SELECTION OF FORMULAE CONCERNING PROTON STORAGE RINGS

G. Guignard
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Abstract

This compilation of formulae concerning proton storage rings covers bunched beams as well as coasting beams. Some formulae are common to both storage rings and proton synchrotrons. Where meaningful, the application of a particular formula to the CERN Intersecting Storage Rings (ISR) is given.
FOREWORD

This report is divided into independent chapters, each of which treats a specific subject. Each chapter is preceded by a list of symbol definitions applicable to that chapter and, where necessary, further definitions are given in the text accompanying a formula. The symbols which are common throughout are listed separately at the beginning. Thus, to establish a symbol's meaning, the reader must first consult the text with the formula, secondly the list at the beginning of the chapter and, if necessary, the list at the beginning of the report.

A short list of basic references, in which the reader can find most of the formulae, is given at the beginning of each chapter. Some formulae, however, have been derived specifically for this report and no references exist for them.

The values of some commonly used constants and of parameters related to the energy and to the working conditions in the CERN storage rings are given in Tables 1 to 10.

The author would welcome any comments or corrections concerning the formulae presented in this report which readers may wish to make.
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LIST OF FREQUENTLY USED SYMBOLS AND THEIR MEANINGS

x \quad \text{horizontal transverse coordinate \,[m]} \\
z \quad \text{vertical transverse coordinate \,[m]} \\
y \quad \text{general transverse coordinate, either horizontal or vertical \,[m]} \\
y' = dy/ds \quad \text{derivative of the general transverse coordinate \,[rad]} \\
s \quad \text{distance along beam axis \,[m]} \\
R \quad \text{average machine radius \,[m]} \\
\theta = s/R \quad \text{azimuthal angle \,[rad]} \\
t \quad \text{time \,[s]} \\
I \quad \text{beam current \,[A]} \\
e \quad \text{electronic charge \,[As]} \\
c \quad \text{velocity of the light \,[ms^{-1}]} \\
\beta \quad \text{ratio of particle velocity to that of light} \\
\gamma \quad \text{ratio of total energy of particle to its rest energy} \\
\eta = \frac{1}{\gamma^2} - \frac{1}{\gamma_t^2} \quad \text{revolution frequency spread per unit of momentum spread, \gamma_t \text{ being the value of } \gamma \text{ at the transition energy where } \eta \text{ changes its sign} \\
p \quad \text{particle momentum \,[GeV/c]} \\
B_x, B_z \quad \text{transverse components of the magnetic induction \,[T]} \\
B_\phi \quad \text{azimuthal component of the magnetic induction \,[T]} \\
B_0 \quad \text{magnetic rigidity \,[Tm]} \\
\beta_{\gamma,i} \quad \text{transverse betatron amplitude function \,[m] in plane \,(y,s). A second index I \, (\beta_{\gamma,i}) \text{ indicates that the amplitude function is taken at one intersection of storage rings} \\
Q_{\gamma} \quad \text{number of betatron oscillations per revolution in transverse plane \,(y,s)} \\
r_e \quad \text{classical electron radius \,[m]} \\
r_p \quad \text{classical proton radius \,[m]} 

The definitions of x, z, s, \theta are summarized in Fig. 1. A prime denotes differentiation with respect to s. The sign ‘ or - on top of a variable indicates its maximum or average value , respectively. The notation |Z| stands for the modulus of Z if Z is complex, and the absolute value of Z if Z is real. The values of some physical constants such as e and c are given in Table 1.

The MKSA unit system is used throughout this report except for the rest energy \,[eV], the momentum of a particle \,[eV/c] and the vacuum pressure \,[Torr]. These three exceptions are retained to respect common usage.
The following abbreviations are used:

- w.r.t. for with respect to;
- r.m.s. for root mean square;
- ISR for Intersecting Storage Rings at CERN;
- SFM for Split Field Magnet, installed in intersection 4 of the ISR.

Fig. 1 Coordinate system
1. **FORMULAE IN CONNECTION WITH THE TRANSVERSE PHASE SPACE**$^{1, 2, 3}$

**Symbols frequently used in this chapter**

- $\alpha_y$: Twiss parameter
- $\nu_y$: phase of the betatron oscillation
- $E_Y$: transverse emittance [m rad]
- $d$: distance along beam axis from the entrance of a given element [m]
- $y$: horizontal or vertical transverse amplitude of the betatron oscillation of a particle [m]
- $y'$: amplitude derivative or betatron oscillation angle [rad]
- $y_p$: transverse position of the phase ellipse centre associated with a momentum deviation [m].

1.1 **Definitions of the basic beam parameters**

The transverse parameters, in which we are interested, are the betatron amplitude function $\beta_y [m]$, the Twiss parameter $\alpha_y$, the phase $\nu_y$, the momentum compaction $\alpha_{p,y} [m]$ and the number of betatron oscillations per revolution $Q_y$. For each transverse plane, these functions are defined by the following relations:

\[
y(s) = \sqrt{\beta_y(s)} E_Y \left[ a \cos \nu_y(s) + b \sin \nu_y(s) \right]
\]

\[
\alpha_y(s) = - \frac{1}{2} \beta'_y(s)
\]

\[
\nu_y(s) = \int_0^s \frac{ds}{\beta_y(s)}
\]

\[
\alpha_{p,y}(s) = \frac{y_y(s)}{\Delta p/p}
\]

\[
Q_y = \frac{1}{2\pi} \int_0^\pi \frac{ds}{\beta_y(s)} = \frac{\nu_y(2\pi R)}{2\pi}
\]

$E_Y$ is given explicitly in Section 1.4.

The first relation defines the $\beta_y$ function, where $a$, $b$ are dimensionless constants depending on the initial conditions. $\pi$ is the machine circumference.

It is worth noting that the average value of the horizontal momentum compaction is directly related to $\gamma_t$ by

\[
\bar{\alpha}_{p,x} = \frac{R}{\gamma_t}
\]
Two other parameters associated with $Q_y$ are often used:

$$Q'_y = \frac{\partial Q_y}{\partial (\Delta p/p)} \bigg|_{\Delta p/p=0} ,$$

$$Q''_y = \frac{1}{2} \frac{\partial^2 Q_y}{\partial (\Delta p/p)^2} \bigg|_{\Delta p/p=0} ,$$

$Q'/Q$ often being called the chromaticity.

Note the special meaning of the prime, which is here a derivative with respect to $\Delta p/p$, and the unusual factor 2 in the definition of $Q''_y$.

For the ISR: the values of all these parameters can be obtained from the computer program AGS^5).

The value of $R$ can be found in Table 2. Table 3 gives $\bar{E}_y$, $\bar{E}_{z,1}$ and $\bar{\alpha}_{p,\chi}$ for the standard ISR working conditions. Table 4 gives $Q_y$ at centre line and $\bar{x} = 40 \text{ mm}$ as well as $Q'_y$ for the presently existing working lines.

1.2 Matrix formalism for betatron oscillations

Let us consider a given element of the machine. It can be characterized by a $2 \times 2$ matrix $M(d)$, which is defined as follows:

$$
\begin{pmatrix}
    y(d) \\
    y'(d)
\end{pmatrix} = M(d) 
\begin{pmatrix}
    y_1 \\
    y'_1
\end{pmatrix},
\quad d \leq \ell,
$$

where $\ell$ is the length of the machine element $[m]$.

Since $d$ is the longitudinal distance $[m]$ from the entrance of the element, $y_1$ and $y'_1$ are the amplitude of the oscillation and its derivative at the entrance to the element.

Let us consider four particular forms of matrix $M(d)$.

i) **Straight section:**

$$M(d) = \begin{pmatrix}
    1 & d \\
    0 & 1
\end{pmatrix}.
$$

ii) **Gradient sector magnet:**

$$M(d) = \begin{pmatrix}
    \text{cf}(\sqrt{K} d) & \frac{1}{\sqrt{K}} \text{sf}(\sqrt{K} d) \\
    \text{sign} \sqrt{K} \text{sf}(\sqrt{K} d) & \text{cf}(\sqrt{K} d)
\end{pmatrix},
$$

where the significance of the symbols depends on the plane of the gradient sector magnet.

In the focusing plane, the following meanings are valid:

$$\text{sf} = \sin, \quad \text{cf} = \cos, \quad \text{sign} = -1 .$$

In the defocusing plane, the meanings are:

$$\text{sf} = \text{sh}, \quad \text{cf} = \text{ch}, \quad \text{sign} = 1 .$$
The parameter $K \, [m^{-2}]$ can be expressed as follows:

\[
K = \frac{|n| + \text{sign} \, \rho^2}{\rho^2} \quad \text{in the bending plane,}
\]

\[
K = \frac{|n|}{\rho^2} = \frac{|G|}{B_0} \quad \text{in the plane perpendicular to the bending.}
\]

$G$ is the sector magnet gradient $[Tm^{-1}]$, $\rho$ is the bending radius $[m]$.

By convention, F-units are focusing in the horizontal plane while D-units are defocusing.

iii) Quadrupole lens:

The matrix $M(d)$ given in ii) is valid, with

\[
\frac{1}{\rho} = 0,
\]

\[
K = \frac{|G|}{B_0},
\]

in both planes.

iv) Edge effect of a gradient magnet:

For a gradient magnet with a linear fringe field, the matrix $M$ in the bending plane will be

\[
M = \begin{pmatrix}
1 & 0 \\
\tan \frac{\epsilon}{\rho} & 1
\end{pmatrix},
\]

and in the other transverse plane

\[
M = \begin{pmatrix}
1 & 0 \\
\frac{1}{\rho} \left[ \frac{b}{6 \rho \cos \epsilon} - \tan \epsilon \right] & 1
\end{pmatrix}.
\]

$\epsilon$ is the angle between the magnet edge and the perpendicular to the beam direction.

$\epsilon$ is taken as positive for both edges of a rectangular magnet (Fig. 2).

$b$ is the length of the fringe field region $[m]$ (Fig. 2).

$\rho$ is the bending radius $[m]$.

For the ISR: the values of $\rho$, $n/\rho$ and $B_0$ are given in Tables 2 and 5.

Since the gradient magnets of the ISR are rectangular, their matrix $M$ is a product of three matrices: one matrix of type ii) for the central part and two matrices of type iv) for the two edges, with $\epsilon$ taken as being equal to half of the bending angle. Assuming $b/\rho$ small, $M$ becomes
\[
M(d) = \begin{pmatrix}
\text{cf}(\sqrt{K} \ d) \pm \frac{\tan \frac{\epsilon}{\rho}}{\sqrt{K}} \ \text{cf}(\sqrt{K} \ d) \\
\text{sf}(\sqrt{K} \ d) \pm \frac{2 \ tan \frac{\epsilon}{\rho}}{\sqrt{K}} \ \text{cf}(\sqrt{K} \ d) \\
\text{sign} \sqrt{K} + \frac{\tan^2 \frac{\epsilon}{\rho}}{\sqrt{K} \ \rho^2} \ \text{sf}(\sqrt{K} \ d) \pm \frac{\tan \frac{\epsilon}{\rho}}{\rho \ \sqrt{K}} \ \text{sf}(\sqrt{K} \ d)
\end{pmatrix}
\]

The parameter definitions are given above and the signs + and − correspond to
the bending plane and to the plane perpendicular to the bending, respectively.

