding the two sheets: sheet I $\rightarrow$ Im $q > 0$; sheet II $\rightarrow$ Im $q < 0$. Thus,
\[ T\text{-invariance } \Rightarrow \quad S(-q*) = S(q)^*, \quad (3.3) \]
\[ \text{unitarity } \Rightarrow \quad S(-q) = S(q)^{-1}. \quad (3.4) \]

Equations (3.3) and (3.4) imply that if $S$ has a pole on the second sheet at $q = q_R$ (Im $q_R < 0$), then there is a pole also at $q = -q_R^*$, and zeros at $q = q_R^*$ and $q = -q_R$. 

Thus a resonance form is
\[ S_{BW}^{el}(q) = \frac{(q - q_R^*)(q + q_R)}{(q - q_R)(q + q_R^*)} \]  
(3.5)

which clearly satisfies the elastic unitarity requirement
\[ |S(q)| = 1 \]
for \( q \) real.

Equation (3.5) is easily rewritten in the Breit-Wigner form
\[ S_{BW}^{el}(q) = \frac{s - \mu^2 - i2\lambda T^q}{s - M^2 + i2\lambda T^q} \],  
(3.6)

where
\[
\begin{align*}
M &= 2 \left( |q_R|^2 + \mu^2 \right)^{1/2} \\
T^q &= -4\text{Im}q_R^*.
\end{align*}
\]  
(3.7)

For a narrow resonance this agrees approximately with the mass and width values defined by the pole-position,
\[ M - i \frac{T^q}{\lambda} = 2 \left( \frac{q^2_R + \mu^2}{\lambda} \right)^{1/2}. \]  
(3.7')

Equation (3.5) is easily modified by multiplying by background factors having \( |s_{BG}| = 1 \). This is a convenient way of preserving unitarity. In contrast, adding background terms to the \( T \)-matrix of Eq. (3.5) can easily spoil unitarity.

To study the \( S^* \) effect, we want to unfold all of the four sheets associated with the \( \Pi \) and the \( K\bar{K} \) thresholds. A simple way is by using the variable\(^{23,61,62}\):
\[ \omega \equiv \frac{q_{\Pi}^* + q_{K}}{\sqrt{m_K^2 - \mu^2}}, \]  
(3.8)

where
\[ q_{\Pi}^2 = \frac{1}{4} \left( M_{\Pi}^2 - 4\mu^2 \right), \quad q_{K}^2 = \frac{1}{4} \left( M_{K}^2 - 4m_K^2 \right). \]

A similar but more complicated variable allowing also for the left-hand cut was used elsewhere\(^{63,64}\) (see also Section 3.4 below). This mapping is illustrated in Fig. 3.10. Using arguments similar to the ones used before, we get
\[ T\text{-invariance: } S(-\omega^*) = S(\omega)^*, \]  
(3.9)
Fig. 3.10 Image of the four-sheeted $s = M^2_{\pi\pi}$ plane under the mapping Eq. (3.8). The physical region starts at $\omega = i$, runs along the circle to $\omega = 1$ (KK threshold), and continues along the positive real $\omega$-axis, $\omega > 1$ to $\omega = +\infty$. Poles (crosses) and zeros (small circles) corresponding to the fit of Ref. 62 are shown (see text).

Elastic unitarity below the $\bar{K}K$ threshold (i.e. $|S(\omega)| = 1$ for $|\omega| = 1$)

$$S(-\omega^{-1}) = S(\omega)^{-1}. \tag{3.10}$$

The important consequence again is the exact relationship between poles and zeros: a pole at $\omega = \omega_R$ is accompanied by a pole at $\omega = -\omega_R^*$ and by zeros at $\omega = -\omega_R^{-1}$ and $\omega = (\omega_R^{-1})^*$ (see Fig. 3.10).

From Fig. 3.8 we see that $S$ has an approximate zero ($\eta_0^0 \approx 0$) at $M_{\pi\pi} = 1020$ MeV, reflecting the fact that the cross-section $\sigma^{(0)}_0(\pi\pi \rightarrow K\bar{K})$ is near its unitarity limit at this energy (Fig. 3.9). By the above arguments this implies the existence of a pole at the same mass but near the borderline between sheets II and IV ($\omega_R^{II}$ on Fig. 3.10). In the $\omega$-plane the zero will have a small positive or negative imaginary part according as the $I = 0$ $\pi\pi$ partial wave passes to the right or to the left of the centre of the Argand diagram.

One easily sees that with a background factor of $S_{BG} = -1$ (corresponding to $\delta_{BG} = 90^\circ$), the pole-zero structure just described will qualitatively reproduce the observed behaviour of $\delta_0^0$ and $\eta_0^0$ very close to the $\bar{K}K$ threshold. We emphasize that it is the sharp well-defined drop in $\eta_0^0$ to nearly zero that has allowed the deduction of this (first) $S^*$-pole in $S_0^0$. 
For an inelastic resonance, however, we do expect a second $S^*$-pole on sheet $II^*$. This is most easily seen by generalizing the elastic Breit-Wigner formula Eq. (3.6) to the two-channel case

$$\sum_{BW}^{\text{inel}}(s) = \frac{M^2 - 2i t_{\pi} q_{\pi} + 2i t_{K} q_{K}}{M^2 + 2i t_{\pi} q_{\pi} + 2i t_{K} q_{K}}$$

(3.11)

where $\Gamma_\pi$ and $\Gamma_K$ are (suitable definitions of) partial widths into $\pi\pi$ and $KK$ (the signs of the $\Gamma_K q_K$ terms are required in order not to have physical sheet poles for $\Gamma_K > 0$). For $\Gamma_K = 0$, the 3rd-sheet pole $\omega_{R}^{III}$ is related to the 2nd-sheet pole by

$$\omega_{R}^{III} = (\omega_{R}^{II})^{-1}.$$  

(3.12)

This would imply the exact cancellation of any structure in $\delta_0$ and $\eta_0$, so it clearly has to be strongly violated. In general, if $\omega_{R}^{III}$ is independent of $\omega_{R}^{II}$, there are four real parameters in the problem (not counting background effects). The inelastic Breit-Wigner Eq. (3.11) only contains three, however. The dynamical significance of such a reduction in the number of parameters is unclear.

Morgan$^{65}$ and Fujii and Fukujita$^{62}$ pointed out seemingly clear evidence for the second 3rd-sheet pole. This is shown in Fig. 3.11 (from Ref. 62). The signal is the size of $\eta_0$ after 1100 MeV to near 1 after the drop: with one pole only, too much inelasticity is predicted.

In Table 3.1 we show several determinations of the sheet $II$-pole. Although we have argued that it is strongly suggested by present data, higher experimental accuracy is needed. This applies even more so to the sheet $III$-pole, which is further away and correspondingly more difficult to pin down.

<table>
<thead>
<tr>
<th>Table 3.1</th>
</tr>
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<tbody>
<tr>
<td>Sheet II pole parameters for the $S^*$ in the $E = M_{\pi\pi}$ plane. MeV units.</td>
</tr>
<tr>
<td>Re ($E_{S^*}^{II}$)</td>
</tr>
<tr>
<td>------------------</td>
</tr>
<tr>
<td>997 ± 6</td>
</tr>
<tr>
<td>982 ± 6</td>
</tr>
<tr>
<td>1012 ± 6</td>
</tr>
<tr>
<td>997</td>
</tr>
<tr>
<td>1007 ± 20</td>
</tr>
<tr>
<td>987 ± 7</td>
</tr>
<tr>
<td>1001 ± 12</td>
</tr>
<tr>
<td>986 ± 5</td>
</tr>
</tbody>
</table>
Fig. 3.11 One-pole fit (A) and two-pole fit (B) of the CERN-Munich data (Refs. 20 and 60) in the work of Ref. 62. The background phase $\delta_0^0$ in the two cases is shown as well.

Accepting the $S^*$ as a resonance, are there further scalar $I = 0$ resonances coupling to $\pi\pi$, $\varepsilon$-states? As we have seen, the partial wave is near its unitarity limit over a large energy interval in the $\rho$ region. Further, as we shall see in Section 4, it is near its unitarity limit again in the $f^0$ region. As emphasized by Morgan [5], however, it does not seem to be consistent to speak about three states ($\varepsilon$, $S^*$, $\varepsilon'$): there simply are not enough loops in the Argand diagram. However, separating out the "background" of the $S^*$ by dividing the physical $S_0^0$ by the four poles and zeros needed to describe the $S^*$ effect, Morgan found the "background" amplitude $a$) to have norm $\approx 1$ and b) to have a resonating phase (Fig. 3.12) corresponding to one $\varepsilon$-resonance with

$$m_\varepsilon = 1100 - 1300 \text{ MeV}, \quad T_\varepsilon = 500 - 700 \text{ MeV}. \quad (3.13)$$

This superbroad state, the resonance interpretation of which it would be very difficult to prove, is chopped into two by the $S^*$, leaving an imaginary part with bumps below and above 1 GeV.
Fig. 3.12 The solid curve is a result of Ref. 20. The dashed curve from Ref. 65 is defined by $\delta_0^0 = \delta_0^0 - \delta_0^0(S^*)$, where $\delta_0^0(S^*)$ is the phase of the simple rational function of $\omega$ having the four poles and zeros fitting the $S^*$ (Ref. 65). The dash-dotted curve for $\eta_0^0(S^*)$ is $\eta_0^0(S^*)$ in good agreement with the "true" $\eta_0^0$.

Morgan also argues that this interpretation is crucial for reasonable SU(3) properties of a (not well established) nonet of scalar particles. The result is far from ideal mixing of $\varepsilon$ and $S^*$, with the Zweig-rule, however, seemingly in good shape. We shall not discuss this aspect further here.

The treatment based on S-matrix elements is easily generalized to take into account the full constraint of two-channel unitarity. This has the advantage over the conventional K-matrix formulation that parameters are directly associated with structures in the data.
The above picture has been criticized by Cerrada et al.\textsuperscript{67}. These authors present a fit (to $\pi\pi \rightarrow \pi\pi$ and $\pi\pi \rightarrow K\bar{K}$, as well as to information on $K\bar{K}$ production in $\bar{p}p$ annihilation), where none of the two-channel eigenphases resonate. They also point to a complication of full two-channel calculations often overlooked: in the $K\bar{K} \rightarrow K\bar{K}$ channel there is a left-hand cut overlapping the pseudo-physical range $4\mu^2 < s < 4m_K^2$, thus spoiling the simple generalized unitarity relation there. In the standard $K$-matrix description, this has the very unfortunate effect of mixing in that complication in all $K$-matrix elements, even though the overlapping is definitely absent in the $\pi\pi \rightarrow \pi\pi$ and $\pi\pi \rightarrow K\bar{K}$ channels. This in itself makes the $K$-matrix formulation not very transparent and is a major reason why we have preferred the $S$-matrix analysis. The analysis of Ref. 67 does find $S^*$-poles, which the authors do not like to ascribe to a resonance, and which do not seem to be too far away from other peoples' poles in the $\omega$-plane. Owing to the singular transformation between $\omega$ and $s$, however, the discrepancy in energy may be enhanced.

Given better data on $K\bar{K}$ production, it should soon be possible to clarify the situation considerably.

3.3 The Roy equations

Dipion production experiments seem only to give reliable information for $M_{\pi\pi} \gtrsim 600$ MeV (see also Section 3.4 below). In Section 3.5 we shall consider results of $K_{e4}$ experiments giving information close to threshold.

In this subsection we shall describe how extra information can be obtained by imposing crossing and analyticity on a data analysis. The idea is well illustrated by a look at Fig. 1.2. Consider the line $t = 0$. Along that line (forward scattering) we have experimental information both in the $s$-channel (on $\pi^+\pi^- \rightarrow \pi^+\pi^-$ scattering, say) and in the $u$-channel ($\pi^+\pi^- \rightarrow \pi^+\pi^+$). Thus, if we are interested in low-energy parameters for comparison with soft meson theory, for example, crossing changes the original problem of extrapolating information to an interpolating one. In this interpolation, singularities at $s = 4\mu^2$ and at $u = 4\mu^2$ must be allowed for: dispersion techniques must be employed.

In the past, a large variety of dispersion techniques have been applied to a large variety of hadronic reactions. In the present case of $\pi\pi$ scattering a seemingly very successful technique employs the so-called Roy equations\textsuperscript{68,69}. Further generalizations of these have subsequently been derived\textsuperscript{70} but so far these have not been studied phenomenologically in detail.

For simplicity let us illustrate the main idea for the case of $\pi^0\pi^0 \rightarrow \pi^0\pi^0$, fully symmetric in $s$-, $t$- and $u$-channels. The amplitude is

$$F(s, t, u) = F(s, t, u) = F(u, t, s) = \frac{1}{3} \left( F_s^{(0)}(s, t, u) + 2 F_s^{(2)}(s, t, u) \right).$$  \hspace{1cm} (3.14)
Fixed-t dispersion relations with two subtractions exist for small $|t|$. For $t = 0$ this follows from the optical theorem Eq. (1.18), using the Froissart bound on $\sigma_T(s)$, $s \to \infty$. For a survey on results proven from axiomatic field theory see, for example, Refs. 1-5.

Thus we may write ($u \equiv 4\mu^2 - s - t$):

$$F(s,t) = \frac{1}{\pi} \int_0^\infty ds' \frac{\text{Im} F(s',t)}{s'^2} \left( \frac{s^2}{s' - s} + \frac{u^2}{s' - u} \right) + C(t).$$ (3.15)

In general, a dispersion relation with two subtractions has a term linear in $s$ as well as the ($t$-dependent) constant $C(t)$. In the present case, however, symmetry in $s$ and $u$ implies that the form (3.15) is general ($s + u = 4\mu^2 - t$).