1.3 Interpolation of the transverse parameters through any element

Computer programs like AGS\(^1\) give the values of \(\alpha_y, \beta_y, \mu_y\) at a discrete number of
points. It can be useful to have formulae giving the values of the parameters at any point
inside the element, knowing their values \(\alpha_{y,1}, \beta_{y,1} [m], \mu_{y,1}\) at the entrance.

\[
\alpha_y(d) = -m_{11} m_{21} \beta_{y,1} - m_{12} m_{22} \frac{1 + \alpha_y^2}{\beta_{y,1}^2} \gamma_{y,1} + (m_{11} m_{22} + m_{12} m_{21}) \alpha_{y,1},
\]

\[
\beta_y(d) = m_{11}^2 \beta_{y,1} + m_{12}^2 \frac{1 + \alpha_y^2}{\beta_{y,1}^2} - 2 m_{11} m_{12} \alpha_{y,1},
\]

\[
\mu_y(d) = \mu_{y,1} + \arctg \frac{m_{12}}{m_{11} \beta_{y,1} - m_{12} \alpha_{y,1}}.
\]

The coefficients \(m_{ij}\) (functions of \(d\)) are the elements of the matrix \(M(d)\) defined in
Section 1.2.
From these general formulae, two special cases can be deduced:

i) **Interpolation in a straight section:**

\[
\alpha_y(d) = \alpha_{y,1} - d \left( 1 + \frac{\alpha_{y,1}^2}{\beta_{y,1}} \right) ,
\]

\[
\beta_y(d) = \beta_{y,1} - 2d \alpha_{y,1} + d^2 \frac{1 + \alpha_{y,1}^2}{\beta_{y,1}} ,
\]

\[
\mu_y(d) = \mu_{y,1} + \arctg \frac{d}{\beta_{y,1} - d \alpha_{y,1}} .
\]

ii) **Interpolation in a gradient sector magnet:**

\[
\alpha_y(d) = \alpha_{y,1} \text{cf}(2 \sqrt{K} d) - \left[ \frac{1 + \alpha_{y,1}^2}{\sqrt{K} \beta_{y,1}} + \text{sign} \sqrt{K} \beta_{y,1} \right] \frac{1}{2} \text{sf}(2 \sqrt{K} d) ,
\]

\[
\beta_y(d) = \beta_{y,1} \text{cf}^2(\sqrt{K} d) - \frac{\alpha_{y,1}}{\sqrt{K}} \text{sf}(2 \sqrt{K} d) + \frac{1 + \alpha_{y,1}^2}{K \beta_{y,1}} \text{sf}^2(\sqrt{K} d) ,
\]

\[
\mu_y(d) = \mu_{y,1} + \arctg \frac{\text{sf}(\sqrt{K} d)}{\sqrt{K} \beta_{y,1} \text{cf}(\sqrt{K} d) - \alpha_{y,1} \text{sf}(\sqrt{K} d)} ,
\]

where the definitions of \( \text{sf}, \text{cf}, \text{sign} \) and \( K \) are explicitly given in Section 1.2.

1.4 **Transverse emittance**

Let us consider one particle of constant energy making an infinite number of turns. All the positions taken by this particle in a transverse phase plane \((y, y')\) at one given point of the machine circumference define an ellipse (Fig. 3) with the following equation:

\[
E_y = \frac{\pi}{\beta_y} \left[ y^2 + (\beta_y y' + \alpha_y y)^2 \right] .
\]

The emittance \( E_y \) is the ellipse area and the factor \( \pi \) has to be included in the numerical value of \( E_y \).

1.5 **Normalized transverse emittance**

The emittance \( E_y \) is an invariant if the energy of the particle does not change. With an energy change, the invariant becomes the normalized transverse emittance \( \epsilon_y [\text{m rad}] \)

\[
\epsilon_y = \frac{E_y}{\beta_y} .
\]

For the ISR: Typical values of \( \epsilon_y \) are given in Table 2. These values correspond to ellipses containing 86.466 % of the particles assuming a gaussian distribution (see also Section 1.6).
1.6 Beam size in the transverse plane

\[ \gamma = \sqrt{\frac{\beta \gamma E_\gamma}{\pi}}. \]

The beam size will be \(2\gamma\).

For a gaussian particle density, the relation \(\gamma = 2\sigma_\gamma \) is used, where \(\sigma_\gamma\) is the standard deviation [m].

For zero divergence \((y' = 0)\), the maximum value of \(y\) will be

\[ y_\phi = \sqrt{\frac{\beta \gamma}{1 + \alpha_\gamma^2}} \frac{E_\gamma}{\pi}. \]

The definitions of \(y\) and \(\gamma\) are summarized in Fig. 3.

1.7 Angular divergence of a beam

\[ \gamma' = \sqrt{\frac{(1 + \alpha_\gamma^2) E_\gamma}{\beta \gamma}}. \]

The total divergence [rad] will be \(2\gamma'\).

At the pulse centre \((y = 0)\), the maximum angular divergence will be

\[ y'_\phi = \sqrt{\frac{E_\gamma}{\pi \beta \gamma}}. \]

The definitions of \(y'_\phi\) and \(\gamma'\) are summarized in Fig. 3.

---

Fig. 3 A two-dimensional beam phase ellipse
2. **FORMULAE IN CONNECTION WITH THE MAGNETIC FIELD**\(^1,^2,^3,^5,^6\)

Symbols frequently used in this chapter

- \( s \) \(_1\) longitudinal position of one given magnetic element \([m]\)
- \( \lambda \) circumference of the machine \([m]\)
- \( \mu_y(s) \) phase of the betatron oscillation (defined in Section 1.1) in plane \((y, s)\).

2.1 **Magnetic rigidity**

The general relation is

\[
B_p = \frac{p}{e}.
\]

For \( p \) in GeV/c and \( B_p \) in Tm, this relation simply becomes

\[
B_p = \frac{10^9}{c} p = 3.335641 \ p, \]

\( \rho \) being the bending radius \([m]\).

Note: \([T] = [Vs\text{m}^{-2}]\).

For the ISR: Table 2 gives the bending radius and Table 5 gives the magnetic rigidity \([Tm]\) for the standard ISR energies.

2.2 **Orbit deformation associated with dipole fields**

\[
y(s) = \frac{\text{sign} \sqrt{\beta_y(s)}}{2 \sin \pi \Omega_y} \int_0^\lambda \sqrt{\beta_y(s_1)} \frac{\Delta B(s_1)}{B_p} \cos \left( \Omega_y \pi - \left| \mu_y(s) - \mu_y(s_1) \right| \right) ds_1,
\]

\( y(s) \) is the orbit amplitude at the position \( s \) of the circumference \([m]\),

\( \Delta B \) is the dipole field \([T]\) orthogonal to the \( y \)-axis and to the \( s \)-axis,

\( \text{sign} \) is +1 for the vertical plane and -1 for the horizontal plane,

\( B_p \) is the magnetic rigidity \([Tm]\) (see Section 2.1).

For dipoles or sections of dipoles which are short compared to the betatron oscillation wavelength, the integral on \( ds_1 \) can be replaced by a summation.

Since the functions \( y(s) \) and \( \Delta B(s) \) are periodic with the machine circumference, it is possible to develop them in Fourier's series. Using the phase \( \mu_y \) (Section 1.1) instead of the coordinate \( s \), the expression for the orbit becomes

\[
y(\mu_y) = \sqrt{\beta_y(\mu_y)} \sum_{\nu=-\infty}^{\infty} \frac{Q_y^2}{Q_y^2 - p^2} c_p \exp \left[ ip \frac{\mu_y}{Q_y} \right],
\]

with

\[
c_p = \frac{1}{2\pi Q_y} \int_0^{2\pi Q_y} \beta_y^{1/2} (\mu_y) \frac{\Delta B(\mu_y)}{B_p} \exp \left[-ip \frac{\mu_y}{Q_y}\right] d\mu_y.
\]

\( \nu \) is an integer, and \( \Delta B \) and \( B_p \) are defined above.
Assuming there is a correction system able to compensate exactly the harmonics of the amplitude function $y(s)$, the average gain in the ratio of the extreme amplitudes after and before correction will be for distributed dipole fields:

$$g(Q_y, n, m) = 1 - \frac{\sum_{p=n}^{m} \frac{1}{|Q_y^2 - p^2|}}{\sum_{p=1}^{\infty} \frac{1}{|Q_y^2 - p^2|}}.$$

$n, m, p$ are integers and $g = \frac{\overline{y}_{\text{corr}}}{\overline{y}}$.

The infinite sum can be calculated analytically using

$$\sum_{p=1}^{\infty} \frac{1}{p^2 - Q_y^2} = \frac{\pi}{2Q_y} \left( \frac{1}{\cot \pi Q_y} \right).$$

2.3 Tune shifts, amplitude beating and horizontal momentum compaction changes associated with quadrupole fields

$$\Delta Q_y = \frac{1}{4\pi} \int_0^\theta \beta_y(s_1) \frac{G(s_1)}{B_p} \, ds_1,$$

$$\Delta \beta_y(s) = \frac{1}{2 \sin 2\pi Q_y} \int_0^\theta \beta_y(s_1) \frac{G(s_1)}{B_p} \cos \left[ 2 \left( Q_y \pi - |\mu_y(s) - \mu_y(s_1)| \right) \right] \, ds_1,$$

$$\Delta \alpha_{p,x}(s) = -\frac{\sqrt{\beta_x(s)}}{2 \sin \pi Q_x} \int_0^\theta \sqrt{\beta_x(s_1)} \frac{G(s_1)}{B_p} \alpha_{p,x}(s_1) \cos \left[ \frac{Q_x \pi - |\mu_x(s) - \mu_x(s_1)|}{B_p} \right] \, ds_1.$$

$G$ is the quadrupole gradient [Tm⁻¹],

$\alpha_{p,x}$ is the horizontal momentum compaction at the quadrupole [m] (see Section 1.1),

$B_p$ is the magnetic rigidity [Tm] (see Section 2.1).

For a short quadrupole, the note mentioned in Section 2.2 equally applies.

2.4 Vertical momentum compaction associated with skew quadrupole fields

The vertical momentum compaction [m] is the ratio of the vertical shift in position to the momentum change. Its mathematical definition is given in Section 1.1.

$$\alpha_{p,z}(s) = \frac{\sqrt{\beta_z(s)}}{2 \sin \pi Q_z} \int_0^\theta \sqrt{\beta_z(s_1)} \frac{G_z(s_1)}{B_p} \alpha_{p,x}(s_1) \cos \left[ \frac{Q_z \pi - |\mu_z(s) - \mu_z(s_1)|}{B_p} \right] \, ds_1.$$

$G_z = dR_z/dx$ is the skew gradient [Tm⁻¹],

$B_p$ is the magnetic rigidity [Tm] (see Section 2.1),

$\alpha_{p,x}(s_1)$ is the horizontal momentum compaction at the skew quadrupole [m] (see Section 1.1).

For a short skew quadrupole, the note mentioned in Section 2.2 equally applies.
The vertical momentum compaction excited by a skew quadrupole manifests itself as a median plane tilt, which is given by

\[ \chi = \arctg \frac{B_{z}z}{B_{x}x}, \]

\( \chi \) being the median plane tilt [rad].

### 2.5 Multipole analysis of a two-dimensional magnetic field

In cartesian coordinates, the two-dimensional magnetic field in a magnet gap can be derived from a complex potential \( P \) [Tm]:

\[ P = V + iU = \sum_{N=1}^{\infty} \frac{1}{N!} B^{(N-1)} w^{N}, \]

where:

\[ w = x + iz, \]

\[ B^{(N-1)} = B_{z}^{(N-1)} + i B_{x}^{(N-1)} \]

\[ = \frac{\partial}{\partial x} (N-1) B_{z}^{(N-1)} \bigg|_{x=z=0} + i \frac{\partial}{\partial x} (N-1) B_{x}^{(N-1)} \bigg|_{x=z=0}. \]

\( N \) is an integer associated with the \( 2N \)-pole term.

The field components are then given by:

\[ B_{x} = -\frac{\partial V}{\partial z} = \text{Im} \left( \frac{\partial P}{\partial w} \right), \]

\[ B_{z} = \frac{\partial V}{\partial x} = \text{Re} \left( \frac{\partial P}{\partial w} \right). \]

In cylindrical coordinates, the complex potential \( P \) can be written:

\[ P = \sum_{N=1}^{\infty} \frac{1}{N!} r^{N} e^{iN\phi}, \]

\( r \) is the distance from the origin [m],

\( \phi \) is the angle between the x-axis and the radius vector [rad].

The real component of \( P \) becomes

\[ V = \sum_{N=1}^{\infty} \frac{r^{N}}{N!} \left[ B_{z}^{(N-1)} \cos N\phi - B_{x}^{(N-1)} \sin N\phi \right], \]

and the field components are given by

\[ B_{r} = \frac{\partial V}{\partial r}, \]

\[ B_{\phi} = -\frac{1}{r} \frac{\partial V}{\partial \phi}. \]
Note that the derivatives can be calculated from

\[
\frac{B_x^{(N-1)}}{N} = -\frac{N!}{\pi N} \int_0^{2\pi} V \sin \phi \, d\phi
\]

\[
\frac{B_z^{(N-1)}}{N} = \frac{N!}{\pi N} \int_0^{2\pi} V \cos \phi \, d\phi
\]

3. **Formulae in connection with the longitudinal phase space**

Symbols frequently used in this chapter:

- \( V \): cavity voltage [V]
- \( \phi_s \): phase of the synchronous particle
- \( \phi \): phase of a particle
- \( h \): harmonic of the radio-frequency system; \( \Delta \phi = -h\Delta \theta \)
- \( M \): number of bunches
- \( \ell \): total bunch length [m]
- \( E_o \): rest energy of proton (Table 1) [eV]
- \( m_o \): rest mass of proton [eV/c^2]

3.1 **Acceleration rate**

The rate of energy increase for the synchronous particle is given by:

\[
\frac{d(B_o)}{dt} = \frac{V \sin \phi_s}{2\pi R}
\]

The phase of the synchronous particle is often replaced by

\[
\Gamma = \sin \phi_s
\]

Using the relation in Section 2.1 for \( p \) in GeV/c and \( B_o \) in Tm, the rate of momentum increase can be written:

\[
\frac{dp}{dt} = k \frac{VR}{R}
\]

where:

\[
k = \frac{c}{2\pi 10^5} = 0.0477135 \text{ [ms}^{-1}]\text{].}

For the ISR: \( R \) and \( V \) are given in Table 2.
3.2 Motion of a nonsynchronous particle

The momentum increase of a nonsynchronous particle differs from the one of the synchronous particle and the former particle will oscillate around the latter in agreement with

$$\frac{d}{dt} \left( \frac{R^2 \gamma E_0}{n c^2 \hbar} \frac{d\phi}{dt} \right) + \frac{eV}{2\pi} (\sin \phi - \sin \phi_s) = 0.$$

The values of $R$, $\gamma$ and $n$ have to be taken for the synchronous particle. When these parameters are constant or slowly varying, the motion equation becomes

$$\frac{d^2\phi}{dt^2} + \frac{4\pi^2 \omega_p^2}{\cos \phi_s} (\sin \phi - \sin \phi_s) = 0.$$

This equation and its first integral also give the variation of the momentum for the nonsynchronous particle, since

$$\Delta p = -\frac{1}{2\pi f_{\text{rev}} h \eta} \frac{d\phi}{dt} = -\sqrt{2} \frac{f_p}{f_{\text{rev}} h \eta} \sqrt{\frac{\cos \phi - \cos \phi_1 + (\phi - \phi_1) \sin \phi_s}{\cos \phi_s}}$$

$f_p$ is the phase oscillation frequency (Section 3.3) $[s^{-1}]$

$f_{\text{rev}}$ is the revolution frequency (Section 3.3) $[s^{-1}]$

$\phi_1$ is one of the two extreme stable phases where $d\phi_1/dt = 0$ (Section 3.6).

For the ISR: $R$ and $h$ are given in Table 2, while $\gamma$ is given for the ISR standard energies in Table 5. The $n$-values given in Table 6 correspond to the currently used working lines. For other lines, $n$ can be calculated starting from $\gamma_t$ given in Table 3. Note that all these figures are only strictly valid on the central orbit.

3.3 Revolution frequency and phase oscillation frequency

The revolution frequency $[s^{-1}]$ of the particles is

$$f_{\text{rev}} = \frac{8c}{2\pi R}.$$  

In the case of small phase oscillation amplitudes around the stable phase $\phi_s$, the oscillation frequency $[s^{-1}]$ is exactly given by

$$f_p = \frac{k}{R} \sqrt{\frac{eV h \eta}{\gamma E_0}} \cos \phi_s,$$

where:

$$k = \frac{c}{(2\pi)^{3/2}} = 1.90349 \times 10^7 \text{[ms}^{-1}], \text{ and}$$

$$\cos \phi_s = \sqrt{1 - \frac{r^2}{2}}, \text{ } \gamma \text{ and } n \text{ being taken for the synchronous particle.}$$

For the ISR: the values of $E_0$, $R$, $V$, $h$, $\beta$, $\gamma$ and $n$ can be found in Tables 1, 2, 5 and 6. The value of the revolution frequency for light, $f_{\text{rev}}^L (\beta = 1)$, is given in Table 2.
3.4 Revolution frequency spread and stability condition

The frequency spread is related to the momentum spread

\[ \frac{\Delta f_{\text{rev}}}{f_{\text{rev}}} = \eta \frac{\Delta p}{p}, \]

where:

\[ \eta = \frac{1}{\gamma^2} \left( \frac{1}{\gamma^2} - \frac{\gamma p_X}{R} \right), \]

\( f_{\text{rev}} \) is defined in Section 3.3.

Looking at the motion equation in Section 3.2, it is clear that the motion is stable and that the RF buckets can exist only if \( \frac{\Delta p}{p}^2 \) is positive, which is equivalent to

\[ \eta \cos \phi_s > 0. \]

For the ISR: the values of \( \eta \) for different conditions are given in Table 6. Since \( \eta \) is always negative, the stability criterion implies that

\[ \frac{\pi}{2} < \phi_s < \pi. \]

3.5 Moving bucket area

The moving bucket area (Fig. 4) is given in the coordinates \( c\Delta p/E_0 = \Delta p/\eta_c c \) and \( \phi \), by the following relation:

\[ A = k \alpha(\Gamma) \sqrt{\frac{eV\gamma}{\hbar|\eta|E_0}}, \]

where:

\[ k = \frac{16}{\sqrt{2\pi}} = 6.38308, \]

\( \alpha(\Gamma) \) is the factor between the moving and stationary bucket areas (Table 7).

For the ISR: the harmonic \( h \) and the voltage \( V \) are given in Table 2 and the values of \( \gamma \) are given in Table 5 for the ISR standard energies. Table 6 gives the values of \( \eta \) for different conditions.

3.6 Separatrix equation and extreme stable phases of a bucket

For stable motion (Section 3.4), the particles must remain inside the bucket. The limit of the bucket in the phase plane (\( \Delta p/m_0c, \phi \)) is called the separatrix (Fig. 4) and is given by the equation

\[ \frac{1}{8\pi^2} \left( \frac{d\phi}{dt} \right)^2 - \frac{\frac{\Delta p}{m_0c}}{\cos \phi_s} (\cos \phi + \phi \sin \phi_s) = - \frac{\frac{\Delta p}{m_0c}}{\cos \phi_s} (\cos \phi_1 + \phi_1 \sin \phi_s) \]

keeping in mind that (Section 3.2)

\[ \frac{\Delta p}{m_0c} = - \frac{\eta c}{\gamma} \int \frac{d\phi}{\Delta f_{\text{rev}}}, \]
\( f_{rev} \) is the revolution frequency (Section 3.3) \([s^{-1}]\),
\( f_p \) is the phase oscillation frequency (Section 3.3) \([s^{-1}]\),
\( \phi_1 \) is one of the extreme phases which can be reached by a stable particle.

The two extreme phases \( \phi_1 \) and \( \phi_2 \) are given by (Fig. 4)

\[
\phi_1 = \pi - \phi_s,
\]

\[
\phi_2 \sin \phi_s \cos \phi_2 = \phi_1 \sin \phi_s \cos \phi_s.
\]

\( \phi_1 - \phi_2 \) is the total phase spread of the bucket. It is equal to \( 2\Delta \phi \) in Section 3.9 only if the bunch fills the bucket, since by definition the bunch is the part of the bucket full of particles.

3.7 Momentum Spread in the Bucket

The extreme momenta which can be reached by a stable particle (Fig. 4) are given by

\[
\frac{\Delta p}{p} = \frac{1}{\beta} \sqrt{\frac{eV}{h \eta \gamma E_0}} \sqrt{\frac{2}{\pi} \cos \phi_s - (1 - \frac{2}{\pi} \phi_s) \sin \phi_s}.
\]

Using the definition of \( \alpha_{p,x} \) given in Section 1.1, the momentum spread of the bucket can give its horizontal width.

For the ISR: \( E_o, h, \beta, \gamma, \eta \) are given in Tables 1, 2, 5 and 6.

It should be kept in mind that the stability criterion (Section 3.4) implies that in this machine

\[
\frac{\pi}{2} < \phi_s < \pi.
\]

Fig. 4 Longitudinal phase space
3.8 Time needed for an adiabatic change of bucket parameters

When stacking in the momentum space of proton storage rings, it is necessary to reduce the bucket area when the pulse arrives close to the stacked beam, in order to avoid perturbing the stack. The necessary time \([s]\) for adiabatic change of bucket parameters is:

\[
T = \frac{1 + \eta_{ad}}{4\pi(1 - \eta_{ad})} \left( \frac{1}{f_{p,2}} - \frac{1}{f_{p,1}} \right).
\]

\(f_{p,1}\) is the phase oscillation frequency (Section 3.3) associated with the initial bucket \([s^{-1}]\), \(f_{p,2}\) is the phase oscillation frequency (Section 3.3) associated with the final bucket \([s^{-1}]\), \(\eta_{ad}\) is the phase-space efficiency of the process, defined as follows:

\[
\eta_{ad} = \frac{1 - \epsilon_p}{1 + \epsilon_p},
\]

\(\epsilon_p\) being the theoretical adiabaticity.

For the ISR: the bucket area is reduced by diminishing the cavity voltage at the end of the accelerating cycle. In this case, the formula for \(T [s]\) becomes:

\[
T = \left( \sqrt{\frac{V_1}{V_2}} - 1 \right) \frac{1}{800 \epsilon}.
\]

\(\epsilon\) is the adiabaticity coefficient as set in the ISR and is related to \(\epsilon_p\) by

\[
\epsilon = \epsilon_p \frac{f_{p,1}}{63.60}.
\]

Typical values for \(\epsilon\) are: 0.015/0.05.

\(V_1\) is the initial accelerating voltage \([V]\), \(V_2\) is the final accelerating voltage \([V]\).

3.9 Debunching time

In the absence of radio-frequency fields, the beam debunches. By definition, the time in which the front of the bunch reaches the tail of the next bunch, for rectangular bunches, is the debunching time \(t_{db} [s]\):

\[
t_{db} = \frac{\pi - \Delta \phi}{2 \pi f_{rev} H|\eta|\Delta p/p}.
\]

\(f_{rev}\) is the revolution frequency (Section 3.3) \([s^{-1}]\),
\(2\Delta \phi\) is the total phase spread of the bunch,
\(2\Delta p\) is the total momentum spread of the bunch \([\text{GeV/c}]\).

For the ISR: the values of \(H\) and \(f_{rev}\) are given in Table 2. The standard ISR values of \(\eta\) are given in Table 6.
3.10 Azimuthal current distribution inside one bunch

For a given number N of charges in a bunched beam and a given number M of bunches, the azimuthal current distribution inside a single bunch can be written as follows for rectangular, parabolic and triangular bunches successively:

\[
I_\theta = I_0
\]
\[
I_\theta = \frac{3}{2} I_0 \left[ 1 - \left( \frac{2R}{L} \right)^2 \theta^2 \right]
\]  \quad \text{for} \ - \frac{L}{2R} \leq \theta \leq \frac{L}{2R},
\]
\[
I_\theta = 2 I_0 \left[ 1 + \text{sign} \left( \frac{2R}{L} \theta \right) \right]
\]

sign is +1 for \( \theta < 0 \) and -1 for \( \theta \geq 0 \).