Roy's observation was that the $t$-dependence of $C(t)$ was given by crossing symmetry:

$$F(s = 0, t = t_0, u = 4\mu^2 - t_0) = F(s = t_0, t = 0, u = 4\mu^2 - t_0).$$ (3.16)

This gives [using the shorthand $F(0,t)$ and $F(t,0)$ for the above left- and right-hand sides of Eq. (3.16)]:

$$F(0,t) = \frac{1}{\pi} \int_0^\infty ds' \frac{\text{Im} F(s',t)}{s'^2} \frac{(4\mu^2 - t)^2}{s' - 4\mu^2 + t} + C(t)$$

$$= F(t,0) = \frac{1}{\pi} \int_0^\infty ds' \frac{\text{Im} F(s',0)}{s'^2} \left( \frac{t^2}{s' - t} + \frac{(4\mu^2 - t)^2}{s' - 4\mu^2 + t} \right) + C(0).$$

In addition

$$F(s = 4\mu^2, t = 0, u = 0) \equiv a_0 = \frac{4}{3} (a_0^{(2)} + 2 a_0^{(2)})$$

or

$$a_0 = C(0) + \frac{1}{\pi} \int_0^\infty ds' \frac{\text{Im} F(s',0)}{s'^2} \cdot \frac{16\mu^4}{s' - 4\mu^2}.$$
Now $C(t)$ can be eliminated and

\[
F(s', t) = a_0 - \frac{4\mu^2}{\pi} \int_0^\infty ds' \frac{\text{Im} F(s', 0)}{s'^2 - 4\mu^2} \left( \frac{s^2}{s' - s} + \frac{u^2}{s' - u} \right) \\
+ \frac{4\mu^2}{\pi} \int_0^\infty ds' \left\{ \text{Im} F(s', 0) \frac{t^2}{s' - t} + [\text{Im} F(s', 0) - \text{Im} F(s', t)] \frac{(4\mu^2 - t)^2}{s' - t - 4\mu^2} \right\}.
\]

(3.17)

This equation (Roy's dispersion relation) is an exact equation which the physical amplitude must satisfy. However, even in the absence of any experimental information, we can easily see that not all amplitudes satisfying it can be serious candidates for $\pi^0\pi^0$ scattering. First, whereas symmetry under $s \leftrightarrow u$ ($t$ fixed) is manifest (by derivation), symmetry under $s \leftrightarrow t$ ($u$ fixed) is not. This means that $\text{Im} F(s', t)$ must satisfy so-called supplementary conditions such that the symmetry holds. Secondly, partial wave amplitudes must satisfy unitarity [implying among other things, positivity of $\text{Im} F(s', 0)$].

Considering first the problem of full crossing symmetry, let us expand $\text{Im} F(s', t)$ in partial waves (for $t < 0$ and low $s'$-values this is outside the physical region; for the domains of convergence of partial wave expansions, see Section 1.1.4):

\[
\text{Im} F(s', t) = \text{Im} f_0(s') + \sum_{\ell \geq 2} (2\ell + 1) \text{Im} f_\ell(s') \mathcal{P}_\ell(\cos \theta')
\]

(3.18)

where

\[
\cos \theta' = 1 + \frac{t}{2s' - 2}, \quad q'^2 = \frac{1}{4} (s' - 4\mu^2).
\]

Only even $\ell$'s enter in $\pi^0\pi^0 \rightarrow \pi^0\pi^0$. The expansion is valid for all $s' \geq 4\mu^2$ provided

\[
0 > t \geq -32\mu^2,
\]

(3.19)

[Fig. 1.2 and Eq. (1.14) -- this result assumes Mandelstam analyticity; see also Refs. 1-5, 68, and 69].
Writing
\[
\text{Im } F(s',t) = \text{Im } f_0(s') + \text{Im } \Phi(s',t)
\]
we can rewrite Eq. (3.17) as
\[
F(s,t,u) = \hat{F}(s,t,u) + \Phi(s,t,u)
\]
with
\[
\hat{F}(s,t,u) = \frac{1}{\pi} \int_0^\infty \frac{ds'}{s'^2} \text{Im } f_0(s') \left( \frac{s^2}{s'-s} + \frac{u^2}{s'-u} + \frac{t^2}{s'-t} \right)
\]
\[
\Phi(s,t,u) = \frac{1}{\pi} \int_0^\infty \frac{ds'}{s'^2} \text{Im } \Phi(s',t) \left( \frac{s^2}{s'-s} + \frac{u^2}{s'-u} - \frac{(4\mu^2-t)^2}{s'-t-4\mu^2} \right)
\]
\[
+ \frac{1}{\pi} \int_0^\infty \frac{ds'}{s'^2} \text{Im } \Phi(s',0) \left( \frac{t^2}{s'-t} - \frac{16\mu^2}{s'-4\mu^2} - \frac{(4\mu^2-t)^2}{s'+t-4\mu^2} \right).
\]

Now \(\hat{F}(s,t,u)\) is manifestly \((s,t,u)\)-symmetric. The phenomenological success of the equations depends on the fact that for \(M_{\pi\pi} \lesssim 1 \text{ GeV}\), a very simple treatment of the \(\Phi\)-term is sufficient.

First notice that \(\hat{F}(s,t,u)\) contains partial waves with arbitrary high \(\ell\)-values, but for \(d\)-waves and higher they are purely real.

In principle, \(\Phi\) could give rise to a complicated description. Certainly this is true at higher energies where it can be shown that the real parts of \(\hat{F}\) and \(\Phi\) must cancel! This implies that the separation has no deep significance. For our present purpose of finding crossing symmetric, unitary amplitudes fitting data below 1 GeV, however, it is useful.

First, \(\Phi\) is small at low energies, in particular
\[
\Phi(s=4\mu^2,t=0,u=0) = \Phi(s=0,t=4\mu^2,u=0) = \Phi(s=0,t=0,u=4\mu^2).
\]

Secondly, one can make use of the important fact that the imaginary parts of partial wave amplitudes with \(\ell \geq 2\) are negligible for \(M_{\pi\pi} \lesssim 1 \text{ GeV}\) (see below). This means that in practice, \(\Phi\) may be treated as a small, real correction below 1 GeV.
To impose unitarity it is most convenient to work with partial waves for amplitudes with definite isospin. Expressions then become very lengthy and we refer to Refs. 68 and 69 for details. The final Roy equations for partial wave amplitudes take the form

\[ \int \mathcal{E}^{(I)}(s) = \int \mathcal{E}^{(I)}(s) + \phi^{(I)}(s), \]  

(3.20)

where for

\[ I = \left( \frac{1}{2}, \frac{1}{2} \right), \quad \hat{\Pi} \mathcal{E}^{(I)}(s) = \left[ \begin{array}{c} a_{0}^{(0)} \\ a_{0}^{(2)} \end{array} \right] \delta_{e0} + \frac{4}{3} \left( 2a_{0}^{(0)} - 5a_{0}^{(2)} \right) q^{2} \sqrt{\frac{1}{6} \delta_{e1}^{2}} \]

\[ + (s - 4\mu^{2}) \sum_{I'} \sum_{I''} \int \mathcal{D} \mathcal{K}_{\mathcal{E}'}^{(I')} (s, s') \mathcal{M}_{\mathcal{E}'}^{(I')} (s'). \]

(3.21)

The kernel functions are explicitly known. Expressions may be found in Ref. 69.

Thus \( \hat{\Pi}^{(I)} \) only depends on the imaginary parts of the s- and p-waves below \( s = \sqrt{s} \) [in practice taken between the \( f^{0}(1270) \) and the \( g(1680) \)]. They sum up to give amplitudes \( \hat{\Pi}^{(I)}(s, t, u) \) satisfying all crossing requirements. The "driving terms" \( \phi^{(I)} \) sum up to give the remainder. These are evaluated from the original Roy dispersion relation, using simple models involving the \( f^{0}(1270) \), \( \phi \)-exchange, \( \rho \) and \( f \) Regge exchange, etc. Even taking rather conservative error estimates, modest changes result for the driving terms since they are small.

In actual calculations we start by concentrating on the equations for \( \lambda = 0, 1 \). The s- and p-waves are typically parametrized such that elastic unitarity is exactly satisfied below the \( K \bar{K} \) threshold. The S* effect is built into \( I = 0 \), and the p-wave contains a \( \rho \)-pole. Before we describe the results in the next subsection, we summarize the set of remarkable "accidents" that make the study of Roy equations so useful for \( \pi \pi \) scattering below 1 GeV:

i) inelasticities can be neglected;

ii) imaginary parts of partial waves with \( \lambda \geq 2 \) can be neglected, and therefore the supplementary conditions can be neglected.

iii) using Bose symmetry, the range (3.19) is sufficient for the equations to rigorously exist up to and a little above the \( K \bar{K} \) threshold;

iv) driving terms are small and can be evaluated with sufficient accuracy using very mild assumptions.
3.4 Results of Roy equation studies

Several more or less equivalent studies exist\(^{63,71-75}\). In general we give results of the BFP work [Basdevant, Follgatt and Petersen\(^{63}\)].

Input into the analysis is:

i) \(\delta_0^0\) for 600 MeV < \(M_{\pi\pi}\) < 1000 MeV (Fig. 3.13);

ii) the S* effect (i.e. \(\eta_0^0\) just above the \(K\bar{K}\) threshold and the sharp rise in \(\delta_0^0\) to 180° at \(M_{\pi\pi} = 2m_K\));

iii) the mass and width of the \(\rho\);

iv) the driving terms evaluated as described.

Unitary, analytic partial wave amplitudes are parametrized as rational forms for the S-matrix elements \(S_\ell^{(1)}\) (\(\ell = 0, 1\)) in a variable \(z\) [somewhat similar to \(\omega\) of Eq. (3.8)] exhibiting the relevant branch points: \(s = 0, s = 4\mu^2, s = 4m_K^2\).

---

**Fig. 3.13** Data points are from Ref. 46. Curves are from the BFP-program (Ref. 63) fitting to these data and fixing the \(I = 0\) s-wave scattering length \(a_0^{(0)}\) (in \(\mu = 1\) units) to the numbers indicated on the figure.
Below we discuss the various kinds of outputs. The possible solutions can be described as a one-parameter family of possibilities. A convenient parameter to use is the I = 0 s-wave scattering length \( a_0^{(0)} \). The precise range which is allowed for \( a_0^{(0)} \) depends on the experimental errors on \( \delta_0^{(0)} \). Using very generous errors accommodating the full range of suggested values, we can find solutions for all values of \( a_0^{(0)} \) in the interval (\( \mu = 1 \) for all scattering length values)

\[-0.05 < a_0^{(0)} < 0.6\,.

Figure 3.13 shows curves for a few sample solutions with \( a_0^{(0)} = 0.17, 0.30, \) and 0.50.

For each value of \( a_0^{(0)} \), the value of \( \delta_0^{(2)} \) is essentially an absolute prediction. In Fig. 3.1 are shown curves corresponding to the curves for \( \delta_0^{(0)} \) in Fig. 3.14. They have

\[\delta_0^{(2)} = -0.0497 + 0.0158 + 0.016\,.

for \( a_0^{(0)} = 0.17, 0.30, \) and 0.50, respectively. They are all in qualitative agreement with experiment. There are systematic uncertainties in the prediction associated with uncertainties in the driving terms. They grow from 0° at threshold to \( \simeq \pm 3° \) at 1000 MeV. The low-energy points of Prukop et al.\(^{57}\) thus seem to violate dispersion relations. The curves also demonstrate the difficulty of obtaining scattering lengths from data, without dispersion theory. Thus, Losty et al.\(^{56}\) find for their data

\[a_0^{(2)} = -0.09 \pm 0.02\,.

The figure shows this conclusion cannot be maintained.

Figure 3.14 shows a conservative estimate of the region allowed by data above 600 MeV for \([a_0^{(0)}, a_0^{(2)}]\). This illustrates the one-dimensional character of the output solutions. The band is often referred to as the universal curve, after Morgan and Shaw who first pointed to its probable existence\(^{76}\). It is given by

\[2a_0^{(0)} - 5a_0^{(2)} = 0.69 \pm 0.04 + 0.96 (a_0^{(0)} - 0.3) + 0.7 (a_0^{(0)} - 0.3)^2. \tag{3.22}\]

Also shown are the predictions of current algebra in the linear approximation, and of the Weinberg values corresponding to absence of the I = 2 \( \sigma \)-term (see Section 1.2 and Fig. 1.4).

The p-wave scattering length turns out to be remarkably stable along the universal curve

\[a_1^{(1)} = 0.040 \pm 0.004, \tag{3.23}\]
Fig. 3.14 Universal curve in the $[a_0^{(0)}, a_0^{(2)}]$ plane. Predictions of current algebra (in the linear approximation) as well as of the Weinberg theory are indicated. Also shown are the impact of the $K_{e4}$ results described in Section 3.5 (see Fig. 1.4). The universal curve was obtained using the BFP program (Ref. 63) fitting to the data on $\delta_0^{(6)}$ of Fig. 3.15 (Ref. 46).

Comparing with Eq. (3.22) we see deviations (as expected) from the linearity rule [Eq. (1.38)] $2a_0^{(0)} - 5a_0^{(2)} - 18\mu^2 a_1^{(1)} = 0$. This puts a limit on the accuracy with which current algebra may be tested. Within the limit, the agreement is tolerable.

The value of $a_1^{(1)}$ [Eq. (3.23)] is about 30% larger than predicted by current algebra.

Not until we bring in information from the new $K_{e4}$ experiment (Section 3.5) can we say much about the breaking of chiral symmetry.

The high degree of stability of the $p$-wave determination is important in connection with an attempt of the CERN-Munich Group$^{78}$ to study the very low $M_{\pi\pi}$ region in a special run on the $\pi^- p \rightarrow \pi^+ \pi^- n$ interaction at 7 GeV/c (beam momentum lower than 17.2 GeV/c is required to have good acceptance in the experiment at low $M_{\pi\pi}$). Results for the $p$-wave are shown in Fig. 3.15 [Basdevant et al.$^{78}$]. The data points give $a_1^{(1)} \approx 0.100$, in very strong disagreement with current algebra -- much worse, however, in quite unacceptably strong disagreement with dispersion calculations, Eq. (3.23). The argument can be looked at separately for the $p$-wave without having to go through the full calculation of the Roy equations (Ref. 78).

We must conclude that provided detection problems have been understood, the simple absorption prescription used to obtain the $\pi\pi$ phase shifts is in error at
Fig. 3.15 Prediction of the low-energy part of the p-wave using Roy equations from Ref. 78. The prediction was based on the low-energy s-waves of Ref. 77. Data points are the low-energy p-waves also of Ref. 77. Discrepancy with dispersion relations is obvious.

low $M_{\pi\pi}$. We noticed already in Section 2.4 that at low $M_{\pi\pi}$ the OPE signal is reduced compared to absorption for the p-wave [Eqs. (2.32)]. Thus a relatively unimportant error in the treatment in the $\rho$ region might render the analysis quite unreliable in the low $M_{\pi\pi}$ region.

The same analysis of the same experiment gave a value of 0.4 for $a_0^{(0)}$. The s-wave is not affected by absorption, so this value might be less subject to error than the p-wave. However, using in the $\langle Y_0^0 \rangle$ moment, a value of $a_1^{(1)}$ as in Eq. (3.23) yields an s-wave outside the unitarity limit! Again we are reluctant to consider the low-energy s-waves of the experiment too relevant to $\pi\pi$ physics. We noted in Section 2.6 that neglected violations of spin coherence and absence of $A_1$-exchange would indeed tend to give spuriously large s-waves. Since $A_1$-exchange appears to have been experimentally established (Section 2.6) we are eagerly awaiting analysis of the polarized target experiment for a possible clarification. Here we simply continue, ignoring all results below 600 MeV.