The expression for \( I_0 \) is

\[
I_0 = \frac{N e \beta c}{M L}
\]

N being the total number of charges.

For gaussian bunches, the current distribution inside a bunch is given by

\[
I_\theta = \frac{N e \beta c}{\sqrt{2\pi} \sigma_s M} \exp \left[ - \frac{R^2 \theta^2}{2\sigma_s^2} \right]
\]

\( \sigma_s \) is the r.m.s. value of the coordinate \( s \) for a gaussian bunch [m].

In this case, a standard definition of the bunch length may be \( L = 4\sigma_s \).

For the ISR: the values of \( R \) and \( \beta \) are given in Tables 2 and 5, respectively. The bunch length is of the order of 7 m and the bunches are approximately parabolic.

3.11 Average current for a circulating beam

The average current \( I \) is

\[
I = k \frac{N \beta}{R}
\]

where:

\[
k = \frac{e c}{2\pi} = 7.64462 \times 10^{-12} [Am],
\]

\( N \) is the total number of charges in the circulating beam.

This formula is valid either for a bunched beam or for a coasting beam.

For the ISR: the values of \( R \) and \( \beta \) are given in Tables 2 and 5, respectively.
3.12 **Bunching factor**

By definition, the bunching factor is the ratio of the average current \( I \) [A] in the circulating beam to the peak current \( \hat{I} \) [A] inside a bunch. This factor can be calculated from the formulae in Sections 3.10 and 3.11 and will depend on the shape of the bunches

\[
B_F = \frac{I}{\hat{I}} = \frac{M \lambda}{S 2\pi R}.
\]

The total length \( \lambda \) of the bunches is taken as 4 times the r.m.s. value for a gaussian shape, as mentioned in Section 3.10.

\( S \) is a shape factor taking the following values:

\[
\begin{align*}
S &= 1 & \text{for rectangular bunches} \\
S &= 1.5 & \text{for parabolic bunches} \\
S &= 1.6 & \text{for gaussian bunches} \\
S &= 2 & \text{for triangular bunches.}
\end{align*}
\]

For the ISR: the bunches are approximately parabolic, so that \( S = 1.5 \). The bunch length is of the order of 7 m.

3.13 **Relative amplitudes of the bunch harmonics**

The relative amplitudes of the bunch harmonics \( q \) are defined as follows:

\[
I_q = \hat{I} \sum_q c_q e^{iq\theta}, \quad q = \text{integer},
\]

\( I_q \) is the azimuthal current distribution of a bunched beam (Section 3.10) [A], \( \hat{I} \) is the peak current inside a bunch (Section 3.10) [A].

The expressions for \( c_q \) depend upon the current distribution inside a single pulse. If the bunches are (Section 3.10)

i) **rectangular**

\[
c_q = D \frac{q^2}{4q^2 + \lambda^2} \sin \frac{q\lambda}{2R},
\]

ii) **parabolic**

\[
c_q = D \frac{4\lambda}{2q^2 + \lambda^2} \left[ \frac{2R}{q\lambda} \sin \frac{q\lambda}{2R} - \cos \frac{q\lambda}{2R} \right],
\]

iii) **gaussian**

\[
c_q = D \frac{\sigma s}{\sqrt{2\pi} R} \exp \left[ -\frac{q^2 \sigma^2}{2R^2} \right],
\]

iv) **triangular**

\[
c_q = D \frac{2R}{2q^2 + \lambda^2} \left[ 1 - \cos \frac{q\lambda}{2R} \right],
\]

where \( D \) is the following series development:

\[
D = \sum_{k=0}^{M-1} \exp \left[ -i q \frac{2\pi}{R} k \right], \quad k = \text{integer}.
\]
If the bunch harmonic number $q$ is a multiple of $h$, $D$ becomes equal to the number of bunches $M$. If, furthermore, the number of bunches is equal to $h$, $D$ is zero for all bunch harmonics which are not a multiple of $h$.

For the ISR: the values of $h$ and $R$ are given in Table 2. The bunch length is given in Section 3.10. The bunches are approximately parabolic, so that formula ii) above applies. Since $M$ is smaller or equal to 20, none of the harmonic amplitudes $c_0$ is zero.

4. FORMULAE IN CONNECTION WITH THE VACUUM CONDITIONS⁸,⁹,¹⁰,¹¹

Symbols frequently used in this chapter

$L$: half distance between two vacuum pumps [m]

$S$: half-lumped pumping speed [m³s⁻¹]

$c_m$: molecular conductance per unit length of the vacuum pipe [m³s⁻¹]

$\sigma$*: ionization cross-section of the gas [m²]

$\eta_d$: net desorption factor, i.e. the number of molecules desorbed per incident ion minus the sticking probability of the ion, which is normally equal to 1

$n$: molecular density of the gas [m⁻³]

$P$: pressure of the gas [Torr]

$g$: gas desorption per tube length, independent of gas density [m⁻¹s⁻¹]

$A$: cross-sectional area of the vacuum tube [m²]

4.1 Relation between pressure and gas density

The molecular density $n$ [m⁻³] of a gas at pressure $P$ [Torr] is given by

$$n = k \frac{P}{T}$$

where:

$$k = N_m \frac{T_{st}}{P_{st} V_m} = 9.65118 \times 10^{24} \text{ [m}^{-3} \text{ K Torr}^{-1}]$$

$N_m$ is the number of molecules per mole (Table 1),

$T_{st}$ is the standard temperature (Table 1) [K],

$P_{st}$ is the standard pressure (Table 1) [Torr],

$V_m$ is the molar volume of an ideal gas under standard conditions (Table 1) [m³],

$T$ is the temperature of the gas [K].

The atomic density $n_a$ [m⁻³] is obtained by multiplying the molecular density $n$ by the number of atoms $N_a$ in a molecule.

For the ISR: using $T = 300$ K, this relation becomes

$$n = 5.21706 \times 10^{22} P$$.
4.2 Gas desorption coefficient and molecular conductance

The gas desorption per tube length $g$ $\left[ \text{m}^{-3}\text{s}^{-1} \right]$ is defined by

$$ g = p_r q. $$

$p_r$ is the perimeter of the vacuum chamber $\left[ \text{m} \right]$, $q$ is the specific gas desorption $\left[ \text{m}^{-2}\text{s}^{-1} \right]$.

The specific molecular conductance $c_m$ $\left[ \text{m}^2\text{s}^{-1} \right]$ is for a long tube of circular section:

$$ c_m = 97 \pi \sqrt{\frac{1}{M}} r^3, $$

for a long tube of elliptical section:

$$ c_m = 97 \pi \sqrt{\frac{1}{M}} a b^2, $$

for a long tube of rectangular section:

$$ c_m = 92.6 \pi \sqrt{\frac{1}{M}} a^2 b \ Y(\delta = a/b), $$

where:

$$ Y(\delta) = \frac{1}{5} \ln \left( \delta + \sqrt{\delta^2 + 1} \right) + \ln \left( \frac{1 + \sqrt{\delta^2 + 1}}{\delta} \right) + \frac{1}{35^2} \left[ 1 + \delta^3 - (1 + \delta^2)^{3/2} \right]. $$

$T$ is the absolute temperature of the gas $\left[ \text{K} \right]$, $M$ is the molecular mass (Table 8), $r$ is the radius of a circular tube $\left[ \text{m} \right]$ $a$ is the half-major axis of an elliptical tube or the half-major side of a rectangular tube $\left[ \text{m} \right]$ $b$ is the half-minor axis of an elliptical tube or the half-minor side of a rectangular tube $\left[ \text{m} \right]$.

For the ISR: some typical values of the perimeter $p_r$ and of the specific conductance $c_m$ for nitrogen molecules at 300 K are summarized in Section 4.3 for different regions of the machine circumference.

For the ISR residual gas composition, the total specific gas desorption $q$ is $4 \times 10^{10}$ $\text{m}^{-2}\text{s}^{-1}$. This value is essentially the $\text{H}_2$ contribution to the specific desorption, since the value of $q$ for $\text{N}_2$ or $\text{O}_2$ is of the order of 1 % of the total value.

4.3 Equilibrium pressure without a particle beam

The geometric model used is a periodic structure with constant specific molecular conductance $c_m$ and regularly spaced pumps (at distance $2l$), all having the same pumping speed $2S$. The model assumes, furthermore, that the gas desorption coefficient $g$ is constant. In this model, the longitudinal coordinate $s$ has its origin at the mid-point between two pumps.
\[ n_0(s) = g \left( \frac{L}{S} + \frac{L^2 - s^2}{2c_m} \right), \]

g and \( c_m \) are explicitly given in Section 4.2.

The equilibrium molecular density \( n_0 \) is that density existing in the vacuum pipe before stacking a beam. The corresponding equilibrium pressure \( P_0 \) can be deduced from the relation given in Section 4.1.

For the ISR: it is shown in Section 4.2 that \( g \) depends on the perimeter \( p_r \) of the vacuum chamber and that \( c_m \) depends on the gas. Some values of \( p_r \), of \( c_m \) for the nitrogen at 300 K and of the distance \( 2L \) are given in the following table for different regions of the ISR machine:

<table>
<thead>
<tr>
<th></th>
<th>2L [m]</th>
<th>( p_r ) [m]</th>
<th>( c_m(N_2) ) [m^3 s^{-1}]</th>
</tr>
</thead>
<tbody>
<tr>
<td>Elliptical chamber</td>
<td>5</td>
<td>2.5</td>
<td>0.355</td>
</tr>
<tr>
<td>0.05 ( \times ) 0.16 m^2</td>
<td></td>
<td></td>
<td>0.05</td>
</tr>
<tr>
<td>Round chamber</td>
<td>10</td>
<td>4</td>
<td>0.503</td>
</tr>
<tr>
<td>0.16 m diameter</td>
<td></td>
<td></td>
<td>0.51</td>
</tr>
<tr>
<td>Experimental region I2</td>
<td>8</td>
<td>4</td>
<td>0.471</td>
</tr>
<tr>
<td>0.15 m diameter</td>
<td></td>
<td></td>
<td>0.42</td>
</tr>
<tr>
<td>Special chamber in</td>
<td>8</td>
<td>1.6</td>
<td>0.343</td>
</tr>
<tr>
<td>injection septum magnet</td>
<td></td>
<td></td>
<td>0.03</td>
</tr>
<tr>
<td>0.04 ( \times ) 0.16 m^2</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The value of the pumping speed for one half the combined station ion pump and sublimation pump is \( S = 0.5 \) m^3 s^{-1}

4.4 Dynamic pressure behaviour in the presence of a beam

The particle beam induces gas desorption via bombardment of the chamber walls by ions created in the residual gas. Neglecting the surface effects, the corresponding dynamic change in the molecular density \( n \) [m^{-3}] is given by

\[
n(s,t) = \sum_{\nu=1}^{\infty} \cos \lambda_\nu s \left[ \frac{C_\nu}{F_\nu} + \left( B_\nu - \frac{C_\nu}{F_\nu} \right) \exp \left( -\frac{F_\nu t}{A} \right) \right],
\]

where:

\[
C_\nu = \left( \frac{2g}{a_\nu^2 \lambda_\nu} \right) \sin \lambda_\nu,
\]

\[
F_\nu = c_\nu \lambda_\nu^2 - \sigma + \frac{I_d}{e},
\]

\[
a_\nu^2 = L + \frac{\sin 2\lambda_\nu L}{2\lambda_\nu},
\]

and \( \lambda_\nu \) are the roots of the transcendental equation \( \lambda_\nu \tan \lambda_\nu L = \frac{S}{c_m} \).
The initial gas distribution \( n(s,0) \) gives for the Fourier's coefficients \( B_v \):

\[
B_v = \frac{1}{a_v} \int_{-L}^{L} n(s,0) \cos \lambda_v s \, ds .
\]

The geometric model used is identical to the model given in Section 4.3 and \( \eta_d \) is also assumed to be constant.

The relation between the density \( n \) and the pressure \( P \) is given in Section 4.1. The coefficients \( g \) and \( c_m \) are given explicitly in Section 4.2.

For the ISR: the latest values of the ionization cross-section at 26 GeV/c are given in Table 8. The total value of \( q \) is given in Section 4.2. The present values of \( L, S \) and \( c_m \) for \( N_2 \) at 300 K are given in Section 4.3.

4.5 Equilibrium pressure in the presence of a beam

A finite equilibrium pressure can only exist for negative exponents in the expression given in Section 4.4. The stability condition in the presence of a beam is thus given by \( F_i > 0 \), i.e.

\[
\eta_d I < (\eta_d I)_{\text{critical}} \quad \Rightarrow \quad \frac{e \, c_m}{\sigma_*} \lambda_1^2 .
\]

\( \lambda_1 \) is the first root of the transcendental equation given in Section 4.4.

When this condition is fulfilled, the equilibrium density is given by

\[
n(s,t^* = \infty) = \sum_{v=1}^{\infty} \frac{C_v}{F_v} \cos \lambda_v s ,
\]

where \( C_v \) and \( F_v \) are given in Section 4.4.

Taking the first term \( v = 1 \) of this series expansion, an approximate pressure bump equation valid for the equilibrium can be written. This yields the relationship between the starting pressure \( P_0 \) [Torr] given in Section 4.3 and the pressure in the presence of a beam \( P_\infty \) [Torr]:

\[
P_\infty = \frac{P_0}{\frac{\sigma_* \eta_d I}{e \, c_m \lambda_1^2}} ,
\]

in which appears the so-called effective pumping speed \( S_{\text{eff}} [m^2 s^{-1}] \)

\[
S_{\text{eff}} = c_m \lambda_1^4 = \frac{g}{P_0} .
\]

The last relationship between the equilibrium pressures with and without beam can be generalized for a system with 2 gas components. The pressures \( P_{\infty,1} \) and \( P_{\infty,2} \) are related to the starting pressures \( P_{0,1} \) and \( P_{0,2} \) by
\[ P_{\infty,i} = \frac{P_o,j \frac{b_{ij}}{S_{\text{eff},i}} + P_o,i \left( 1 - \frac{b_{ij}}{S_{\text{eff},j}} \right)}{\left( 1 - \frac{b_{ij}}{S_{\text{eff},i}} \right) \left( 1 - \frac{b_{ij}}{S_{\text{eff},j}} \right) - \frac{b_{ij}}{S_{\text{eff},i}}} \]

with:

\[ i = 1, 2 ; \ j = 1, 2 \text{ and } i \neq j . \]

The coefficients \( b_{ij} \) are defined by

\[ b_{ij} = \frac{\sigma_{*,i}}{e} \eta_{d,ij} I \quad i,j = 1, 2 . \]

\( \sigma_{*,j} \) is the ionization cross-section of the gas number \( j [m^2]. \)

\( \eta_{d,ij} \) gives the desorption yield of gas \( i \) by an ion of gas \( j. \)

For the ISR: the following approximate values of \( S_{\text{eff}} \) can be used for estimating the values of \( \eta_d \) in an elliptical chamber (0.05 \times 0.16 m²):

\[ S_{\text{eff},H_2} \approx 0.028 \text{ m}^2 \text{s}^{-1} \]

\[ S_{\text{eff},CO} \approx 0.0075 \text{ m}^2 \text{s}^{-1} , \]

\( \eta_d \) being given by the relation (see above)

\[ \eta_d = \frac{e S_{\text{eff}}}{\sigma_* I} \left( 1 - \frac{P_o}{P_{\infty}} \right) . \]

\( P_o \) and \( P_{\infty} \) are measured and \( \sigma_* \) is given in Table 8.

4.6 Pressure decay when dumping the beam

When the beam is dumped, the pressure decreases again. The starting decay rate is simply related to the equilibrium pressure in the presence of the beam \( P_{\infty} \) by

\[ \dot{P}_d = - \frac{\sigma_* \eta_d I}{e A} P_{\infty} . \]

\( \dot{P}_d \) is the decay rate of the pressure just after dumping the beam [Torr s⁻¹].

4.7 Clearing current associated with electron production

The ionization of the residual gas produces electrons, which give rise to a clearing current. In addition to this, the ions produced may sputter metal atoms from the vacuum chamber walls, which further increase the clearing current seen by collecting electrodes. The global effect is

\[ I_{cl} = I_{cl}^0 \left( 1 + \frac{\delta \Phi_*}{\frac{e}{2} \nu s_{t, re} - \Delta \Phi_*} \right) , \]
where:

\[ I_{c1}^O = \sigma_{*N_2} \ n_e d I, \]

\[ I_{c1}^O \] is the total clearing current seen on the collecting electrodes [A],

\[ I_{c1}^O \] is the clearing current due to primary electrons only [A],

\( \delta \) and \( \Delta \) are the sputtering coefficients for the residual gas and for the sputtered atoms, respectively,

\( \bar{v} \) is the mean velocity of the sputtered atoms [m s\(^{-1}\)],

\( s_t \) is the sticking probability of the sputtered atoms,

\( \tau_+ \) is the ionization cross-section of the sputtered atoms [m\(^2\)],

\( \sigma_{*N_2} \) is the ionization cross-section of the nitrogen gas [m\(^2\)],

\( d \) is the length over which electrons are cleared [m],

\( \tau \) is the radius of the vacuum tube [m],

\( n_+ \) is the equivalent density of nitrogen gas for ionization [m\(^{-3}\)].

The molecular density \( n_+ \) is related to the \( N_2 \) equivalent ionization pressure \( P_+ \) [Torr] by (Section 4.1)

\[ n_+ = k \frac{P_+}{T}. \]

\( k \) is given explicitly in Section 4.1,

\( T \) is the absolute temperature of the residual gas [K].

The pressure \( P_+ \) is the pressure of nitrogen giving the same ionization current as the actual residual gas. For a mixture of different gases, the expression of \( P_+ \) is

\[ P_+ = \frac{1}{\sigma_{*N_2}} \sum_i \left( P_i \sum_j \sigma_{*j} \right). \]

\( i \) is numbering the gases of the mixture,

\( j \) is numbering the atoms in a molecule of the ith gas. If there are \( m \) identical atoms B in a molecule, the summation on \( j \) should contain every one and can be replaced by \( \sigma_{*B} \),

\( P_i \) is the pressure of the ith gas [Torr],

\( \sigma_{*j} \) is the ionization cross-section for protons on the jth atom [m\(^2\)].

Since the denominator in the expression for \( I_{c1} \) may vanish at some given current \( I \), the clearing current and the density of sputtered atoms will tend to infinity. This critical beam current is given by

\[ I_{critical} = \frac{\pi \ x \ \bar{v} \ s_t \ \text{re}}{2 \Delta \lambda}. \]

For the ISR: using the value of \( \sigma_{*N_2} \) given in Table 8 and \( T = 300 \) K, the clearing current due to primary electrons is

\[ I_{c1}^O = 2.63799 \ P_+ d I. \]

For the typical ISR residual gas composition, \( P_+ \) is calculated from the above mentioned formula, so that:

\[ P_+ = 0.64 \ P_g, \]

\( P_g \) being the gauge pressure [Torr].
4.8 Beam neutralization factor

Electrons produced by ionization of the residual gas (Section 4.7) become trapped in the potential well of the particle beam and as a result the beam becomes partly neutralized. If \( \rho_c \) is the total charge density in the beam and \( \rho_p \) the density of the protons alone, the definition of the neutralization factor \( \eta_e \) is

\[
\rho_c = \rho_p (1 - \eta_e) .
\]

5. FORMULAE IN CONNECTION WITH LUMINOSITY\(^{12,13,14,15}\)

Symbols frequently used in this chapter

- \( L \) luminosity \([\text{m}^{-2}\text{s}^{-1}]\)
- \( L_0 \) luminosity for centred beams \([\text{m}^{-2}\text{s}^{-1}]\)
- \( h_{\text{eff}} \) effective height associated with two vertically centred beams colliding vertically at zero angle \([\text{m}]\)
- \( w_{\text{eff}} \) effective width associated with two horizontally centred beams colliding horizontally at zero angle \([\text{m}]\)
- \( \sigma_{1,2} \) r.m.s. value for the vertical densities of particles in beams 1 and 2 \([\text{m}]\)
- \( \tau_{1,2} \) r.m.s. value for the horizontal densities of particles in beams 1 and 2 \([\text{m}]\)
- \( \psi \) horizontal colliding angle \([\text{rad}]\)

5.1 Effective sizes of thin beams

For horizontal and vertical Gaussian distributions with r.m.s. values \( \tau_{1,2} \) and \( \sigma_{1,2} \), the effective sizes are

\[
\begin{align*}
    w_{\text{eff}} &= \sqrt{2\pi} \sqrt{\tau_1^2 + \tau_2^2} \\
    h_{\text{eff}} &= \sqrt{2\pi} \sqrt{\sigma_1^2 + \sigma_2^2} .
\end{align*}
\]

These relations apply to the case of Gaussian beams colliding at zero angle horizontally and vertically, respectively.

The values of \( \sigma_{1,2} \) and \( \tau_{1,2} \) have to be taken for \( b_{z,1} \) and \( b_{x,1} \), respectively (see Section 1.6).

Note: \( 2\sqrt{\pi} = 3.544908 \), \( \sqrt{2\pi} = 2.506628 \).

5.2 Horizontal standard deviation of a stacked beam

When the beam is stacked in the longitudinal phase space (Fig. 4), its shape is approximately rectangular in the variable \( x \). For a rectangular horizontal distribution with Gaussian tails, the standard deviations \( \tau_{1,2} \) are given by
\[ \tau_{1,2} = \frac{1}{2\sqrt{5}} \sqrt{\frac{w_{1,2}^3 + 3\sqrt{2\pi} \delta_{1,2} \sqrt{w_{1,2}^3 + 2\delta_{1,2}^2 \sqrt{w_{1,2}^3 + 12\sqrt{2\pi} \delta_{1,2}^3}}}{w_{1,2} + \sqrt{2\pi} \delta_{1,2}}}, \]

\( w_{1,2} \) is the total width of the rectangular part of the distribution \([\text{m}]\),

\( \delta_{1,2} \) is the r.m.s. value of the Gaussian tails of the distribution \([\text{m}]\).

Note: \( 2\sqrt{5} = 3.464102, \sqrt{2\pi} = 2.506628 \).

5.3 General definition of the luminosity

By definition, the luminosity is the time-averaged integral over the interaction volume \( \Omega \) of the number of reactions per unit time and unit volume

\[ L = \frac{1}{\sigma_r} \frac{\tau_{\text{rev}}}{\tau_{\text{rev}}} \int \int_{\Omega} \frac{d^2N}{dtd\Omega} \, dtd\Omega, \]

\( \tau_{\text{rev}} \) is the revolution time of the particles \([\text{s}]\) (see Section 3.3),

\( \sigma_r \) is the total cross-section of the reaction \([\text{m}^2]\),

\( \Omega \) is the interaction volume where the two beams collide \([\text{m}^2] \), \( d\Omega = ds \, dx \, dz \).

The number of reactions per unit time and unit volume satisfies the following relation associated with the Lorentz transformation of the variables:

\[ \frac{1}{\sigma_r} \frac{d^2N}{dtd\Omega} = \rho_1(x, z, s, t) \rho_2(x, z, s, t) \, c \, E(\beta_1, \beta_2, \psi), \]

where the relativistic factor \( E \) is given by

\[ E(\beta_1, \beta_2, \psi) = \sqrt{\beta_1^2 + \beta_2^2 + 2\beta_1 \beta_2 \cos \psi - \beta_1^2 \beta_2^2 \sin^2 \psi}. \]

When \( \beta_1 = \beta_2 \), this factor becomes

\[ E(\beta, \psi) = 2\beta \cos \frac{\psi}{2} \sqrt{1 - \beta^2 \sin^2 \frac{\psi}{2}}. \]

\( \rho_{1,2}(x, z, s, t) \) are the particle densities per unit volume, which can also depend explicitly on time as, for instance, in the case of a bunched beam.