A final prediction of the Roy equation studies concerns the higher partial waves (Table 3.2). At very low energies there is some dependence on $a_0^{(0)}$ (which becomes overwhelming for $\ell \geq 4$). Apart from that, the predictions are very accurate all the way to 1 GeV, where they have been confirmed experimentally. At 1 GeV only half the size of $\delta_2^{(0)}$ is due to the tail of the $f^0(1270)$, the remainder coming from the crossed channels.
In the \( \rho \) region and below, the experimental \( \delta_2^{(0)} \) is larger than the prediction, but detection problems make the determination uncertain\(^{46}\). The small d-wave turned out to have an unexpectedly large influence on the s-wave determination. This is due to a geometrical "accident" illustrated in Fig. 3.16 (Ref. 46). The

<table>
<thead>
<tr>
<th>( M_{\pi\pi} ) (MeV)</th>
<th>( \delta_2^{(0)} ) (degrees)</th>
<th>( \delta_2^{(2)} ) (degrees)</th>
<th>( \delta_3^{(1)} ) (degrees)</th>
</tr>
</thead>
<tbody>
<tr>
<td>400</td>
<td>0.07 ± 0.01</td>
<td>0.00 ± 0.01</td>
<td>0.002 ± 0.001</td>
</tr>
<tr>
<td>600</td>
<td>0.9 ± 0.1</td>
<td>-0.1 ± 0.1</td>
<td>0.04 ± 0.02</td>
</tr>
<tr>
<td>800</td>
<td>3.5 ± 0.5</td>
<td>-0.5 ± 0.2</td>
<td>0.2 ± 0.1</td>
</tr>
<tr>
<td>1000</td>
<td>11 ± 2</td>
<td>-1.0 ± 0.6</td>
<td>0.8 ± 0.2</td>
</tr>
</tbody>
</table>

Fig. 3.16 Properties of phase-shift solutions from Ref. 46 obtained by fitting \( J \leq 2 \) moments for \( -t < 0.2 \) GeV\(^2\) in two typical mass bins: (a) 720 < \( M_{\pi\pi} < 840 \) MeV. The \( \chi^2 \) versus \( \delta_2^{(0)} \) plots show the sensitivity of the analysis to the value of \( \delta_2^{(0)} \). The continuous and dashed lines are respectively the \( \chi^2 \) profiles for \( \delta_2^{(0)} \) fixed as required by (A) the Roy equations (Ref. 46) and (B) the data. The lower diagrams are the Argand plots of \( f_0^{(0)} \) obtained with and without s-wave unitarity imposed [ ising choice (A) for \( \delta_2^{(0)} \) ]. The \( I = 2 \) s-wave contribution (OS) is shown together with the \( I = 0 \) s-wave unitarity circle. The line OP indicates the direction of \( f_1^{(1)} \).
best information on the $s$-wave comes from the $(\gamma^2)$ dominated by $s$-$p$ interference. However, to obtain the $s$-$p$ interference, the $p$-$d$ interference first has to be subtracted. The figure shows how a very small error in that subtraction can lead to a systematic bias of $10^\circ$ in the $s$-wave. The final band in Fig. 3.3 corresponds to using the theoretical $d$-waves of Table 3.2 (Refs. 46 and 63).

### 3.5 Results from $K_{\ell 4}$ decay

Long ago it was pointed out by Pais and Treiman that the decay

$$K^+ \to \pi^+ \pi^- e^+ \nu$$

would provide information on $\pi\pi$ scattering near threshold in an extremely clean way\(^9\). The only assumptions needed are those of time-reversal invariance, $\Delta I = \frac{1}{2}$ rule and $V-A$ form of weak interactions.

In practice the experimental difficulties are enormous: the branching ratio is only $3.7 \times 10^{-5}$, the effect of $\pi\pi$ scattering is a minor correction, but there are five variables in the problem:

- $M_{K\nu} = \text{invariant dilepton mass}$;
- $M_{\pi\pi} = \text{invariant dipion mass}$;
- $\theta_{\ell} = \text{angle of } e^+ \text{ in the dilepton c.m. with respect to the dipion line of flight}$;
- $\theta_{\pi} = \text{angle of } \pi^+ \text{ in the dipion c.m. with respect to the dilepton line of flight}$;
- $\phi = \text{angle between the dipion plane and the dilepton plane as seen in the } K^+ \text{ c.m.}$

The most general form (consistent with $V-A$) of the transition matrix element is given by

\[
\langle \pi^+(p_+) \pi^-(p_-) | J^\lambda_{\nu} + J^A_{\nu} | K^+(k) \rangle = \frac{1}{m_K} \left[ F \cdot P_{\lambda} + G \cdot Q_{\lambda} + R \cdot k_{\lambda} \right] + \frac{H}{m_K^2} \epsilon_{\lambda\mu\nu\sigma} k^{\mu} P_{\nu} Q^{\sigma},
\]

where

$$P \equiv p_+ + p_-, \quad Q = p_+ - p_-.$$

The form factors $F$, $G$, $R$, and $H$ depend in general on $M_{\pi\pi}$, $M_{K\nu}$, $\theta_{\pi}$; $R$ gets multiplied by the electron mass in the experimental decay distribution and may be neglected. For $F$, $G$, and $H$ the $\theta_{\pi}$-dependence and the phases are given by\(^8\) (neglecting $d$-waves and higher)
\[ F(M_{\pi\pi}, M_{\pi\nu}, \Theta_{\pi}) = \int_{\delta_{0}^{(0)}}^{\delta_{1}^{(1)}} (M_{\pi\pi}) e^{i \delta_{0}^{(0)}(M_{\pi\pi}) + \int_{\delta_{1}^{(1)}}^{\delta_{1}^{(1)}} (M_{\pi\pi}) e^{i \delta_{1}^{(1)}(M_{\pi\pi})} \cos \Theta_{\pi} \]

\[ G(M_{\pi\pi}, M_{\pi\nu}, \Theta_{\pi}) = \int_{\delta_{0}^{(0)}}^{\delta_{1}^{(1)}} (M_{\pi\pi}) e^{i \delta_{0}^{(0)}(M_{\pi\pi})} \]

\[ H(M_{\pi\pi}, M_{\pi\nu}, \Theta_{\pi}) = \int_{\delta_{0}^{(0)}}^{\delta_{1}^{(1)}} (M_{\pi\pi}) e^{i \delta_{0}^{(0)}(M_{\pi\pi})} \]

with \( f_s, f_p, g_p, h_p \) real form factors.

Pais and Treiman pointed out that \( \delta_{0}^{(0)}(M_{\pi\pi}) - \delta_{1}^{(1)}(M_{\pi\pi}) \) could be obtained in two independent ways by studying fine details of the \((\theta_{\pi}, \phi)\) correlation.

In the recent very high statistics (30,000 events) Geneva-Saclay experiment, the experimental acceptance is such that although it is well known, the Pais-Treiman scheme is not directly applicable. Instead, a more direct and powerful (but slightly less model-independent) analysis is performed. The \( M_{\pi\pi} \) dependence of \( f_s, f_p, g_p, h_p \) is studied and found to be negligible.

In Fig. 3.17 are shown the determinations of \( \delta_{0}^{(0)} - \delta_{1}^{(1)} \) compared to low-energy predictions of four solutions of the Roy equations, fitting the data on \( \delta_{0}^{(0)} \) above 600 MeV (Fig. 3.13). Allowing for systematic errors, the figure gives

\[ \alpha_{0}^{(0)} = 0.26 \pm 0.05 \]  

Fig. 3.17 Determinations of \( \delta_{0}^{(0)} - \delta_{1}^{(0)} \) from the Geneva-Saclay kaon experiment (Ref. 81). Also shown are predictions from the Roy equations (curves) with numbers giving \( a_{0}^{(0)} \) in \( \mu = 1 \) units. Curves were obtained using the BFP program (Ref. 63) fitting to the data on \( \delta_{0}^{(0)} \) of Ref. 46 (see Figs. 3.13 and 3.14).
The result quoted in Ref. 81 is \( a^0_0 = 0.28 \pm 0.05 \) and is based on slightly older Roy equations solutions.

It thus seems, at long last, that a fairly clear picture for low-energy \( \pi \pi \) scattering has been established. It also appears unlikely that an even better \( \kappa_{\pi\pi} \) experiment will be performed in the foreseeable future.

Using the result (3.26) together with the universal curve (Fig. 3.14) gives a reasonably narrow region in the \( [a^0_0, a^2_0] \)-plane. A small but clear disagreement with the Weinberg prediction is indicated. However, remembering that off-shell extrapolation uncertainties are present, the disagreement need not be too serious. Certainly the predictions of other breaking schemes, such as indicated in Fig. 1.4, are completely ruled out.

4. THE REGION \( M_{\pi\pi} > 1 \text{ GeV} \) PHASE-SHIFT AMBIGUITIES

4.1 The method of Ochs and Wagner\(^{20,42}\)

Figure 4.1 shows the \( \pi\pi \) mass spectrum of the CERN-Munich \( \pi^- p \to \pi^+ \pi^- n \) 17.2 GeV/c experiment\(^{19-22}\). It is instructive to see how corrections for efficiencies emphasizes the g-meson signal: poor efficiency for \( |\cos \theta| = 1 \) tends to strongly suppress high \( L \) components in the angular distribution.

![Mass spectrum of the CERN-Munich data at 17.2 GeV/c (Refs. 19-22) on \( \pi^- p \to \pi^+ \pi^- n \). Histogram is raw data, error-bars are data corrected for detection efficiency.](image)

**Fig. 4.1** Mass spectrum of the CERN-Munich data at 17.2 GeV/c (Refs. 19-22) on \( \pi^- p \to \pi^+ \pi^- n \). Histogram is raw data, error-bars are data corrected for detection efficiency.
Figures 4.2 show the data on which all analyses above the f^0(1270) region have been based: unnormalized t-channel moments averaged over 0.01 ≤ |t| ≤ 0.15 GeV^2 ≈ 7μ^2. Experimentally, t-channel moments with m ≥ 2 are consistent with zero in that t-interval (see Fig. 2.4).

To directly extend the kind of analysis described in Section 2 would be very complicated owing to the necessity for including λ-values at least up to λ = 3.

Fig. 4.2  t-channel moments of Refs. 19-22 integrated over 0.5 μ^2 ≤ |t| ≤ 7 μ^2.
Also the data hardly justify such a complicated analysis. Instead, the method proposed by Ochs and Wagner\textsuperscript{20,42} makes use of a number of simplifying circumstances true at lower \(M_{\pi\pi}\) values, thereby enabling results to be obtained from those \(t\)-averaged moments of Fig. 4.2:

i) Spin coherence as in the absorption model (Section 2).

ii) Phase coherence in the strong form of factorization:

\[
L_0 = N \times M_{\pi\pi} (2\ell + 1)^{1/2} \frac{T_{\pi\pi}^{\lambda} (M_{\pi\pi})}{\sqrt{q_{\pi\pi} (M_{\pi\pi})}}
\]

[neglecting the \(t\)-dependence of \(q(\text{off})\); see Eq. (2.29)]. The normalization \(N\) is independent of \(\lambda\) and \(M_{\pi\pi}\) as in Eq. (2.29). For \(m = 1\) the assumption is

\[
\frac{L_{\lambda}^{(-)}}{L_0} = C_\lambda (M_{\pi\pi}),
\]

with \(C_\lambda\) real and smoothly varying with respect to \(M_{\pi\pi}\). The \(\lambda\)-dependence of \(C_\lambda\) turns out to be in accordance with the Williams model prediction (Fig. 4.3):

\[
C_\lambda (M_{\pi\pi}) = \sqrt{\ell (\ell + 1)} \cdot C (M_{\pi\pi}).
\]

However, the \(M_{\pi\pi}\) dependence of \(C(M_{\pi\pi})\) is such that \(C\) goes to zero with increasing \(M_{\pi\pi}\) faster than predicted by the Williams model, where \(C(M_{\pi\pi}) = \frac{M_{\pi\pi}^{-1}}{\pi} \cdot \frac{\sqrt{2}}{2} \). This, of course, seems fortunate for the \(\pi\pi\) analysis at high \(M_{\pi\pi}\) (but may be a further worry for the analysis at low \(M_{\pi\pi}\), Section 3.4).

**THE AMPLITUDE RATIO** \([L_{\lambda}(L+1)]^{1/2} \frac{L_{\lambda}^{(-)}}{L_0}\)

\[\begin{array}{c}
\text{Fig. 4.3 The } M_{\pi\pi} \text{ dependence of } C(M_{\pi\pi}) = C_\lambda (M_{\pi\pi}) / \sqrt{\ell (\ell + 1)} \text{ from Ref. 82.}
\end{array}\]
iii) The experimental absence of moments with $m > 1$ is exploited. This means that

$$\begin{align*}
\langle \pm \rangle (t) &\equiv 0 \quad \left| \langle \pm \rangle (t) \right| = \left| \langle - \rangle (t) \right|
\end{align*}$$

[see, for example, Eq. (2.33) for a proof at the s+p level].

It is easy enough to point to effects which could upset the validity of assumptions (i) to (iii). They do, however, seem to describe the data very well. A fortunate aspect of $\pi\pi$ studies at higher values of $M_{\pi\pi}$ is the expected dominance of OPE over background. In the Williams model this follows, for example, from Eq. (2.32) by the appearance of $M_{\pi\pi}$ in the OPE signal and the absence of that factor in the background. As noticed above, the data tend to make this effect even stronger\[20,82\]. For the very same reason, analyses at low $M_{\pi\pi}$ values may be in doubt, as emphasized repeatedly (Section 3.4).

Even allowing for $A_2$-exchange (or $\omega$-exchange in non-charge-exchange reactions) the conclusion that OPE dominates over background at high $M_{\pi\pi}$ follows. First, for a fixed value of $M_{\pi\pi}$, $A_2$-exchange will dominate over $\pi$-exchange as the incident momentum increases because

$$\alpha_{A_2}(\theta) \approx \frac{1}{2} > \alpha_{\pi}(\theta) \approx 0.$$
However, for a fixed beam momentum the $M_{\pi\pi}$-dependence of a given exchange $E$ is expected to be of the form

$$\frac{d\sigma}{dt} \propto \left( M_{\pi\pi}^2 \right)^{\alpha} M_{\pi\pi}(s) - 2 \alpha E(t)$$

This expression follows from a general Mueller type of analysis of the inclusive reaction $\pi N \rightarrow X N$, using duality ideas and finite missing mass ($M_{\pi\pi}$) sum rules [for a simplified derivation see, for example, the article by Kajantie\(^{83}\); also see the review by Hoyer\(^{84}\)]. For the dipion resonances, $\rho$, $f$, $g$, etc., the relevant $\alpha_M(0)$ is expected to be $\frac{1}{2}$, whereas for the non-resonant $\pi\pi$ background we expect $\alpha_M(0) = \alpha_p(0) = 1$. It is seen that for large $M_{\pi\pi}$, pion exchange ($\alpha_E = 0$) dominates over $A_2^-$ (or $\omega^-$) exchange ($\alpha_E = \frac{1}{2}$) by a factor $M_{\pi\pi}$ in the amplitude.

Further confidence in the Ochs-Wagner scheme comes from the fact that results derived from it below 1 GeV nicely agree with the more model independent analysis of Estabrooks and Martin\(^{46}\).

We close this subsection by showing the results of the energy-dependent analysis of Ochs and Wagner\(^{20,42}\) (Fig. 4.4). We postpone discussions of this to Section 4.3.

### 4.2 Phase-shift ambiguities

Above 1 GeV, $\pi\pi$ scattering is known to be inelastic, both $K\bar{K}$ and $4\pi$ inelasticities being experimentally well established. In inelastic regions, knowledge of the differential cross-section at any one energy is insufficient information to obtain the partial wave amplitudes. Just how serious this "phase-shift ambiguity" problem is has recently been studied theoretically by a large number of authors\(^{85,86}\). The results obtained have been quite strong; even with infinite accuracy, there exists a continuum of ambiguities at each energy. Imposing inelastic unitarity and $\cos \theta$ analyticity does not change the conclusion, nor does knowledge of the forward phase from total cross-sections (via the optical theorem) remove the continuum ambiguity. (In $\pi\pi$ physics, total cross-sections are essentially unknown anyway.)

The elegant theorems do not tell us to what extent such phase-shift ambiguities are really troublesome in practice. As we shall see, however, indications are that they are of overwhelming practical importance. The ambiguities can be reduced but never solved by having experimental information on other channels, such as $\pi^+\pi^- \rightarrow \pi^0\pi^0$, $\pi^-\pi^0 \rightarrow \pi^-\pi^0$, $\pi^+\pi^- \rightarrow K\bar{K}$, etc. In practice, such information is available only to a very limited extent. To solve the problem completely in a purely experimental fashion is possible in principle, but seems totally out of the question in practice. We would need to study reactions such as $4\pi \rightarrow 4\pi$, or else to perform very subtle double scattering experiments.
Roughly two kinds of "theoretical" solutions exist.

In one \(^{87}\) we attempt to predict partial wave amplitudes for large \(\ell\) in terms of information on nearby \(\cos \theta\) singularities obtained using crossing. The difficulty is that the method is not very reliable and at most works for very high \(\ell\). To have practical importance, however, we must be able to predict for sufficiently low \(\ell\) that the partial waves in question feature clearly in the data. The prospect for \(\pi \pi\) seems extremely weak.

In the other "theoretical" method, we use fixed \(t\)-analyticity. Even in the case of \(\pi \pi\) scattering we shall show in Section 5 that uncertainties seem very firmly under control. The technique has proved itself in practice in analyses of \(K^\pm p\) and in particular of \(\pi N \rightarrow \pi N\) by Pietarinen with remarkable success \(^{86,88-97}\). We shall treat this technique in Section 5.

Here we first discuss in a simplified way the ambiguities in terms of zeros, as was done by Gersten \(^{98}\) and by Barrelet \(^{99}\), as well as the suggestion of Barrelet to require smoothness of the zero trajectories. This seems to give rise to a first clarification. As we shall argue, however, the technique is not capable of solving the problem in quantitative terms.

Let us write the amplitude at fixed energy as

\[
T(\cos \theta) = \sum_{\ell=0}^{\infty} (2\ell+1) T_\ell P_\ell(\cos \theta),
\]

where for simplicity we ignore isospin. Then

\[
T_\ell = \frac{\ell \delta_\ell^2 - 1}{2i}, \quad \text{Im} \ T_\ell = |T_\ell|^2 \frac{1-\rho_\ell^2}{2} \gg |T_\ell|^2
\]

\[
\frac{d\sigma}{d\Omega} = \frac{4}{q^2} |T|^2
\]

[see Eqs. (1.17)-(1.20)].

Now suppose the partial wave series can be exactly truncated at \(\ell = L\). Then there are \(2(L+1)\) real parameters in the problem. Experimentally, moments \(\langle Y^0_{L'} \rangle\) with \(0 \leq L' \leq 2L\) can be measured giving \(2L+1\) parameters, and, assuming the total cross-section is known, there are just as many measurable parameters as there are real and imaginary parts (!).

In reality there are many discrete ambiguities, and in the limit \(L \to \infty\) they become continuum ones.
For truncation at \( \lambda = L \), the amplitude is a polynomial of order \( L \) in the variable \( z = \cos \theta \). Thus
\[
T(z) = c e^{i\phi} \prod_{i=1}^{L} (z - z_i^*)^{c_i}, \quad c, \phi \in \mathbb{R}
\]
where some of the \( z_i^* \)'s could be real. From the experimental cross-section we get
\[
|T|^2 = c^2 \prod_{i=1}^{L} (z - z_i^*)(z - z_i^*)^*. \quad (4.5)
\]

The ambiguity arises because we cannot know whether \( z_i \) or \( \bar{z}_i^* \) belongs to \( T \); both are zeros of \( |T|^2 \). If there are \( P \) complex zeros with \( 0 \leq P \leq L \), this gives rise to a \( 2^P \)-fold ambiguity of which some may be ruled out by the unitarity inequalities (4.2).

Without becoming too specialized, we can give the following two arguments against the above treatment:

i) For large \( \lambda \) we expect the \( \lambda \)-dependence of \( |T_\lambda| \) to be
\[
|T_\lambda| \sim R^{-\lambda}
\]
where (Section 1.1.4)
\[
R \sim 1 + \frac{c}{\lambda}, \quad c > 0.
\]

Then interference between \( T_0 \) and \( T_\lambda \) implies that the moments \( \{|T_\lambda|^2\} \) have exactly the same \( \lambda \)-dependence! Therefore it is quite unallowed (at large \( \lambda \)) to truncate the partial wave expansion at \( \lambda = L \) and the moment expansion at \( \lambda = 2L \); both should be truncated at the same point. This would severely change our counting of parameters above.

ii) The moment expansion of the cross-section only converges within a certain ellipse in the \( \cos \theta \) plane (Section 1.1.4). Only the finite number of zeros within that ellipse can be found reliably by the expansion. Truncating at a high \( L \)-value, we will certainly find \( L \) zeros, but the majority of those would have no physical significance at all; they are artifacts of the expansion.

Notice that in the elastic region we can afford to truncate the partial wave series and the moments at the same place and still get an essentially unique solution. This is so because each complex partial wave amplitude \( T_\lambda \) is described by one real parameter \( \delta_\lambda \). The so-called trivial ambiguity \( T + T^* \) is easily removed if there is a clear resonance around. Singular cases where a twofold ambiguity can occur have been studied. They appear to have no real phenomenological relevance, since a minimal amount of smoothness in energy will exclude them.\(^{100} \)}
4.3 Barrelet zero analysis

The idea of Barrelet\textsuperscript{99}) that complex zeros should be smooth functions of energy and that this would help phase-shift analyses, has been applied in many reactions.

In the present case of $\pi^+\pi^- + \pi^+\pi^-$, two analyses based on this idea are available, by Münner\textsuperscript{101}) and by Estabrooks and Martin\textsuperscript{82}). They are so similar that we shall treat them together, pointing to differences in details when relevant. Both employ the method of Ochs and Wagner\textsuperscript{20,42}) to obtain extrapolated on-shell moments of the $\pi\pi$ differential cross-section. Actually, this way of describing the situation is a simplification. On-shell moments are (unfortunately) not really obtained directly in either analysis. Rather, the on-shell $\pi\pi$ partial wave amplitudes themselves (except for an over-all phase) are sought for. Owing to interference between OPE and background, the different solutions do not even reproduce exactly the same cross-section. That is, in principle off-shell $\pi\pi$ scattering contains more information than on-shell $\pi\pi$ scattering. In practice, this information is, however, extremely model-dependent. Also, numerically the various solutions do reproduce the same $d\sigma/d\Omega$ for $\pi^+\pi^-$ scattering to a much higher accuracy than experimental errors would suggest.

It would be extremely useful if analysts would kindly publish their on-shell moments with error estimates. As we shall argue later, these constitute the really useful information extracted from the data. Any one particular phase-shift solution from Refs. 82 and 101 is very much more uncertain.

Let us then assume that the $\pi^+\pi^-$ differential cross-section is given. All analyses truncate at $\ell = 3$. This is consistent with the experimental fact that moments ($Y_0^L$) with $L > 6$ are zero within errors. As discussed above, however, it does not at all follow that the partial wave expansion can be truncated at $\ell = 3$. If, on the other hand, the first four partial waves contain the leading resonances, such as is the case below the $f$-region, it is perhaps plausible that higher waves are very much smaller, thus making the truncation allowed.

It follows that the analysis involves three complex zeros: $z_1(M_{\pi\pi})$, $z_2(M_{\pi\pi})$, and $z_3(M_{\pi\pi})$. They are labelled according to the energy at which they come close to the physical region. In the $\rho$ region this is true only of $z_1$; in the $f^0(1270)$ region, of $z_2$ as well, etc.

Figures 4.5 show results for $\text{Im} z_1(M_{\pi\pi})$ from the four solutions of Münner\textsuperscript{101}) (CM74) and from the Ochs and Wagner solutions (CM73) energy-dependent (EDA) and energy-independent (EIA)\textsuperscript{42}). The figure is from the detailed discussion of zero-trajectories by Shimada\textsuperscript{102}). Figure 4.6 shows both real and imaginary parts of $z_1(M_{\pi\pi})$ from the Estabrooks and Martin (EM) solutions\textsuperscript{82}), where a particular sign (corresponding to their solution A) is given for the imaginary parts.
Fig. 4.5 Zero trajectories (Ref. 102) from the analysis of Münner [□(--), △(+-), ○(--), ▲(++) (Ref. 101) and of Ochs and Wagner (Refs. 20 and 42) [CM73 EIA: energy-independent analysis (□); and CM73 EDA: energy-dependent analysis (△)].

Fig. 4.6 Zero trajectories of the Estabrooks and Martin analysis (Ref. 82). Only one sign (corresponding to their solution A) is given for the imaginary part.
It is evident from the figures how smoothness of $\text{Im } z_i$ as a function of $M_{\text{NN}}$ only leaves a few discrete possibilities, corresponding to bifurcations at each zero of $\text{Im } z_i$.

When strong peripheral resonances are present, it can be shown that zero trajectories must approach the physical region with a negative imaginary part (Refs. 82, 101, and 102).

If only waves $T_{\ell}$ and $T_{\ell-1}$ need be considered, we use
\[(2\ell-1) P_{\ell-1}(z) \cdot Z = \ell P_{\ell}(z) + (\ell-1) P_{\ell-2}(z)\]
to write
\[(2\ell-1) P_{\ell-1}(z) \approx \frac{\ell P_{\ell}(z)}{z}\]
for $|z| > 1$. Thus
\[(2\ell+1) P_{\ell}(z) \cdot T_{\ell} + (2\ell-1) P_{\ell-1}(z) \cdot T_{\ell-1} \approx P_{\ell}(z) \left[ (2\ell+1) T_{\ell} + \frac{\ell}{z} T_{\ell-1} \right]\]
which is zero for
\[z = -\frac{\ell}{2\ell+1} \frac{T_{\ell-1} T_{\ell}^*}{|T_{\ell}|^2} .\]
If $T_{\ell-1}$ is somewhere after resonance and $T_{\ell}$ somewhere before resonance, we clearly have
\[\text{Im } z < 0 .\]
Thus at low energies the sign of $\text{Im } z_i(M_{\text{NN}})$ for $i = 1, 2, 3$ is believed to be known. This leaves four discrete remaining possibilities (Figs. 4.5 and 4.6).

Männer classifies the possibilities according to the signs of $\text{Im } z_1, \text{Im } z_2, \text{Im } z_3$ at $M_{\text{NN}} = 1500$ MeV. Since $\text{Im } z_3 < 0$ (below 1.8 GeV) this gives the four possibilities: ---, ++, --, +--.

Männer further fixes the over-all phase by requiring that the resonating partial wave $[d$-wave in the $f^0(1270)$ region and $f$-wave in the $g(1680)$ region] be given by an inelastic Breit-Wigner. This results in semi-energy independent solutions. The Breit-Wigner assumption is probably rather safe for the $d$-wave. For the very inelastic $f$-wave, on the other hand, we shall see that the assumption is probably not valid. In the energy region between the $f^0$ and the $g$, the over-all phase is very unstable.
It appears from Fig. 4.5 that the zero-technique is capable of excluding the Ochs-Wagner solution: it switches from \( \rightarrow \) to \( \rightarrow \) around 1500 MeV. Indeed the Ochs-Wagner solution gives a bad fit to the moments (especially to \( \langle x^2 \rangle \)) at that energy\(^{22}\).

Figures 4.7 to 4.10 show the Männer solutions.

Figures 4.11 show the EM solutions (as depicted in Ref. 82). They are obtained by making an energy-dependent fit to the zero-trajectories rather than to the partial wave amplitudes. The over-all phase is adjusted to roughly conform to Breit-Wigner behaviour of leading resonances. For \( M_{\pi\pi} < 1.25 \), auxiliary solutions with \( \text{Im} z_1 < 0 \) are shown. These are found to violate data on \( \pi^0\pi^0 \) production. We shall come back to comparing with data on other reactions below.

The four EM solutions A, B, C, D correspond in that order to the four Männer solutions \( \rightarrow \), \( \rightarrow \), \( \rightarrow \), \( \rightarrow \).\(^{22}\)

The two last solutions (EM \( \rightarrow \), D, or Männer \( \rightarrow \), \( \rightarrow \)) have some (resonant?) s-wave activity in the g region. This implies at \( M_{\pi\pi} = 1.7 \text{ GeV} \) a very small cross-section at 90° for \( \pi^+\pi^- \rightarrow \pi^0\pi^0 \) [Fig. 4.12 by Shimada\(^{102}\)]. This appears to be in rather striking contrast to the \( \pi^0\pi^0 \) data. Figure 4.13 shows raw angular distributions from the CERN-IHEP-Karlsruhe-Pisa-Vienna discovery of the h-meson\(^{15}\). For this reason (and for simplicity of discussion) we shall concentrate on solutions EM-A/Männer \( \rightarrow \) and EM-B/Männer \( \rightarrow \), of which the latter shows a \( \rho'(1650) \rightarrow 2\pi \) signal whereas the first does not. Both are in qualitative agreement with the \( \pi^0\pi^0 \) data (not corrected for poor efficiency at \( \cos \theta \approx 1 \)).

Attempts at resolving the ambiguity between the two solutions will be the main subject of the last two sections. Without such a resolution, we have learned surprisingly little of what we wanted to learn. Only in the \( f^0(1270) \) region is the situation reasonably clear: the s-wave is very large (unitarity violations for solutions B/\( \rightarrow \) we do not take seriously for the moment; see Section 6) and there is no trace of a p-wave resonance. This, however, was known several years ago\(^6\).