The indices 1 and 2 are associated with beam 1 and beam 2.

Some specific applications of this general formula are given in Sections 5.4 to 5.7.

Note: \( L \, [\mu\text{b}^{-1}\text{s}^{-1}] = 10^{-34} \, L \, [\text{m}^{-2}\text{s}^{-1}] \).

5.4 Luminosity for two coasting beams with a non-zero colliding angle

Assuming a vertical Gaussian distribution of the particles and a horizontal colliding angle, the luminosity for two coasting beams is given by

\[ L = E(\beta_1, \beta_2, \psi) \frac{I_1 I_2}{\sigma^2 c \beta_1 \beta_2 \sin \psi} \exp \left[ -\frac{1}{2} \frac{z^2}{\sigma_1^2 + \sigma_2^2} \right], \]

\( z \) is the vertical beam separation \([\text{m}]\).

The indices 1 and 2 are associated with beam 1 and beam 2.
The effective height $h_{\text{eff}}$ is defined in Section 5.1 and is assumed not to vary over the interaction length. The relativistic coefficient $E(\beta_1, \beta_2, \psi)$ is given in Section 5.3.

With the assumption that $\beta_1 = \beta_2 = 1$, the expression for $L$ simplifies to

$$L = k \frac{I_1 I_2}{h_{\text{eff}} \tan \left(\frac{\psi}{2}\right)} \exp \left[-\frac{1}{2} \frac{z^2}{\sigma_1^2 + \sigma_2^2}\right],$$

where:

$$k = \frac{1}{e^2 c} = 1.299427 \times 10^{29} \left[\text{A}^{-2}\text{m}^{-1}\text{s}^{-1}\right].$$

This formula is also valid for a vertical colliding angle and a horizontal Gaussian distribution provided that the vertical variable $z$, $h_{\text{eff}}$ and $\sigma_1, \sigma_2$ are replaced by their horizontal equivalents $x$, $w_{\text{eff}}$ and $\tau_1, \tau_2$, respectively.

This formula applies directly to the case where one of the two beams is bunched, if the interaction length is small compared with the bunch length and if the average current (Section 3.11) of the bunched beam is used.

For the ISR: the different values of $\psi$ associated with the SPM are given in Table 10. For the nominal value of $\psi = 14.773^\circ$ and for $\beta_1 = \beta_2 = 1$, $L$ reduces to

$$L = 1.0024 \times 10^{36} \frac{I_1 I_2}{h_{\text{eff}}} \exp \left[-\frac{1}{2} \frac{z^2}{\sigma_1^2 + \sigma_2^2}\right].$$

5.5 Luminosity for two bunched beams with a non-zero colliding angle

Assuming transverse Gaussian distributions for the particles, a finite horizontal colliding angle and a bunch length which is large compared to the interaction length, the luminosity for two bunched beams is given by

$$L = E(\beta_1, \beta_2, \psi) \frac{2\pi R}{M \beta_1 \beta_2} S' \frac{I_1 I_2}{e^2 c} \frac{1}{h_{\text{eff}}} \sin \psi \exp \left[-\frac{1}{2} \frac{z^2}{\sigma_1^2 + \sigma_2^2}\right],$$

where $z$ is the vertical beam separation [m], $M$ is the number of bunches in the two beams, $\beta$ is the full bunch length [m].

The indices 1 and 2 are associated with beam 1 and beam 2.

The effective height $h_{\text{eff}}$ is defined in Section 5.1 and is assumed not to vary over the interaction length. The relativistic coefficient $E(\beta_1, \beta_2, \psi)$ is given in Section 5.3.

$S'$ is a shape factor depending on the form of the bunches (Section 3.10):

- $S' = 1$ for rectangular bunches,
- $S' = 1.2$ for parabolic bunches,
- $S' = 1.234$ for Gaussian bunches,
- $S' = 1.335$ for triangular bunches.

In the case of Gaussian bunches, the bunch length has been taken as 4 times the r.m.s. value (Section 3.10). These values of $S'$ are true if the bunches of each beam are synchronous.
As in Section 5.4, this expression for \( L \) simplifies when \( \beta_1 = \beta_2 = 1 \) and remains valid for a vertical colliding angle provided that the vertical variables \( z, h_{\text{eff}} \) and \( \sigma_{1,2} \) are replaced by their horizontal equivalents \( x, w_{\text{eff}} \) and \( \tau_{1,2} \).

5.6 Luminosity for two coating beams with zero colliding angle

Assuming horizontal and vertical Gaussian distributions of the particles, it is possible to calculate the luminosity for two coating beams with zero colliding angle when the interaction length is given. Since the interaction length can be arbitrarily large, it may be important in this specific case to take into account the variations of the beam sizes over this length:

\[
L = L_0 \exp \left\{ -\frac{1}{2} \left[ \frac{z^2}{\sigma_1^2 + \sigma_2^2} + \frac{x^2}{\tau_1^2 + \tau_2^2} \right] \right\},
\]

where:

\[
L_0 = (\beta_1 + \beta_2) \frac{I_1 I_2}{e^2c \beta_1 \beta_2 h_{\text{eff}} w_{\text{eff}}} 28z, I F \left[ \frac{\arctg \frac{d}{28z, I}}{28z, I}, \sqrt{1 - \left( \frac{y, I}{\beta_1, I} \right)^2} \right]
\]

for \( \beta_1, I \geq \beta_2, I \),

\[
L_0 = (\beta_1 + \beta_2) \frac{I_1 I_2}{e^2c \beta_1 \beta_2 h_{\text{eff}} w_{\text{eff}}} 28x, I F \left[ \frac{\arctg \frac{d}{28x, I}}{28x, I}, \sqrt{1 - \left( \frac{y, I}{\beta_1, I} \right)^2} \right]
\]

for \( \beta_1, I \leq \beta_2, I \),

\( F(\phi, k) \) being the elliptic integral of the first kind\(^{7,16}\).

\( x \) is the horizontal beam separation [m],
\( z \) is the vertical beam separation [m],
\( d \) is the interaction length [m].

The indices 1 and 2 are associated with beam 1 and beam 2. \( \beta_1, I \) and \( \beta_2, I \) are the minima of the betatron amplitude functions, which are assumed to be at the centre of the interaction region. The effective sizes \( h_{\text{eff}} \) and \( w_{\text{eff}} \) defined in Section 5.1 are taken at these minima, and the size variation is quadratic.

If the beam sizes do not vary over the interaction length, \( L_0 \) simplifies to

\[
L_0 = (\beta_1 + \beta_2) \frac{I_1 I_2 d}{e^2c \beta_1 \beta_2 h_{\text{eff}} w_{\text{eff}}}
\]

The above formula directly applies to the case where one of the two beams is bunched provided the average current (Section 3.11) of the bunched beam is used.

5.7 Luminosity for two bunched beams with zero colliding angle

Assuming horizontal and vertical Gaussian distributions of the particles, the luminosity for two bunched beams with zero colliding angle is given by

\[
L = (\beta_1 + \beta_2) \frac{aR}{M} \frac{I_1 I_2}{e^2c \beta_1 \beta_2 h_{\text{eff}} w_{\text{eff}}} \exp \left\{ -\frac{1}{2} \left[ \frac{z^2}{\sigma_1^2 + \sigma_2^2} + \frac{x^2}{\tau_1^2 + \tau_2^2} \right] \right\},
\]
x is the horizontal beam separation [m],
z is the vertical beam separation [m],
N is the number of bunches.
The indices 1 and 2 are associated with beam 1 and beam 2. The effective sizes \( h_{\text{eff}} \) and \( w_{\text{eff}} \) are defined in Section 5.1 and are assumed not to vary over the interaction length, which is equal to the bunch length.

Compared with the case described in Section 5.5, this expression for L is not depending on the bunch shape and is only valid if the bunches of each beam are synchronous.

5.8 Beam-beam rate at one intersection

\[
B = \sigma_m L
\]

\( B \) is the beam-beam rate \( [s^{-1}] \),
\( \sigma_m \) is the monitor constant at this intersection \( [m^2] \).

5.9 Measurement of the effective sizes of the beam

For two beams colliding at a large horizontal angle, the only relevant beam size is the height as shown in Sections 5.4 and 5.5. On the contrary for collinear beams, both the height and the width are important (Sections 5.6 and 5.7). In both situations, however, the same method can be used for measuring the effective sizes of the beam, which are defined in Section 5.1. For Gaussian distributions, the effective sizes are given by integrating the beam-beam rate w.r.t. the beam separation:

\[
h_{\text{eff}} = \frac{1}{\hat{B}(z=0)} \int_0^\infty B(z) \, dz,
\]

\[
w_{\text{eff}} = \frac{1}{\hat{B}(x=0)} \int_0^\infty B(x) \, dx,
\]

\( \hat{B}(z=0) \) is the maximum beam-beam rate appearing in \( z = 0 \) \( [s^{-1}] \),
\( \hat{B}(x=0) \) is the maximum beam-beam rate appearing in \( x = 0 \) \( [s^{-1}] \),
\( B(x) \) is the beam-beam rate as a function of the horizontal beam separation \( x \) \( [s^{-1}] \),
\( B(z) \) is the beam-beam rate as a function of the vertical beam separation \( z \) \( [s^{-1}] \).

For the ISRS: this method of measurement of \( h_{\text{eff}} \) is commonly used. In addition to this, there is a beam profile monitor using a sodium curtain. Measuring the full widths \( D_1 \) and \( D_2 \) \( [m] \) with this monitor at the half-maxima of the vertical beam profiles for beams 1 and 2, the effective height can be estimated from

\[
h_{\text{eff}} = \sqrt{2\pi} \sqrt{0.191 \ (D_1^2 + D_2^2) - 1.1 \times 10^{-6} \ \frac{\beta_z^2 L}{\beta_{z,m}^2}}.
\]

The constant \( 1.1 \times 10^{-6} \) comes from an experimental estimation of the instrument's resolution (valid for \( h_{\text{eff}} \), \( D_1 \) and \( D_2 \) in m). \( \beta_{z,m} \) is the value of \( \beta_z \) at the monitor position. If only one beam is measured (\( D_m \)), it is necessary to assume that both beams have roughly the same height \( D_1 = D_2 = D_m \) in the above formula.
5.10 Luminosity loss due to orbit distortions for a non-zero colliding angle

Orbit distortions can appear as median plane tilts and longitudinal slopes at the intersections. If the horizontal colliding angle is different from zero, both these mechanisms reduce the luminosity for vertically aligned beams as follows:

\[
\frac{\Delta L}{L} = -5.0 \frac{1}{h_{\text{eff}}^2} \left[ (\tau_1 X_1)^2 + (\tau_2 X_2)^2 \right],
\]

\[
\frac{\Delta L}{L} = -4.1 \frac{(\tau_1^2 + \tau_2^2) \cos^2 (\psi/2)}{h_{\text{eff}}^2 \sin^2 \psi} (\phi_1 - \phi_2)^2 ,
\]

\(\Delta L/L\) is the luminosity loss,

\(X_{1,2}\) are the horizontal median plane tilts for beams 1 and 2 [rad],

\(\phi_{1,2}\) are the longitudinal slopes in the plane \((z,s)\) for beams 1 and 2 [rad],

\(\tau_{1,2}\) are given explicitly in Section 5.2 for a stacked beam [m],

\(h_{\text{eff}}\) is given explicitly in Section 5.1 [m].

These formulae are still valid for a vertical colliding angle, vertical median plane tilts and longitudinal slopes in the plane \((x,s)\) if \(\tau_{1,2}\) and \(h_{\text{eff}}\) are replaced by \(\sigma_{1,2}\) and \(w_{\text{eff}}\), respectively.

For the ISR: possible values for \(\psi\) are given in Table 10.

5.11 Luminosity loss due to orbit distortions for a zero colliding angle

From the two effects mentioned in Section 5.10, only the second is important for a zero colliding angle, since the beam is generally circular in such an intersection \((\alpha_{p,x} = 0)\). For vertically aligned beams, the reduction of luminosity will be

\[
\frac{\Delta L}{L} = -2.6 \times 10^{-3} \frac{\sigma^2}{h_{\text{eff}}^2} (\phi_1 - \phi_2)^2,
\]

d is the interaction length [m],

\(\phi_{1,2}\) are the longitudinal slopes in the plane \((z,s)\) for beams 1 and 2 [rad].

This formula gives the luminosity loss for longitudinal slopes in the plane \((x,s)\) provided that \(h_{\text{eff}}\) is replaced by \(w_{\text{eff}}\).
6. FORMULÆ IN CONNECTION WITH CURRENT LOSS RATE AND STORED ENERGY

6.1 Current loss due to the total luminosity

The current loss in s\(^{-1}\) is given by

\[
\frac{1}{I} \frac{dI}{dt} = k \frac{\beta L N \sigma_{pp}}{R I},
\]

with

\[
k = \frac{\varepsilon c}{2\pi} = 7.6446 \times 10^{-12} \text{ [Am]},
\]

N is the number of intersections,
L is the luminosity [m\(^{-2}\)s\(^{-1}\)],
\(\sigma_{pp}\) is the total proton-proton cross-section [m\(^2\)].

In practical cases, the current loss is very often given in ppm min\(^{-1}\) so that the formula becomes

\[
\frac{1}{I} \frac{dI}{dt} = 4.5868 \times 10^{-4} \frac{\beta L N \sigma_{pp}}{R I}.
\]

For the ISR: using R, N and \(\sigma_{pp}\) given in Table 2, the current loss is, in s\(^{-1}\),

\[
\frac{1}{I} \frac{dI}{dt} = 1.6673 \times 10^{-4} \frac{\beta L}{I},
\]

and, in ppm min\(^{-1}\),

\[
\frac{1}{I} \frac{dI}{dt} = 10^{-4} \frac{\beta L}{I}.
\]

6.2 Current loss due to the nuclear scattering on the residual gas

The current loss in s\(^{-1}\) is given by

\[
\frac{1}{I} \frac{dI}{dt} = c n_{NS} \sigma_{n,N},
\]

\(\sigma_{n,N}\) is the total nuclear cross-section for protons on nitrogen [m\(^2\)],
n\(_{NS}\) is the equivalent density of nitrogen atoms for nuclear scattering [m\(^{-3}\)].

The atom density \(n_{NS}\) is related to the \(N_2\) equivalent nuclear scattering pressure \(P_{NS}\) [Torr] by (see Section 4.1)

\[
n_{NS} = k N_a \frac{P_{NS}}{T},
\]

where:

\[
k N_a = 1.93024 \times 10^{25} \text{ [m}^{-3} \text{ K Torr}^{-1}],
\]

k is explicitly given in Section 4.1,
\(N_a\) is the number of atoms in the \(N_2\) molecules, i.e. \(N_a = 2,\)
T is the temperature of the residual gas [K].

The pressure \(P_{NS}\) is the pressure of nitrogen giving the same beam losses due to nuclear scattering as the actual residual gas.

For a mixture of different gases, the expression of \(P_{NS}\) is
\[ P_{\text{NS}} = \frac{1}{2\sigma_{n,N}} \sum_{i} \left( p_i \sum_{j} \sigma_{n,j} \right), \]

\( i \) is numbering the gases in the mixture,

\( j \) is numbering the atoms in a molecule of the \( i \)th gas. If there are \( m \) identical atoms B in a molecule, the summation on \( j \) should include all of them and can be replaced by \( m \sigma_{n,B} \).

\( p_i \) is the pressure of the \( i \)th gas [Torr],

\( \sigma_{n,j} \) is the total nuclear cross-section for protons on the \( i \)th atom \([\text{m}^2]\).

For the ISR: using \( \sigma_{n,N} \) given in Table 9 and \( T = 300 \) K, the current loss is, in \( s^{-1} \),

\[ \frac{1}{T} \frac{dI}{dt} = 7.3298 \times 10^2 P_{\text{NS}}, \]

and, in ppm min\(^{-1}\),

\[ \frac{1}{T} \frac{dI}{dt} = 4.3979 \times 10^{10} P_{\text{NS}}. \]

The values of \( \sigma_{n,j} \) at the ISR energies are given in Table 9 for different atoms. \( P_{\text{NS}} \) is computed by the ISR control computer according to the above formula. For the ISR residual-gas composition, \( P_{\text{NS}} \) is given by

\[ P_{\text{NS}} = 0.37 P_g, \]

\( P_g \) being the gauge pressure [Torr].

### 6.3 Stored energy in the beam

The total energy stored in a particle beam is given by

\[ E_s = 10^4 \frac{I_{CP}}{f_{\text{rev}} \beta}, \]

\( f_{\text{rev}} \) is the revolution frequency \([s^{-1}]\) (Section 3.3),

\( E_s \) is the stored energy \([J]\).

For the ISR: \( \beta \) is given in Table 5 for the standard energies and the value of the revolution frequency \( f_{\text{rev}} \) for particles travelling at the speed of light is given in Table 2.
7. FORMULAE IN CONNECTION WITH BEAM BLOW-UP BY SCATTERING AND
BEAM DAMPING BY STOCHASTIC COOLING\textsuperscript{17,18,19,20,21,22)}

Symbols frequently used in this chapter

$\sigma_y$ r.m.s. value of the radial or vertical beam dimension [m]

$w$ stack width [m]

$E_y$ transverse emittance as defined in Section 1.4 [m rad]

$\bar{\sigma}_{p,x}$ average value of the horizontal momentum compaction as defined in Section 1.1 [m]

$E_o$ rest energy of proton [GeV]

$E_e$ rest energy of electron [GeV]

7.1 Multiple scattering effect on transverse beam dimensions

The rate of fractional r.m.s. beam dimension increase [s\(^{-1}\)] due to multiple scattering in the residual gas is given by

$$\frac{1}{\sigma_y} \frac{d\sigma_y}{dt} = k \frac{E_y}{p^2 E_y} n_{MS},$$

where:

$$k = 4\pi^2 \left( \frac{E_e}{c} \right)^2 \frac{r_e^2 c G_N}{1.08543 \times 10^{-23} \left[ (GeV/c)^2 \text{ m}^2 \text{ s}^{-1} \right]},$$

and

$$\sigma_y = \frac{1}{2} \sqrt{\frac{E_y G_y}{\pi}}.$$

$G_N$ is the absolute gas factor for the nitrogen atom (Table 1). This factor is given for any atom j by

$$G_j = Z_j^2 \left( \frac{5.836 \times 10^6}{(A_j Z_j)^{1/3}} \right),$$

$Z_j$ is the charge of the atom j ($Z = 7$ for N),

$A_j$ is the atomic number of the atom j ($A = 14$ for N),

$n_{MS}$ is the equivalent density of nitrogen atoms for multiple scattering [m\(^{-3}\)].

The atom density $n_{MS}$ is related to the N\(_2\) equivalent multiple scattering pressure $P_{MS}$ [Torr] by (see Section 4.1)

$$n_{MS} = k N_a \frac{P_{MS}}{T},$$

where:

$$k N_a = 1.93024 \times 10^{25} \left[ \text{m}^{-3} \text{ K Torr}^{-1} \right],$$

$k$ is explicitly given in Section 4.1,

$N_a$ is the number of atoms in the N\(_2\) molecules, i.e. $N_a = 2$,

$T$ is the temperature of the residual gas [K].
The pressure $P_{MS}$ is the pressure of nitrogen giving the same multiple scattering effect as the actual residual gas. For a mixture of different gases, the expression of $P_{MS}$ is

$$P_{MS} = \frac{1}{2N} \sum_i \left( p_i \sum_j g_j \right),$$

i is numbering the gases in the mixture,
j is numbering the atoms in a molecule of the $i$th gas. If there are $m$ identical atoms B in a molecule, the summation on $j$ should include all of them and can be replaced by $m c_B$, $P_i$ is the pressure of the $i$th gas [Torr], $G_j$ is the absolute gas factor for the $j$th atom. The values of $G_j$ are given in Table 1 for different atoms.

Particle losses on an aperture limit can be calculated knowing the rate of change of the standard deviations of the particle distributions and knowing the form of these distributions. Frequently, a Gaussian distribution is chosen for the vertical plane and a rectangular distribution with Gaussian tails is chosen for the horizontal plane.

Note: For two vertically centred beams with the same Gaussian vertical distribution and colliding vertically at zero angle, the effective height is related to the emittance by

$$h_{eff} = \sqrt{\frac{E_z}{E_z,1}}.$$ 

For the ISR: using the relation between $P_{MS}$ and $P_{MS}$ with $T = 300$ K, the rate of fractional standard deviation increase [$s^{-1}$] is

$$\frac{1}{\sigma_y} \frac{d\sigma_y}{dt} = 0.698380 \frac{E_y}{I} \frac{P_{MS}}{E_y}.$$

Table 2 gives the normalized emittance values. The values of $\bar{E}_y$ are given for different working conditions in Table 3. $P_{MS}$ is calculated by the ISR control computer according to the above formula. For the ISR residual-gas composition, $P_{MS}$ is given by

$$P_{MS} = 0.20 P_g ,$$

$P_g$ being the gauge pressure [Torr].

7.2 Intra-beam scattering effect on vertical beam dimension

This formula, giving the rate of vertical dimension change, takes into account the angular distribution of collision momenta inside a wide and uniform stack\(^{18}\):

$$\frac{1}{\sigma_z} \frac{d\sigma_z}{dt} = k \frac{I \sqrt{\frac{E_z}{E_x}, \frac{E_z}{E_x}}}{wp^5 E_z^{1/2}} F_1 \left( \frac{\sigma_p z}{\sigma_{p,x}}, \frac{\sigma_p s}{\sigma_{p,x}} \right) \ln C ,$$

where:

$$k = 4\pi^2 e \frac{r^2}{p} \left( \frac{E_0}{c} \right)^3 = 4.2201 \times 10^{-16} [(\text{GeV/c})^5 \text{ m}^2 \text{ A}^{-1} \text{ s}^{-1}].$$
The rate of fractional standard deviation change is given in s⁻¹.

The parameters appearing in the function \( F_1 \) and the variable \( C \) are defined as follows:

\[
\frac{\sigma_{P, Z}}{\sigma_{P, X}} = \frac{\sigma_{Z, X}}{\sigma_{X, Z}} = \sqrt{\frac{E_x}{E_Y}}
\]

\[
\frac{\sigma_{P, S}}{\sigma_{P, X}} = \frac{E_x}{\sqrt{\sigma_{P, X}}} \quad \text{valid for } w \gg \sigma_{X}
\]

\[
C = k_1 \frac{E_z^{3/2} p^2}{\sqrt{B_z}}
\]

with

\[
\sigma_y = \frac{1}{2} \sqrt{\frac{E_y}{\pi}} \quad \text{(see also Section 1.6)},
\]

and

\[
k_1 = \frac{1.36}{8^{3/2} E_0^{3/2} r_p} = 2.2597 \times 10^{16} \, [\text{GeV}^{-2} \text{ m}^{-1}] .
\]

The function \( F_1 \) is given in Fig. 5.

This formula contains the assumption that \( E_x \) changes only slowly with time, that the current losses remain small and that the rate of the vertical dimension change does not exceed \( \sim 0.15 \text{ h}^{-1} \).

The note in Section 7.1 concerning the effective height also applies here.

For the ISR: the values of \( \sigma_y \) and \( \sigma_{P, X} \) are given in Table 3. The relativistic parameters are given for standard energies in Table 5 and the normalized emittance is given in Table 2.