Before we discuss ways of resolving ambiguities, let us comment on the extent of continuum ambiguities (or uncertainties) beyond the over-all phase one, and the (four) discrete ones.

Figures 4.14 to 4.17 are the results of the EM analysis prior to smoothing of the zeros. The over-all phase is fixed by requiring the phase of the d-wave to be that of a Breit-Wigner. Clearly this is very unreasonable in the g-region, but it has the virtue of making the over-all phase smooth. The very large scatter in the Argand diagrams is then a measure of the remaining ambiguity.
Fig. 4.7 Solution (---) of Ref. 101 for (a) the $I = 0$ s-wave and the p-wave, and (b) for the $I = 0$ d-wave and the f-wave. Numbers are $M_{\pi\pi}$ values in GeV.

Fig. 4.8 Solution (----) of Ref. 101. Same notation as in Fig. 4.7.
Fig. 4.9 Solution (+−−) of Ref. 101. Same notation as in Fig. 4.7.

Fig. 4.10 Solution (+++) of Ref. 101. Same notation as in Fig. 4.7.
Fig. 4.11 Estabrooks and Martin's solutions A, B, C, D (Ref. 82). Smoothed version. Dashed line with triangles corresponds to an alternative possibility below 1.3 GeV.

Fig. 4.12 Predictions (Ref. 102) of $\pi^+\pi^- + \pi^0\pi^0$ differential cross-sections based on the four Møller solutions (Ref. 101) at $M_{\pi\pi} = 1.7$ GeV.
Fig. 4.13 Experimental $\pi^0\pi^0$ angular distributions evaluated in the Gottfried-Jackson (t-channel) frame for various mass intervals in the CERN-HEP-Karlsruhe-Pisa-Vienna experiment (Ref. 15) on $p\pi^-\pi^0 n$ at 40 GeV/c. No corrections for efficiencies were made. Dashed curves show contributions from pure spin $J = 2$ and $J = 4$. 
Fig. 4.14 Estabrooks and Martin's solution A (EMA) as given by the tables of Ref. 82. Same notation as in Fig. 4.7.

Fig. 4.15 EMB. Same notation as in Fig. 4.14.
Fig. 4.16 EMC. Same notation as in Fig. 4.14.

Fig. 4.17 EMD. Same notation as in Fig. 4.14.
Whether the term ambiguity is appropriate or not depends on higher moments. As experimental accuracy increases it is conceivable that higher moments would continue to be negligible for a long time. In that case, the truncation at \( k = 3 \) is much better than anticipated, and the scatter in the Argand diagrams reflects the experimental uncertainties on moments. In this situation we would speak of uncertainties rather than ambiguities. A perhaps more likely situation is that an increase of experimental accuracy would lead to the detection of higher moments, making it necessary to allow for more partial waves, etc. In that case the present scatter would not be removed by increased experimental accuracy, one would have a true continuum ambiguity to fight.

Whatever the origin, the existing scatter in Figs. 4.14 to 4.17 shows that we have a very large continuum freedom in practice. In the next two sections, we shall describe how the techniques of fixed-\( t \) amplitude analysis seems capable of removing both the scatter, the over-all phase ambiguity, and the discrete ambiguity.

5. **Techniques of Fixed-Momentum Transfer Analysis**

5.1 **General properties of forward \( \pi^+\pi^- + \pi^\mp \pi^\mp \) scattering**

The amplitude

\[
F^{+-}(s, t) = \frac{4}{3} F_0^S + \frac{1}{2} F_1^S + \frac{1}{6} F_2^S = \frac{4}{3} F^0_t + \frac{1}{2} F^1_t + \frac{1}{6} F^2_t
\]

(5.1)

describes, for \( t = 0 \), forward \( \pi^+\pi^- \rightarrow \pi^\pm\pi^\mp \) scattering and for \( s \leq 0 \) (\( u \geq 4\mu^2 \)) \( \pi^+\pi^- + \pi^\mp\pi^\mp \) scattering.

We choose to discuss this amplitude in some detail, partly for its general interest, partly because in practice it is the most difficult amplitude to control, and in particular because a very simple and nearly rigorous discussion is possible.

It is convenient to introduce the crossing symmetric variable

\[
\nu = \frac{s - u}{4 \mu}
\]

(5.2)

which (for \( t = 0 \)) is equal to pion lab. energy for a stationary target (whatever the relevance of such a variable is in practice).

Then \( F^{+-}(\nu) \) (specifying to \( t = 0 \)) is an analytic function in the \( \nu \)-plane cut along \((-\infty, -\mu) \) and \((\mu, \infty) \) (Fig. 5.1).
Fig. 5.1 Singularity structure of $F^{+-}(\nu)$ in the $\nu$-plane. Also shown is the zero $\nu = \nu_0$ between $\nu = -\mu$ and $\nu = +\mu$ as well as the points $\nu = \pm \nu_H$ corresponding to $E_{HH} = 1.8$ GeV in the $s$-channel and in the $u$-channel. The integration contour $C$ is described in the text.

From the optical theorem [Eq. (1.18)] we get

$$\Im m F^{+-}(\nu + i\epsilon) = \frac{g_{\pi\pi}^2}{16\pi} \sigma_{tot}^{++}(s)$$

$$= \frac{\mu \sqrt{\nu^2 - \mu^2}}{16\pi} \sigma_{tot}^{++}(s)$$ \hspace{1cm} (5.3)

for $\nu > \mu$, and

$$\Im m F^{+-}(-\nu - i\epsilon) = + \frac{\mu \sqrt{\nu^2 - \mu^2}}{16\pi} \sigma_{tot}^{--}(s)$$ \hspace{1cm} (5.4)

again for $\nu > \mu$. Here $\sigma_{tot}^{++}(s)$ are the total cross-sections for $\pi^+ \pi^-$ scattering at $s = 2\nu + 2\mu^2$ [Eq. (5.2)].

Also we have at the thresholds

$$F^{+-}(\mu) = \frac{4}{3} (a_0^{(0)} + \frac{1}{2} a_0^{(2)}) \mu \simeq 0.09 > 0$$

$$F^{+-}(-\mu) = a_0^{(2)} \mu \simeq -0.02 < 0$$

(Section 3.4).

Thus $F^{+-}(\nu)$ has a zero on the real axis in the interval $(-\mu, \mu)$. Without knowledge of the signs of $a_0^{(1)}$ we may prove that $F^{+-}(\nu)$ can have at most two zeros. For simplicity we assume the signs given above. We also assume that the zero position is known (it can clearly be calculated very accurately from the known low-energy parameters). Let the zero-position be $\nu = \nu_0$ with $\nu_0 \in (-\mu, \mu)$.

By considering the phase variation of the function

$$F^{+-}(\nu) / (\nu - \nu_0)$$
around the contour $C$ of Fig. 5.1, and using positivity of the imaginary parts [Eqs. (5.3) and (5.4)] as well as the Froissart bound for the total cross-section, it is not difficult to establish that

$$F^+(-\nu) / (\nu - \nu_0)$$

can have no zeros. This will allow us to write down simple relations [Eqs. (5.10) to (5.12) below].

First, let us discuss the standard picture of the asymptotic behaviour of $F^+(\nu)$. This is based on the $t$-channel decomposition Eq. (5.1). For the $I_t = 0$ part, we assume dominance by $P$ and $f$ exchange Regge pole; for $I_t = 1$ by reggeized $\rho$-exchange, whereas $I_t = 2$ exchange is assumed negligible asymptotically.

Thus

$$F^+(\nu) \sim \frac{F^+(\nu)}{\nu \to \infty} F_P(\nu) + F_f(\nu) + F_S(\nu) \quad (5.5)$$

$$F^+(\nu) \equiv F^{++}(\nu) \sim \frac{F_P(\nu)}{F_P(\nu)} + F_f(\nu) - F_S(\nu)$$

with

$$F_P(\nu) \sim i \frac{\sigma_{tot}(\nu)}{\nu} \lesssim \frac{\mu}{\mu} \frac{\sigma_{tot}(\nu)}{\nu}$$

$$F_f(\nu) \sim \frac{\beta_f(\nu)}{\nu} \left( \frac{\nu}{\nu_0} \right)^{\alpha_f(\nu)}$$

$$F_S(\nu) \sim \frac{\beta_S(\nu)}{\nu} \left( \frac{\nu}{\nu_0} \right)^{\alpha_S(\nu)}$$

This corresponds to the case where the $\sigma_{tot}^{\pm}(\nu)$ have a common finite value at infinite energy. As will become clear below (Section 5.2), a modification by logarithms is irrelevant for our present purpose.

In the standard duality picture with no exotic states, we expect exchange degeneracy of the $\rho$- and $f$-trajectories with

$$\alpha_f(\nu) = \alpha_S(\nu) \approx \frac{1}{2}$$

as well as of the residue functions:

$$\beta_f(\nu) = \beta_S(\nu) \quad (5.7)$$

If strong exchange degeneracy [Eq. (5.8)] holds, $F^{+-}$ becomes imaginary much faster as $\nu \to \infty$ than as $\nu \to -\infty$:

$$F_f(\nu) + F_S(\nu) \quad \text{purely imaginary (} \pi^+ \pi^- \to \pi^+ \pi^- \text{)}$$

$$F_f(\nu) - F_S(\nu) \quad \text{purely real (} \pi^+ \pi^- \to \pi^+ \pi^- \text{)}$$
Knowing the asymptotic behaviour of \( F^{+-} \) we can write down a twice subtracted dispersion relation:

\[
\text{Re} F^{+-}(\nu) = -\frac{\nu^2 - \mu^2}{\pi} P \int_0^\infty \frac{d\nu'}{\nu' \nu^2 - \mu^2} \left\{ \frac{\text{Im} F^{+-}(\nu')}{\nu' - \nu} + \frac{\text{Im} F^{++}(\nu')}{\nu' + \nu} \right\} \\
+ \frac{1}{12} (2 a_0^{(0)} - 5 a_0^{(1)}) \nu + \left( \frac{1}{6} a_6^{(0)} + \frac{7}{12} a_6^{(1)} \right) \mu .
\]

(5.9)

It is our a priori knowledge of the existence of an equation like Eq. (5.9) which supplies the missing phase information needed in addition to the cross-section \(|F^{+-}|^2\) in order to get the full amplitude.

It is important to be aware that analyticity is very much more than mere smoothness. Analyticity such as expressed by the dispersion relation has a fundamental non-local element in it. In practice this implies that an amplitude analysis exploiting analyticity needs to be performed at all energies at once. This clearly leads to a difficult computational problem. In Section 5.3 we shall describe some powerful general techniques which have been developed. In the present case of \( \pi^{+}\pi^- \) forward scattering, a very neat presentation is possible: the phase-modulus representation.

The knowledge that \( F^{+-}(\nu) \) has only the one zero at \( \nu = \nu_0 \) allows us to conclude that

\[
\text{log} \left( \frac{F^{+-}(\nu)}{\nu - \nu_0} \right) = \text{log} \left| \frac{F^{+-}(\nu)}{\nu - \nu_0} \right| + i \text{arg} F^{+-}(\nu)
\]

(5.10)

is analytic in the \( \nu \)-plane cut along \((-\infty, -\mu), (\mu, \infty)\). Thus, a dispersion relation for that logarithm would give information on the modulus \(|F^{+-}|\) in terms of the phase (= imaginary part of logarithm). It is clearly more convenient to turn the dispersion relation "inside out". We can do this by writing it for the function

\[
G(\nu) \equiv (\nu^2 - \mu^2)^{1/2} \text{log} \left( \frac{F^{+-}(\nu)}{\nu - \nu_0} \right),
\]

(5.11)

using just above the cuts

\[
(\nu^2 - \mu^2)^{1/2} = \mp i \sqrt{\nu^2 - \mu^2}
\]

for \( \nu > \mu \). This gives

\[
\text{arg} F^{+-}(\nu) = \sqrt{\nu^2 - \mu^2} P \int_0^\infty \frac{d\nu'}{\nu' \nu^2 - \mu^2} \left\{ \frac{\text{log} \left| \frac{F^{+-}(\nu')}{\nu' - \nu_0} \right|}{\nu' - \nu} + \frac{\text{log} \left| \frac{F^{++}(\nu')}{\nu' + \nu_0} \right|}{\nu' + \nu} \right\}
\]

(5.12)
with
\[ |F^{\pi\pi}(\nu)|^2 = 2\mu(\nu + \mu) \frac{d\sigma^{\pi\pi}}{d\Omega}. \]

This shows explicitly that knowledge of the cross-section at all energies yields the phase uniquely. In fact, we have information not yet exploited. Thus no asymptotic phase information has been inserted, and our knowledge of the phase in the elastic region has not been used.

In practice, life is very much more complicated. We shall solve the various complications one by one.

Perhaps the most serious difficulty is that we do not know (or know very little about) the cross-section above 1.8 GeV. In the next subsection we shall argue that although this may lead to a very large phase ambiguity in principle, it only leads to a surprisingly small one in practice provided high-energy \( \pi\pi \) cross-sections are "as well behaved" as high-energy cross-sections are for other hadronic processes.

5.2 The effect of unknown high-energy cross-sections

Imagine two analytic model amplitudes for \( F^{\pi\pi}(\nu) \), say \( F_1(\nu) \) and \( F_2(\nu) \), which are "well behaved" and both fit our knowledge of forward \( \pi^- \pi^+ \to \pi^- \pi^+ \) and \( \pi^+ \pi^- \to \pi^+ \pi^- \) scattering. How different can their phases be?

To give an estimate of this, let us consider the problem somewhat simplified:

i) \( F_1 \) and \( F_2 \) both have one zero situated at \( \nu = \nu_0 \); \( \nu_0 \) is known, \( \nu_0 \in (-\mu, \mu) \).

ii) \( |F_1(\nu)| \equiv |F_2(\nu)| \) for \( -\nu_H \leq \nu \leq -\mu \) and \( \mu \leq \nu \leq \nu_H \), where \( \nu_H \approx 84 \mu \) corresponds to 1.8 GeV.

iii) Our crucial smoothness assumption we take to be that above the "resonance region"

\[ \left| \frac{d\log|\sigma_{\pi\pi}(\nu)|}{d\log\nu} \right| < M. \tag{5.13} \]

\( M \) is a pure number independent of units for \( \sigma \) or \( \nu \) and independent of the base of the logarithm. Figure 5.2 shows a plot of \( \log \sigma_{\pi\pi} \) versus \( \log r_L \) for \( \pi N \) scattering. We see that the "resonance" or "bump" region extends to \( p_L \approx 2 \) GeV and that we may take \( M < \frac{1}{2} \) above that energy.

At 1.8 GeV we are not yet above the energy where resonances appear as separate bumps. Below we shall estimate separately the effect of the condition (5.13) and that of the \( h \) meson.
Writing
\[ F_1(\nu) = |F_1(\nu)| e^{i\Phi_1(\nu)} \quad j \quad F_2(\nu) = |F_2(\nu)| e^{i\Phi_2(\nu)} \]
we want to derive a bound on
\[ \Psi(\nu) \equiv \Phi_1(\nu) - \Phi_2(\nu) \]
in terms of assumptions about i) \( \Gamma_{h \to \pi^+} \) or ii) \( M \) in (5.13).

To this end let us define the function \( h(\nu) \) by
\[ h(\nu) \equiv \left( \log \frac{|F_2(\nu)|}{|F_1(\nu)|} \right)/(1 - (\nu/\mu)^2)^{1/2} \quad (5.14) \]

By the assumptions, \( h(\nu) \) is regular in the \( \nu \)-plane cut along \((-\infty, -\nu_H)\) and \((\nu_H, \infty)\), i.e. \( h(\nu) \) is regular in the "measurement regions" \( \mu \leq |\nu| \leq \nu_H \). In that region we have
\[ \Psi(\nu) = \sqrt{(\nu/\mu)^2 - 1} \cdot h(\nu) \quad (5.15) \]

An unsubtracted dispersion relation for \( h(\nu) \) exists:
\[ h(\nu) = \frac{1}{\pi} \int_{\nu_H}^{\infty} d\nu' \left\{ \frac{\text{Im} h(\nu')}{\nu' - \nu} - \frac{\text{Im} h(-\nu')}{\nu' + \nu} \right\} \quad (5.16) \]
For $|\nu| \geq \nu_H$ we have

$$\text{Im } h(\nu) = \log \left| \frac{F_1(\nu)}{F_2(\nu)} \right| \sqrt{(\frac{\nu}{\mu})^2 - 1}$$

$$\sim \frac{1}{\nu} \log \left( \frac{\sigma_{\text{tot}}^{(1)}(\nu)}{\sigma_{\text{tot}}^{(2)}(\nu)} \right), \quad \nu \rightarrow \infty,$$

where $\sigma_{\text{tot}}^{(1)}$ and $\sigma_{\text{tot}}^{(2)}$ are the total cross-sections implied by $F_1$ and $F_2$, respectively. For $\nu > \nu_H$ they are the $\pi^+\pi^-$ total cross-sections, and for $\nu < -\nu_H$ they are the $\pi^+\pi^+$ total cross-sections.

Now let us estimate the phase uncertainty caused by an "unknown" resonance, using, as an example, the $h$ meson. More precisely, we consider the case where $F_1$ has no $h$-meson bump whereas $F_2$ has one. However, we assume that above the $h$ region we have

$$F_2(\nu) \approx F_4(\nu)$$

and that asymptotically the equality becomes exact. Violations of Eq. (5.18) are treated separately below, using the condition (5.13). Condition (5.18) implies that

$$\int_{\text{h-region}} d\nu \log |F_1(\nu)| = \int_{\text{h-region}} d\nu \log |F_2(\nu)|.$$  

Thus, away from the $h$ region itself we can approximate the dispersion integrals (5.16) by a dipole form. This gives for the phase difference $\gamma_h$ between $F_1$ and $F_2$ due to the $h$ meson,

$$\left| \gamma_h(\nu) \right| \approx \frac{1}{3} (2\ell + 1) \chi_h \left( \frac{m_h}{2\mu} \right)^2 \frac{\nu}{\nu_h} \frac{1}{(\nu - \nu_h)^2}.$$  

Here $\ell = 4$, $\chi_h$ is the branching ratio of $h \rightarrow 2\pi$, and we assume $\chi_h < 33\%$ (the LSV model would give $\chi_h \approx 25\%$; Table 1.1). Inserting numbers we find that

$$\left| \gamma_h \right| < 1^0 \text{ at } M_{\pi^\pm} = 1400 \text{ MeV},$$

$$< 5^0 \text{ at } M_{\pi^\pm} = 1600 \text{ MeV},$$

$$< 12^0 \text{ at } M_{\pi^\pm} = 1800 \text{ MeV}.$$  

We conclude that even though the $h$ meson is quite close to the end of the data region $M_{\pi^\pm} = 1800$ MeV, and even though we have taken it to be a prominent resonance, it gives a surprisingly small uncertainty in dispersion theory evaluations of the phase of the forward $\pi^+\pi^-$ amplitude.
For the non-resonant part of the unknown high-energy cross-section, we next proceed by using the condition (5.13). First we write

\[
\text{Im} \, h(\nu) = \frac{\mu}{\nu} \, s^+(\nu) + \frac{\mu}{\nu} \, s^-(\nu) \tag{5.22}
\]

\[
\text{Im} \, h(\nu) = -\frac{\mu}{\nu} \, s^+(\nu) + \frac{\mu}{\nu} \, s^-(\nu) .
\]

Then

\[
h(\nu) = \frac{2\mu}{\pi} \left[ \int_{\nu_H}^{\infty} \frac{d\nu'}{\nu'^2 - \nu^2} \right] + \nu \int_{\nu_H}^{\infty} \frac{d\nu'}{\nu'\left(\nu'^2 - \nu^2\right)} .
\]

Condition (5.13) then implies

\[
\left| \frac{d}{d \log \nu} \, s^+(\nu) \right| < M, \quad \nu > \nu_H .
\]

We can then derive bounds, first on \(|h'(\nu)|\) using partial integration, and subsequently on \(|h(\nu)|\) by making use of the fact that \(\psi\) is small below 1 GeV. A very crude estimate for the integrals gives

\[
|\psi(\nu)| \leq \frac{4}{3} \left(\frac{\nu}{\nu_H}\right)^3 \left(1 + \left(\frac{\nu}{\nu_H}\right)^2\right) \cdot M . \tag{5.23}
\]

Taking \(M < \frac{1}{2}\) we get

\[
|\psi| < 3.5^\circ \quad \text{at} \quad M_{\pi\pi} = 1400 \text{ MeV},
\]

\[
< 8.5^\circ \quad \text{at} \quad M_{\pi\pi} = 1600 \text{ MeV},
\]

\[
< 19^\circ \quad \text{at} \quad M_{\pi\pi} = 1800 \text{ MeV} . \tag{5.24}
\]

Again we can conclude that phase ambiguities coming from lack of knowledge about high-energy cross-sections are surprisingly small.

5.3 Analytic approximation theory

For practical evaluations, the phase representation method outlined is not the most convenient one [for an actual application using a modified version, see Common\(^{104}\)]. For t negative zeros of the amplitude are less easy to control and the right- and left-hand cuts overlap. Also, information is available in the form of values of a statistical nature at a finite number of energies. This means that some smoothing is necessary prior to any principal value integration. And principal value integrations tend to be lengthy with computers.

In this section we outline some very powerful analytic approximation and representation methods. These have been developed from early works by Cutkosky\(^{105}\), Cislli\(^{106}\), and others\(^{36},^{37}\). For the present type of applications a definite
scheme has been set up by Pietarinen\(^{86,90-97}\). This has proved exceedingly useful in practice. The technique is remarkable in that it combines mathematical elegance with practical applicability. Principal value integration is totally circumvented, with corresponding savings in computing time.

We start by considering a simplified example.

Let \( F(\nu) \) be analytic in the \( \nu \)-plane cut along \((\nu_0, \infty)\) and finite throughout (i.e. \( F \) has only a right-hand cut). Further, assume \( F \) has only discrete singularities on the real axis of square root type or smoother.

We start by considering the problem of finding analytic approximants to \( F \) when "experimental" information on the real and imaginary parts of \( F \) is available at a finite number of points. In practice, we shall have to cope with the more difficult situation that only information on the modulus of \( F \) is available. We return to this later.

Thus assume we have measurements on \( \text{Re} \ F \) at points

\[
\nu_1, \ldots, \nu_M
\]

giving results

\[
f_1, \ldots, f_M
\]

with standard deviations

\[
e_1, \ldots, e_M
\]

also assume we have measurements on

\( \text{Im} \ F \) at points

\[
\nu_{M+1}, \ldots, \nu_N
\]

with results

\[
f_{M+1}, \ldots, f_N
\]

with standard deviations

\[
e_{M+1}, \ldots, e_N
\]

It may happen that some of the points \( \nu_i \) with \( i > M \) coincide with some of the points \( \nu_j \) with \( j \leq M \).

Let \( \phi(\nu) \) be any analytic function with analyticity properties similar to \( F \). We want in some sense (to be defined) to find the optimal approximant for the given information.

The usual \( \chi^2 \) for any \( \phi \) is defined by

\[
\chi^2(\phi) = \sum_{k=1}^{M} \left( \frac{\text{Re} \phi(\nu_k) - f_k}{e_k} \right)^2 + \sum_{k=M+1}^{N} \left( \frac{\text{Im} \phi(\nu_k) - f_k}{e_k} \right)^2.
\]

(5.25)

Infinitely many functions with \( \chi^2 = 0 \) exist. In general they will be extremely un-smooth outside the data region, and we would like to consider them as totally
"unacceptable" candidates for a physical amplitude. We want to introduce our a priori bias that amplitudes are smooth, and declare that the optimal approximant is the "smoothest" function that has an acceptable $\chi^2$.

To this end we want to define a measure $\Phi(\phi)$ of the lack of smoothness of $\phi$; $\Phi(\phi)$ is called the penalty functional or the convergence test function. The optimal approximant is then defined to be the one that minimizes

$$X(\phi) = \chi^2(\phi) + \Phi(\phi). \tag{5.26}$$

How do we define $\Phi(\phi)$ and how do we minimize $X(\phi)$? To understand this, it is useful to focus on particular representations of $\phi(\nu)$.

Consider the mapping

$$\nu \rightarrow \chi = \frac{a - (\nu_0 - \nu)}{a + (\nu_0 - \nu)}^{1/2}, \quad a \text{ real } > 0,$$

mapping the cut $\nu$-plane to the interior of the unit circle $|z| \leq 1$ (Fig. 5.3).

---

**Fig. 5.3** Image of the $\nu$-plane cut along $(\nu_0, \infty)$ under the mapping $\nu \rightarrow z$. 

---
Any function \( \Phi(v) \) with analyticity properties as specified above may be represented by the following Taylor series converging throughout the unit circle, including \( |z| = 1 \) (i.e. \( \text{Im } v = 0 \)):

\[
\Phi(v) = \Phi(z(v)) = \sum_{n=0}^{\infty} a_n z^n.
\]

(5.27)

Pietarinen argues that a good measure for lack of smoothness is given by

\[
\Phi(\Phi) \equiv \lambda \sum_{n=0}^{\infty} (n+1)^3 a_n^2,
\]

(5.28)

where \( \lambda \) is some real parameter to be specified below.

It may seem that the form (5.28) requires too stringent smoothness. Indeed we wanted to approximate amplitudes \( F(v) \) allowed to have square-root singularities. For such a function, however, it can be shown that

\[
|a_n| \lesssim \lambda^{-\frac{1}{2}} n^{-\frac{3}{2}}, \quad \text{for } n \to \infty,
\]

(5.29)

where \( \lambda^{-\frac{1}{2}} \) is a measure of the strength of the singularity. But it is clear that the behaviour (5.29) would yield a divergent result for \( \Phi \) as defined by (5.28). The point is that this is exactly what we want! In fact to make sure that \( F(v) \) really does have a square-root singularity at some point would require infinite statistics, in which case \( \chi^2 \) would also diverge. We can show that when \( X(\phi) \) is at minimum, \( \Phi \approx \chi^2 \). Thus, as the experimental information improves, both \( \chi^2 \) and \( \Phi \to \infty \) together. All of these heuristic arguments can be made more precise. The effect of the scheme is thus for the approximants to allow, at most, square-root singularities, and only to the extent they are really forced upon the approximant by the data.

It should also be intuitively clear from the above that the parameter \( \lambda \) in (5.28) is not a free one. It is related to the strength of the "worst" square-root singularity, and as such it is given by data. Below we shall quote a formal condition that determines \( \lambda \).

First let us consider the problem of finding the coefficients \( \{a_n\} \) that give the minimal value of \( X(\phi) \). We assume a (trial) value for \( \lambda \). Clearly \( X(\phi) \) is bilinear in the unknowns \( \{a_n\} \) by virtue of Eqs. (5.25), (5.27), and (5.28). Therefore the minimum is unique. The condition that

\[
\frac{\partial}{\partial a_i} X = 0
\]

(5.30)

leads to a set of linear equations for the \( \{a_n\} \). In practice, all sums are truncated at some value \( n = P \). By virtue of \( \Phi \), the results are completely stable against increasing \( P \), provided \( P \) is reasonably large, i.e. of the same order of
magnitude as the number $N$ of data. Equation (5.30) then becomes $P$ independent linear equations for the $P$ unknowns $a_1, \ldots, a_P$.

The solution is unique and found by a simple matrix inversion. The inverted matrix is the error correlation matrix, and the diagonal elements are estimates of the square of the standard deviation on the $a_n$'s: $\sigma^2(a_n)$. We may derive the following condition on $\lambda$ (maximum likelihood):

$$\lambda \sum_{n=0}^{\infty} (n+1)^3 \tilde{a}_n^2 = \sum_{n=0}^{\infty} \left[ 1 - \lambda (n+1) \sigma^2(a_n) \right],$$

(5.31)

where the $\tilde{a}_n$'s are the solutions of the linear equations and where the sums may also be truncated at $P$. In practice, it is not difficult to find a $\lambda$ so that Eq. (5.31) is satisfied: we make a few trials and proceed by iteration. An even simpler procedure is to start by a large value of $\lambda$ and decrease $\lambda$ until $\chi^2$ is acceptable.

Once the coefficients $\{\tilde{a}_n\}$ for the optimal approximant $\tilde{\phi}(\nu)$ is found, $\tilde{\phi}$ may be evaluated very easily as a polynomial, truncating Eq. (5.27). Using nested multiplication, this polynomial evaluation is extremely much faster than a principal value integration on a computer.

5.4 Practical fixed-$t$ and fixed-$u$ analysis

This is a brief account of the current status of the work by Progatt and myself. In Section 5.5 below I shall discuss the results and compare them with the somewhat similar works by Common and by Johnson, Martin and Pennington.

The present account deviates a little from our published version in ways I shall indicate.

Let us start by considering forward and near-forward $\pi^+ \pi^- \rightarrow \pi^+ \pi^-$ scattering in the $s$-channel and $\pi^+ \pi^- \rightarrow \pi^+ \pi^+$ scattering in the $u$-channel.

To set up a good representation for the amplitude $F^{\pm^-}(\nu,t)$ at fixed-$t$ we must allow for the following complications:

i) At $|\nu| \rightarrow \infty$, $F^{\pm^-}$ itself diverges for non-vanishing total cross-section.

ii) We have learned in Section 5.2 that it is important to request the high-energy part of the amplitude to be very much smoother than the low- and intermediate-energy part. For the forward amplitude this was not explicitly taken into account in Ref. 107.

iii) $F^{\pm^-}$ has both a right-hand cut and a left-hand cut; they overlap for $t < -4\mu^2$. 
To cope with these complications we write
\[ F^+(\nu, t) = F_P^-(\nu, t) + F_f^-(\nu, t) + F_s^-(\nu, t) + \mathcal{F}^{++}(\nu, t), \]  
(5.32)
where individual terms have right-hand cuts and left-hand cuts. For the first three we only give the asymptotic forms
\[
\begin{align*}
F_P^-(\nu, t) & \sim \beta_P^-(t) e^{-\frac{\pi}{2} \alpha(t)} \sin\frac{\pi}{2} \alpha(t) \\
F_f^-(\nu, t) & \sim \beta_f^-(t) e^{-\frac{\pi}{2} \alpha(t)} \left( \frac{s}{s_0} \right)^{\alpha(t)} \\
F_s^-(\nu, t) & \sim \beta_s^-(t) e^{-\frac{\pi}{2} \alpha(t)} \left( \frac{s}{s_0} \right)^{\alpha(t)} .
\end{align*}
\]  
(5.33)

The actual forms chosen have acceptable low-energy behaviours for all t.

Trajectories are fixed as \( \text{GeV} = 1 \):
\[
\begin{align*}
\alpha_P^-(t) &= 1 + 0.3 \cdot t \\
\alpha_f^-(t) &= 0.5 + 0.9 \cdot t .
\end{align*}
\]  
(5.34)

The residue functions were parametrized as
\[
\begin{align*}
\beta_P^-(t) &= -\frac{\sigma^+(\nu)}{32\pi} e^{bt} , \quad b \approx 3-4 \text{ GeV}^{-2} \\
\beta_f^-(t) &= \alpha_f \left( -0.7 T^r \left( 1 - \alpha(t) \right) \cos\frac{\pi}{2} \alpha(t) + b_f \cdot t \right) \\
\beta_s^-(t) &= \alpha_s \left( 0.7 T^l \left( 1 - \alpha(t) \right) \sin\frac{\pi}{2} \alpha(t) + b_s \cdot t \right) .
\end{align*}
\]  
(5.35)

Apart from the parameters \( a \) and \( b \), the \( \rho^- \) and \( f^- \)-residue functions are of the LSV form \(^{13}\).

From Section 5.2 we know that the specific form of \( F^+ \) at high energies is unimportant as long as it is smooth. The above ansatz allows us to test this, and further has the advantage of ensuring standard Regge behaviour.

For the remainder, \( \mathcal{F}^{+-}(\nu, t) \), we write
\[ \mathcal{F}^{+-}(\nu, t) = \mathcal{F}_R^{+-}(\nu, t) + \mathcal{F}_L^{+-}(\nu, t) , \]  
(5.36)

where \( \mathcal{F}_R^{+-} \) and \( \mathcal{F}_L^{+-} \) only have a right-hand \( \sigma^- \) cut and a left-hand cut, respectively. They are represented as functions of the variable \( z(\nu) \) introduced above using
\[
\nu_o = \nu_o(t) = \nu + \frac{t}{4\mu} ,
\]  
and \( \mu = 1.8 \text{ GeV}^{1/2} \), giving a maximal spread of the energy interval \( 600 \text{ MeV} < \mu < 1800 \text{ MeV} \) on the unit circle.
Then
\[
\mathcal{I}^+_{R\pm}(\nu,t) = \left(1 + Z(\nu t)\right)^{2-2\alpha(t)} \mathcal{P}(\nu,t) \mathcal{I}^+_{R\pm}(z(\nu t))
\]
\[
\mathcal{I}^+_{L\pm}(\nu,t) = \left(1 + Z(-\nu t)\right)^{2-2\alpha(t)} \mathcal{P}^+_{R\pm}(z(-\nu t))
\]
(5.37)
where
\[
\mathcal{P}_{R\pm}(z) = \sum_{h=0}^{P} \alpha^{(t \pm)} n^h,
\]
(5.38)
and where \(\mathcal{P}(\nu,t)\) is a simple rational function of \(z\) having poles just outside the unit circle at positions appropriate to the three prominent resonances \(\rho, f,\) and \(g\) and using PDG values for masses and total widths.

For \(\mathcal{P}\) we have taken \(P = 20\) for \(\pm\) and \(P = 10\) for \(++\). The function \(\mathcal{P}(\nu,t)\) is quite inessential but may be thought to leave a smoother object \(\Psi_{++}\) to be represented by a polynomial.

The penalty function is taken as
\[
\mathcal{O}(F^{++}) = \lambda_{++} \sum_{n=0}^{P_{++}} \left(\alpha_{n}^{++}\right)^2 (n+1)^3 + \lambda_{+-} \sum_{n=0}^{P_{-+}} \left(\alpha_{n}^{+-}\right)^2 (n+1)^3.
\]
(5.39)

Note that the factor \(\left[1 + z(\pm \nu)\right]^{2-2\alpha(t)}\) makes \(\mathcal{I}^{++}\) vanish at \(|\nu| \to \infty\) as a (daughter-) Regge pole with intercept one unit lower than \(\alpha(t)\). This means that we are posing very stringent smoothness requirements on \(F^{++}\) at high energies as we wanted. However, we are not at all committed to definite values of the parameters entering into the residue functions (5.35).

A similar analysis was carried out at fixed \(u\)-values. The treatment is quite similar in spirit. Simplifications occur, however, in that \(F^{++}(\nu,u)\) is symmetric in \(\nu\). Also the asymptotic form corresponds to exotic exchange. We therefore require the amplitude to vanish [using factors similar to \((1 + z)^{2-2\alpha}\) in Eqs. (5.37)]. This immediately ensures that our backward high-energy amplitudes are very smooth.

### 5.5 Results of the fixed-\(t/u\) analysis

Having described the method of analysis alone, let us now start by listing the numerical input.

For \(t < 0\) (\(u < 0\)) there are unphysical domains on the cuts:
\[
\nu_{\nu}(t) = \mu + t/4 \mu \leq \nu \leq \nu_{\nu}(t) = \mu - t/4 \mu
\]
on the right-hand cut and a similar one on the left-hand cut. On these, partial wave expansions are expected to converge well for the imaginary part provided $|t| (|u|)$ is not too large (see Section 1). For the real part no useful information is expected to be obtainable.

For the whole region below the $K\bar{K}$ threshold we use BFP-type solutions to the Roy equations \(^{75}\) (Section 3.4). Different values of $a_0^b$ were tried but the results were found to be insensitive for $a_0^b$. Principal results use $a_0^b = 0.3 \mu^{-1}$ (see Section 3.5). The s- and p-waves fit data above 600 MeV. For the evaluation of the imaginary part in unphysical regions, both d- and f-waves (as calculated from the Roy equations) were retained (Table 3.2).

The analysis was carried out at $t$ (u) values equal to 0, -0.2, -0.4, -0.6, -0.8, and -1.0. For the larger $|t| (|u|)$ values, results are not reliable. However, for the subsequent constrained phase-shift analysis it seems that we depend essentially on the $t = 0$ and the $u = 0$ amplitudes (see Section 6 below).

For $M_{\pi \pi} > 1$ GeV we are always in the physical region. Here we only use experimental information on the modulus of $|F^{+-}|$ as deduced from any one of the four EM solutions (Section 4 and Ref. 82). As emphasized already, despite the noisy character of these phase-shift solutions, the modulus reconstructed from them is quite smooth as is in fact obvious from the way they were derived.

With information on the modulus rather than on the real or imaginary part, $X(\xi)$ in Eq. (5.26) is no longer bilinear in the coefficients $\{a_n^{+-}, a_n^{++}\}$. Thus a more involved strategy for finding the optimal solution is required. In practice we used a definite phase-shift solution as the starting value. In the neighbourhood of that, the problem was then linearized (for details, see Ref. 107), and the procedure was iterated until stability was obtained. The results were found to be completely independent of the starting values we tried.

Information on the $I = 2$ u-channel cut ($\pi^+ \pi^+ \rightarrow \pi^+ \pi^+$ scattering) is important. Below 1 GeV we used the predictions of the BFP solution so that crossing is guaranteed. Above 1 GeV we used smooth forms joining on to the result of Hoogland et al. \(^{55}\) at 1500 MeV and extrapolating to 1800 MeV. Various inelasticities for the $I = 2$ s- and d-waves (consistent with the experimental cross-section) were tried.

For $t = 0$ a rather detailed analysis is now available. Figure 5.4 shows results of attempts by Froggatt and myself to estimate the extent of the phase ambiguity in $F^{+-}$ below 1.8 GeV. As expected from the discussion of Section 5.2, the results are fairly insensitive to even quite large changes in parameters such as $\sigma_T(\omega)$, etc., in Eqs. (5.35). Thus the extreme right and left versions in Fig. 5.4 correspond to $\sigma_T(\omega) = 20$ mb and $\sigma_T(\omega) = 5$ mb, respectively (other parameters were changed as well to get those two solutions). Taking larger or smaller values of $\sigma_T(\omega)$ appears to give unacceptable fit to the data.
These findings are nicely confirmed in the recent work of Johnson, Martin and Pennington\(^{108}\). They employ standard dispersion-relations, fix the high-energy contribution by a parametrized Regge form, use the modulus of \(F^{\pi^+\pi^-}\) between 1 GeV and 1.8 GeV, and parametrize and fit the phase to the dispersion relation. This leaves a spurious singularity at 1.8 GeV which should not be important below 1.7 GeV, say. The technique is fairly straightforward in this case since \(|F^{\pi^-}|\) is nowhere small (if \(|F^{\pi^-}|\) for some \(t\)-value \# 0 develops a zero, the phase might be a complicated thing to parametrize). Their result for the forward amplitude is shown in Fig. 5.5. There is complete agreement with Fig. 5.4.

A rather different result, however, has been obtained by Common\(^{106}\). The phase of his forward amplitude is systematically larger than in Figs. 5.4 and
Fig. 5.5 Forward amplitude $F^{+-}$ from Ref. 108. Notation similar to Fig. 5.4.

Fig. 5.6 The backward $\pi^+\pi^- \to \pi^-\pi^+$ amplitude $F^{+-}$ at $u = 0$ from Ref. 107.
5.6 by an amount exceeding the estimate of Section 5.2. We suggest that this
is due to lack of smoothness of high-energy amplitudes. Indeed such a high degree
of smoothness was not explicitly imposed on his analysis.

Finally, Fig. 5.6 shows the backward amplitude \( f^{+-}(v,u) = 0 \) of Ref. 107.

In the next section we shall describe the implications of these results on
\( \pi^\pi \) phase-shift solutions.

6. UNIQUENESS? DISCUSSION - CONCLUSIONS

6.1 Constrained phase-shift analysis\(^{107} \)

Having obtained fixed-\( t \) and fixed-\( u \) amplitudes, we are ready to carry out a
constrained phase-shift analysis.

We emphasize that this is not a partial wave projection of the fixed-\( t \) and
fixed-\( u \) amplitudes, energy by energy. In practice this would yield results having
a poor fit to the "experimental" moments and having poor energy smoothness for the
\( \pi^\pi \) partial waves. The reason is that there is no built-in continuity in \( t \) (\( u \))
in the analysis of the preceding section, and for the larger values of \(|t| \) (\(|u|\))
the results are not very stable.

Instead we propose to carry out a "standard" energy-independent phase-shift
analysis of the "experimental" moments, constraining, however, the undetermined
phase to the fixed momentum transfer amplitudes. Roughly, then, the \( t = 0 \) ampli-
dude sets the over-all phase, whereas the \( u = 0 \) amplitude picks out the "right"
discrete solution. The remaining amplitudes at \( t, u \neq 0 \) are partly redundant and
anyway much suppressed by the "error" used (see below).

Let us start by defining the "experimental" moments \( \{A_L\} \):

\[
|F^{+-}(M_{\pi\pi}, \cos \Theta)|^2 \equiv \sum_{L=0}^{\infty} (2L+1) A_L(M_{\pi\pi}) P^L(\cos \Theta).
\]  

(6.1)

The \( \{A_L\}'s \) are obtained, energy by energy, by using on the left-hand side the
four EM phase-shift solutions A, B, C, D (Section 4). This gives rise to seven
"experimental" \( A_L \)'s, \( A_0, \ldots, A_6 \). These are (almost) independent of the solution
used on the left-hand side of Eq. (6.1).

We then write

\[
F^{+-}(M_{\pi\pi}, \cos \Theta) = \sum_{L=0}^{L_{\text{max}}} (2L+1) \hat{f}^{+-}(M_{\pi\pi}) P^L(\cos \Theta),
\]

(6.2)

where we take \( L_{\text{max}} = 6 \).
The 14 real parameters,
\[ \text{Re} f_0^{+}, \text{Im} f_0^{+}, \ldots, \text{Re} f_6^{+}, \text{Im} f_6^{+}, \]
are then determined by minimizing the quantity
\[ \chi^2 \equiv \chi^2 \left( \{ A_L \} \right) + \chi^2 \left( \{ f^{+} \} \right) + \Phi \left( \{ f^{+} \} \right) + C(\eta^0)^4 \] (6.3)
We now describe the meaning of these three terms:

\[ \chi^2 \left( \{ A_L \} \right) \] is simply the \( \chi^2 \) for any set \( \{ f^{+} \} \) to the "experimental" moments \( A_0, \ldots, A_6 \) with errors taken from the off-shell moments \( \{ A_L \} \) of Fig. 4.2.

\[ \chi^2 \left( \{ F (\cos \theta_1) \} \right) \] is the \( \chi^2 \) to the fixed-\( t \) and fixed-\( u \) amplitudes: for the fixed value of \( M_{\Pi\Pi} \) being considered, the \( t \)- and \( u \)-values, \( u = t = 0.0, -0.2, \ldots, -1.0, \) define 12 \( \cos \theta \) values \( \{ \cos \theta_1 \} \) and 12 complex amplitudes. The \( \chi^2 \) is evaluated by (1) taking an error of 0.03 for the \( t/u = 0 \) amplitudes, and \( \infty \) for the others; (2) iterating the complete analysis with errors between 0.03 and 0.10 for all \( t/u \)-values and so that \( \chi^2 \left( \{ A_L \} \right) \leq 7. \)

The term \( \Phi \) is chosen to ensure a reasonable convergence of the sum Eq. (6.2). Specifically
\[ \Phi \left( \{ f^{+} \} \right) = \sum_{\ell=0}^{\ell_{\text{max}}} \left[ (\text{Re} f^{+}_\ell)^2 + (\text{Im} f^{+}_\ell)^2 \right] C_{\ell} \] (6.4)
For \( \ell \to \infty \) we expect the \( C_{\ell} \)'s to grow exponentially with \( \ell \) at a rate given by the distance to the nearest singularities in the cos \( \theta \) plane (see Sections 1 and 5.3). We found it was possible to take \( C_{\ell} \) with \( \ell > 3 \) so large that the resulting \( f^{+}_\ell \) with \( \ell > 3 \) were negligible, and yet have decent \( \chi^2 \) to \( \{ A_L \} \) and \( \{ F (\cos \theta_1) \} \) and decent smoothness in energy of the partial waves.

Finally the term \( C(\eta^0)^4 \) was used with \( C = 5 \) to cure the problem with \( s \)-wave unitarity seen in the M"{a}nner ++ and EM(B) solutions (Figs. 4.8a, 4.1, 4.15a) as well as in our published solution. Our current results are shown in Fig. 6.1.

To obtain the even partial waves of definite isospin \( I = 0 \), we must "correct" for the \( I = 2 \) part in the \( \pi^+ \pi^- \) amplitude [see Eqs. (1.6)]. Fortunately the weight of the \( I = 2 \) part is small so that uncertainties on that play a minor role.

6.2 Discussion of the solution

Compared to the phase-shift solutions of Section 4, the one in Figs. 6.1 must be considered to be quite remarkably smooth. This is so even though no Breit-Wigner type of information was introduced for the \( f^0 \) or the g resonances. In fact we see that the g-loop in the F-wave has a strong background phase on top
Fig. 6.1 Result of energy-independent phase-shift analysis constrained to analytic fixed-\(t\) and fixed-\(u\) amplitudes.

of the pure inelastic Breit-Wigner behaviour. Although somewhat uncertain in detail owing to the fact that the \(g\) is close to the end of the interval of the analysis, this general behaviour seems definitely required by the essentially pure imaginary nature of \(F^{+}\) at these energies.

The fact that \(F^{+}\) is purely imaginary is qualitative support for the idea of strong \(\rho\)-\(f\) exchange degeneracy (see Section 5.1), although in fact the amplitudes at high energies are fully consistent with a fair breaking of that: \(\rho\)-exchange stronger than \(f\)-exchange.

The solution of Figs. 6.1 appears to be a smooth version of the EM class B or Männner ++- types. Below we briefly comment on some distinctive features and difficulties.

6.2.1 The \(\rho'(1650)\)

Although no evidence is found for a \(\rho'\) in the \(f^0(1270)\) region, there is very clear evidence for a \(\rho'\) in or just below the \(g\)-region. It is natural to associate
this with the object seen in photoproduction\textsuperscript{109,110} and in $e^+e^-$ experiments\textsuperscript{111}. The overwhelming decay mode appears to be into the $4\pi$ channel, and in fact until recently there was no clear evidence for a $\rho' + 2\pi$ decay. The solution shown in Fig. 6.1, on the other hand, requires a branching ratio of perhaps 20-30\%.

Recently\textsuperscript{110}, very clear evidence for the $2\pi$ mode was in fact observed in photoproduction. Figures 6.2 and 6.3 show the $4\pi$ and the $2\pi$ signals observed in the FNAL experiment. A naive look at the two bumps would suggest a $\rho' + 2\pi$ branching ratio much smaller than 10\%. However, there appears to be a clear conflict between the shapes of the two bumps, the $4\pi$ signal showing far bigger spread in energy than the $2\pi$ one. Clearly, a careful analysis of the data is required before determining whether the $\rho' + 2\pi$ seen in the phase-shift solution is compatible with the photoproduction data or not.

6.2.2 Other charge states

Information on scattering in other charge states would be extremely valuable. So far, the accuracy of such information is much inferior to the corresponding information on $\pi^+\pi^- + \pi^+\pi^-$. An interesting discussion of the effect of the limited amount of data has been given by Shimada\textsuperscript{102}). He points out that a combined

![Graph](image)

Fig. 6.2 Mass distribution of four charged particle final states from the FNAL photoproduction experiment (Ref. 110).
phase-shift analysis of the reactions $\pi^-\pi^0 \rightarrow \pi^-\pi^0$ and $\pi^+\pi^- \rightarrow \pi^0\pi^0$ would be most useful. For the time being we must be content with checking that any phase-shift solution is at least compatible with whatever meagre information exists. In Figs. 4.12 and 4.13 we have already seen how $\pi^+\pi^- \rightarrow \pi^0\pi^0$ appeared to rule out solutions $++$ and $--$.

A further interesting remark by Shimada concerns the behaviour of the p-wave in the $M_{\pi}\pi = 1300-1350$ region. The EM-B solution has a positive real part for the p-wave at that energy (Fig. 4.15a). This is demonstrated by Shimada to be incompatible with the (very limited) data on $\pi^-\pi^0$ (112). However, the "B-like" solution of Froggatt and myself in Fig. 6.1 is fully compatible with the $\pi^-\pi^0$ data. This incidentally illustrates the fact that data on these other charge modes are in principle and in practice unable to give uniqueness by themselves. It should be obvious, on the other hand, that good data would be extremely valuable extra constraints on fixed-t/u analyses combined with constrained phase-shift analysis.

6.2.3 s-wave $K\bar{K}$ production

Shimada (102, 113) also carried out a phase-shift analysis of the existing data on $K\bar{K}$ production in the $f^0(1270)$ region and a little above $60, 114$). He found a clear s-wave signal. This seems to represent a difficulty for B-type solutions, which have $I = 0$ s-waves saturating unitarity at all energies above the $f^0$. The
condition at $M_{\pi\pi} = 1250$ MeV derived from the $K\bar{K}$ data is that $\eta_0^2 \leq 0.8$. The trouble is now completely cured by our new solution due to the last term in Eq. (6.3) (cf. Fig. 6.1).

As emphasized by Shimada"\textsuperscript{102,113}"), these various difficulties are not present with A-type of solutions. So why is solution A not a better candidate? The work of Froggatt and myself gives the indirect answer that solution A did not emerge from the fixed-$t$ ($-u$) analysis, and we have argued that this analysis should be rather stable. In particular, the very large difference between the imaginary parts of the forward and backward amplitudes at, say, $M_{\pi\pi} = 1500$ MeV (see Figs. 5.4-5.6) indicate that the p-wave at that energy must have a fairly large imaginary part. This immediately rules out solution A.

An alternative and more direct way of investigating the problem was given by Johnson, Martin and Pennington"\textsuperscript{108}". They consider certain smoothed versions of the four EM solutions with the over-all phase given by their own dispersion relation calculation of the forward $\pi^+\pi^-$ amplitude (Fig. 5.5). The question now is: will backward dispersion relations be satisfied for these solutions?

The crucial region to look at is clearly the $p'$ region. Figure 6.4 shows first the modulus of $P^\pi\pi$ at 1550 MeV as a function of $t$. We see the dips in

![Fig. 6.4](image-url)

Fig. 6.4 Modulus and phase relative to forward phase of $P^\pi\pi$ and $M_{\pi\pi} = 1.55$ GeV from Ref. 108.
|F^+|^2 at the three t-values corresponding to $\text{Re } z_i^*; i = 1, 2, 3$. As explained in Section 4, solutions A and B differ in the sign of $\text{Im } z_i^*$. This gives rise to the different phase behaviours as functions of t given in Fig. 6.4. At $u = 0$ ($t = -2.5 \text{ GeV}^2$) the difference is large enough that it should manifest itself in a dispersion relation.

This indeed turns out to be so. Figures 6.5 show real parts of $F^+$ for $t, u = 0$ as a function of $M_{\pi\pi}$. The "data" are the real parts calculated from the phase-shift solution, and the curve is the result of the dispersion relation. We see that solution B satisfies the dispersion relation nicely, whereas solution A is ruled out.

![Solutions A and B](image)

Fig. 6.5 Dispersion relation fits to EMA and EMB from Ref. 108. The phase at $t = 0$ is used for the over-all phase. One sees a disagreement with backward dispersion relations for EMA.

6.3 Concluding remarks

In these lectures we have tried to produce a credible case for $\pi\pi$ phase-shift analysis. During the 1970's the field has made enormous progress. We now have a coherent and mature picture from threshold to 1.8 GeV.

In the threshold region the new $K_{e4}$ data appear to settle the long-standing discussion concerning $\pi\pi$ scattering lengths. Combined with Roy equation studies (Sections 3.4 and 3.5) the data support current algebra, although phenomenological numbers tend to be $\approx 30\%$ larger than predicted. The chiral symmetry-breaking scheme of Weinberg and of Gell-Mann, Oaks and Renner is in much better shape than any other simple breaking scheme, although the agreement with experiment is not perfect.
Concerning our second main theme, that of daughter spectroscopy, we seem to have evidence \textit{against} the first daughter trajectory. The only possible exception seems to be with the $\varepsilon$. However, there are reasons for considering the large $s$-wave below 1 GeV to be the combined tails of an $\varepsilon(1100-1300)$ and an $S^*$ near the $K\bar{K}$ threshold, or possibly for not having an $\varepsilon$ at all (Section 3.2).

With the $\varepsilon(1100-1300)$ and the phase-shift solution of Figs. 6.1 picked out by fixed-$t$ and fixed-$u$ analyses, we furthermore seem to have evidence \textit{in favour} of a second daughter trajectory. Compatibility of a $\rho'(1650) \rightarrow 2\pi$ signal as strong as required, with photoproduction data, has yet to be cleared up. Interesting discussions on the relation between daughter spectroscopy and zero-trajectories in various charge states has been given by Shimada\textsuperscript{102} and by Eguchi, Fukugita and Shimada\textsuperscript{115}.

I have chosen to finish these concluding remarks by briefly discussing how different kinds of experiments could help to clarify and extend the picture.

\textbf{a)} We have already pointed to the usefulness of data on scattering in other $\pi\pi$ charge states such as $\pi^+\pi^- \rightarrow \pi^0\pi^0$ and $\pi^-\pi^+ \rightarrow \pi^-\pi^0$. Combined with fixed-$t$ and fixed-$u$ analyses of data on $\pi^+\pi^- \rightarrow \pi^+\pi^-$, a strongly overconstrained situation would emerge provided data were of comparable quality.

\textbf{b)} Low-energy studies on $\pi^+\pi^- \rightarrow \pi^+\pi^+$ or $\pi^-\pi^+ \rightarrow \pi^-\pi^+$ would no longer appear to be very useful. The Roy equations are capable of predicting the scattering from $\pi^-\pi^- \rightarrow \pi^+\pi^-$ data more accurately than could be hoped for from the experiment. On the other hand, I = 2 studies at higher energies would be most welcome.

\textbf{c)} Studies of inelastic reactions would be useful. In particular there seems to be a strong need for clarifying how unitarity is saturated (or how it avoids being oversaturated) by the elastic reaction $\pi\pi \rightarrow K\bar{K}$, $\pi\pi \rightarrow 4\pi$, etc.

\textbf{d)} An interesting new class of experiments have recently become available. Figure 6.6 shows results on $\pi^-\pi^+$ total cross-sections as a function of $s = \sqrt{s}$ (Refs. 116 and 117). The results were obtained by studying the inclusive reaction

$$\pi^+p \rightarrow \rho X$$

as a function of missing mass and momentum-transfer squared $t$ to the proton. Clearly the cross-section extrapolated to $t = m^2$ gives the $\pi^+\pi^-$ total cross-section. We see that the prediction of factorization of pomeron-exchange (valid at asymptotic energies provided total cross-sections are constant)

$$\sigma_{\pi^-\pi^+} = \sigma_{\pi^-\pi^+} \sigma_{\rho} \sigma_{\rho} / \sigma_{\rho\rho} \approx 14 \text{mb},$$
Fig. 6.6 $\sigma_{\text{tot}}(\pi^-\pi^+) \, \text{from data on } \pi^+n + pX \text{ from Ref. 116.}$

Low-energy data are from Ref. 117. Curve is prediction of P-exchange factorization (asymptotically, only).

is qualitatively brought out by the very uncertain data points. Interesting as it is, the total $\pi^+\pi^-$ cross-section can have only a limited influence on our knowledge of $\pi\pi$ phase-shifts. First, we have seen that the phase as predicted below 2 GeV by analyticity is quite insensitive to the value of $\sigma_{\text{tot}}(\pi^+\pi^-)$. Secondly, although $\sigma_{\text{tot}}(\pi^+\pi^-)$ as measured below 2 GeV would allow an absolute determination of the forward phase, it is clear from the near imaginarity of $F^{++}$ at $t = 0$ that the measurements would have to be incredibly accurate to be really useful.

A far more interesting total cross-section to measure would be $\sigma_{\text{tot}}(\pi^+\pi^+)$. It is something of a mystery how it can develop from only $\approx 5 \, \text{mb} \text{ at } 1500 \, \text{MeV}$ to $14 \, \text{mb} \text{ higher up. Also } F^{++}$ at $t = 0$ is far from imaginary, so $\sigma_{\text{tot}}(\pi^+\pi^+)$ measured at low energies would be extremely useful. A possible reaction might be the inclusive

\[ \Upsilon^+ p \rightarrow \Lambda^0 X^{++} \]

looking at the $\Lambda^0 \rightarrow p + \pi^-$ decay for identification of the $\Lambda$-mass and of the momentum transfer to the $\Lambda$. 
REFERENCES


See, for example, S.L. Adler and R.F. Dashen, Current algebras and applications to particle physics (Benjamin, New York, 1968).


37) S. Ciulli, C. Pomponiu and I. Sabba-Stefănescu, Phys. Reports 17C, 133 (1


42) W. Ochs, Die Bestimmung von $\pi\pi$-Streuphasen auf der Grundlage einer Ampli analyse der Reaction $\pi^-p \rightarrow \pi^-\pi^+n$ bei 17 GeV/c Primär Impuls, Thesis Ludwig-Maximilians-Universität, Munich, 1973.

43) P. Estabrooks and A.D. Martin, Phys. Letters 41B, 350 (1972); and Inter Conf. on $\pi\pi$ Scattering and Associated Topics, Tallahassee, 1973 (eds. P.K. Williams and V. Hagopian), AIP Conf. Proc. 13, New York (1973),


67) M. Cerrada et al., University of Madrid preprint, Grupo de Altas Energias, JEN 76/1 (1976).
78) J.L. Basdevant et al., preprint Université Pierre et Marie Curie, Paris, PAR-LPTHE 75.11 (1975).


95) E. Pietarinen, Karlsruhe University preprint TKP 5/75 (1975).


97) E. Pietarinen, Karlsruhe University preprint TKP 4/75 (1975).


103) Particle Data Group, nN two-body scattering data, LBL-63 (1973).


114) A.J. Paubicki et al., Argonne preprint ANL/HEP 75-09, 1 April 1975.