### 7.3 Intra-beam scattering effect on transverse beam dimensions and momentum spread

The theory referred to in this section takes into account the energy spread within the beam and the linear part of the dependence of the scattering angle on the coordinates of the betatron oscillations. It then gives not only the vertical dimension change but also the horizontal blow-up and the increase in momentum spread [s⁻¹]:

\[
\frac{1}{\sigma_z} \frac{\mathrm{d} \sigma_z}{\mathrm{d} t} = \frac{A}{2} F_2(a, b, c)
\]

\[
\frac{1}{\sigma_x} \frac{\mathrm{d} \sigma_x}{\mathrm{d} t} = \frac{A}{2} F_2 \left( \frac{1}{a}, \frac{1}{b}, k \right) + \frac{1}{1 + \frac{E_x E_x}{w^2}} F_2 \left( i, \frac{1}{a}, j \right)
\]

\[
\frac{1}{\Delta p/p} \frac{\mathrm{d} (\Delta p/p)}{\mathrm{d} t} = n A F_2 \left( i, \frac{1}{a}, j \right) \left[ 1 - \frac{1}{\sqrt{1 + \frac{E_x E_x}{w^2}}} \right],
\]

\( n \) is equal to ½ and 1 for bunched and unbunched beams, respectively.
\[ F_1 \left( \frac{\sigma_{px}}{\sigma_{px}}, \frac{\sigma_{as}}{\sigma_{px}} \right) \text{ as a function of } \frac{\sigma_{px}}{\sigma_{px}} \text{ with } \frac{\sigma_{as}}{\sigma_{px}} \text{ as parameter} \]

**Fig. 5** Function \( F_1 \) for estimating the intra-beam scattering effect
(formula in Section 7.2)

The parameters \( a, b, c, i, j \) and \( k \) appearing in the function \( F_2 \) and the variable \( A \) are defined as follows:

\[ a = \frac{\gamma \sqrt{E_z} \bar{\alpha}_{px} \sqrt{E \bar{E}_{xx}} + w^2}{w \sqrt{E \bar{E}_{xx} ar{E}_{zz}}} \]

\[ b = \frac{\sqrt{\bar{E}_{xx}}}{\bar{E}_{xx} E_z} \]

\[ c = \frac{k_1 p \bar{E}^{3/4}}{E_0 \bar{E}_z} \]
\[
\begin{align*}
  i &= \frac{w x}{\gamma p_x \sqrt{E_x x_x + w^2}}, \\
  j &= k_1 \frac{\beta w \sqrt{E_x x_x}}{\alpha p_x \sqrt{E_x x_x + w^2}} (E_x x_x)^{1/4}, \\
  k &= k_2 \frac{p_x}{E_o} \sqrt{E x x / \beta x} (E_x x_x)^{1/4}, \\
  A &= k_2 \frac{I \alpha p_x}{p x \sqrt{E x x_x x}}.
\end{align*}
\]

where:

\[
\begin{align*}
  k_1 &= \frac{1}{\sqrt{2 \pi} / p} \sqrt{\pi} = 2.41886 \times 10^9 \text{ [m}^{-1/2}] , \\
  k_2 &= \frac{2 \pi}{e} \frac{r^2}{p} \frac{E}{E_o} = 7.15851 \times 10^{-17} \text{ [m}^2 \text{ GeV}^2 \text{ A}^{-1} \text{ s}^{-1}].
\end{align*}
\]

The function \( F_2(a, b, c) \) cannot be evaluated analytically and it is necessary to use the computer calculated curves in Figs. 6 and 7. To reduce the range of numerical values for \( F_2 \), one can use the following relations:

\[
F_2(a, b, c) = F_2(b, a, c)
\]

\[
F_2(a, b, c) + \frac{1}{a^2} F_2 \left( \frac{1}{a}, \frac{b}{a}, \frac{c}{a} \right) + \frac{1}{b^2} F_2 \left( \frac{1}{b}, \frac{a}{b}, \frac{c}{b} \right) = 0.
\]

The formula in Section 7.3 for the vertical dimension is not directly comparable in its algebraic form with the preceding formula in Section 7.2, because the functions \( F_1 \) and \( F_2 \) are not defined in the same way. Nevertheless, we can say that both these formulae (7.2 and 7.3) agree within ~15 % for the vertical dimension change.

The note in Section 7.1 concerning the effective height also applies here.

It should be noted that there is a mistake of a factor of 2 in the reference paper\(^9\) and this is the reason why we have taken the factor A/2 instead of A in the formulae of Section 7.3. The two first relations give the rate of change of the standard deviations as in Section 7.1 and the third one gives the momentum spread increase. Therefore, it is possible to calculate the losses of particles in a limited aperture provided the distributions are known.

For the ISR: the values of \( \beta_y \), \( \beta_z \), and \( \alpha_{p_x} \) are given in Table 3. The relativistic parameters are given for standard energies in Table 5 and the emittance is given in Table 2.
Fig. 6  Function $F_2$ for estimating the intra-beam scattering effect
(Formula in Section 7.3)
7.4 Rate of stochastic cooling of momentum spread

A horizontal pick-up system is used to detect the radial position error of the centre of gravity of a short sample of particles with respect to a nominal radial position. This error is interpreted as an error in momentum with respect to the nominal momentum. A feedback system corrects a fraction $g$ of this error further downstream as the sample passes a wide-band cavity, which accelerates or decelerates it accordingly.

The cooling rate for momentum spread is $^{21}$\,[s$^{-1}$]

$$\frac{1}{\tau_p} = \frac{2N_f}{N} \left[ g - \frac{\beta_x^2}{2} \left( 1 + \frac{\beta_x E_{x,x}}{4\alpha_{p,x}^2 \langle \Delta p^2 \rangle} + \frac{\Delta v^2}{\langle \Delta p^2 \rangle} \right) \right]$$

where:

- $\beta_x$ and $\alpha_{p,x}$ are taken at the pick-up position,
- $N_f$ is the frequency bandwidth of the feedback system [s$^{-1}$],
- $N$ is the number of particles in the ring,
- $g$ is the fraction of observed momentum error corrected per passage through the system,
\( \Delta p \) is the momentum deviation from the nominal value of the single particle [GeV/c],
\( \langle \Delta p \rangle \) is the average momentum deviation in a sample,
\( \langle \Delta p^2 \rangle \) is the mean square momentum deviation over particles in many samples, i.e. over all particles in the machine,
\( \nu \) is the r.m.s. electronic noise of the amplifier system, expressed at the pick-up level as a fictitious momentum error of the sample,
\( p \) is the nominal momentum [GeV/c].

Some other interesting parameters can be defined:

i) Equivalent length of a sample [m]

\[
\ell_{e} = \frac{c}{2} \frac{1}{W_f}.
\]

ii) Number of particles in a sample

\[
N_s = \frac{1}{2e} \frac{1}{\nu^2} W_f.
\]

iii) Optimum gain \( g_o \) which permits the fastest cooling

\[
g_o = \frac{1}{1 + \frac{\alpha_{x,x} + \alpha}{4 \Delta p^2_p \frac{\Delta p^2}{p^2}}}
\]

and, when the ratio of the noise power to the signal power is large

\[
g_o \approx \frac{\Delta p^2}{\nu^2}.
\]

7.5 Fraction of error corrected in a momentum cooling system

The momentum cooling fraction \( g \) (Section 7.4) associated with an available power \( P \) [W] in the wide-band cavity is

\[
g = g_o \sqrt{\frac{P}{P_{tot}}},
\]

\( g_o \) is the optimum gain (Section 7.4),

\( P_{tot} \) is the total power required for the optimum gain \( g_o \) [W]. For a large ratio of the noise power to the signal power, \( P_{tot} \) is given by

\[
P_{tot} = \frac{g_o^2 + e^2}{12 N_s R_n} \left( \frac{c \Delta p_{tot}}{e} \right)^2
\]

where:

\( N_s \) is the number of particles in a sample (Section 7.4),

\( R_n \) is the impedance of the wide-band cavity [\( \Omega \)],

\( \Delta p_{tot} \) is the total momentum width of the beam for a rectangular distribution [eV/c].
7.6 Noise to signal ratio in a momentum cooling system

The ratio of the noise power to the signal power (Section 7.4) is given by the difference signal current from the horizontal pick-up and by the electronic noise of the amplifier system

\[ \frac{v^2}{\langle \Delta p^2 \rangle} = 3 \frac{I_v^2}{I_{sc}^2}, \]

\( I_v \) is the noise current of the amplifier system [A],
\( I_{sc} \) is the signal current due to the longitudinal Schottky noise [A].

These currents are given by

\[ I_{sc} = \sqrt{2e I W_f}, \]
\[ I_v = \frac{1}{R_p} \left( 10^{n/20} e \right) k_B T W_f, \]

\( W_f \) is the frequency bandwidth of the feedback system [s⁻¹],
\( R_p \) is the impedance of the pick-up [Ω],
\( k_B \) is the constant of Boltzmann [eV K⁻¹] (Table 1),
\( T \) is the absolute temperature [K],
\( n \) is the noise figure [dB].

7.7 Horizontal betatron oscillation damping in a momentum cooling system

The momentum cooling system described in Section 7.4 can also damp the horizontal betatron oscillations. The resulting cooling rate for the radial amplitude \( x \) is

\[ \frac{1}{T_x} = \frac{W_f}{N} \left[ g_x \cos (\Delta \mu_x) - \frac{g_x^2}{2} \left( 1 + \kappa_x + \frac{4 \alpha_{p,x}(P.U.)}{\beta_x(P.U.)} \frac{\langle \Delta p^2 \rangle}{p^2} \right) \right], \]

where:

\[ g_x = \frac{\alpha_{p,x}(P.U.)}{\alpha_{p,x}(P.U.)} \sqrt{\beta_x(P.U.)}, \]

\( g, N \) and \( W_f \) are defined in Section 7.4, as well as \( \frac{\langle \Delta p^2 \rangle}{p^2} \),
\( \Delta \mu_x \) is the horizontal phase shift from the pick-up to the cavity gap,
\( \kappa_x \) is the ratio of the noise power to the signal power, in which the signal current is the pick-up current due to horizontal betatron oscillations,
P.U. and gap mentioned in brackets indicate that the functions have to be taken at the pick-up and cavity positions, respectively.

The horizontal stochastic cooling is maximum if \( \Delta \mu_x \) is equal to an odd number of half betatron wavelength, but appears as soon as \( \cos (\Delta \mu_x) < 0 \) and \( g_x |\cos (\Delta \mu_x)| \) is larger than the second order term.
7.8 Rate of stochastic cooling of vertical betatron oscillations

A wide-band vertical difference pick-up is used to observe vertical errors of the
centre of gravity of a short longitudinal sample of particles. A feedback system corrects
a fraction $g_z$ of this error further downstream as the sample passes a fast vertical kicker.

Assuming that the vertical statistical error $\langle z \rangle$ of the sample at any passage through
the system is insignificantly reduced by the correction done during previous passages, the
cooling rate for vertical betatron oscillations is\(^{21}\) \([s^{-1}]\)

$$\frac{1}{\tau_z} = \frac{W_f}{N} \left[ g_z - \frac{g_z^2}{2} (1 + \kappa_z) \right],$$

$N$ and $W_f$ are defined in Section 7.4,
$\kappa_z$ is the ratio of the white noise power to the signal power, in which the signal current
is the pick-up current due to vertical betatron oscillations,
$g_z$ is the fraction of observed vertical error corrected per passage through the system.

The parameters $\kappa_S$ and $N_S$ can be defined as in Section 7.4, but the optimum gain per-
mittting the fastest cooling is now given by

$$g_{z,0} = \frac{1}{1 + \kappa_z},$$

and, when the ratio of the noise power to the signal power is large

$$g_{z,0} \approx \frac{1}{\kappa_z}.$$

7.9 Necessary power for optimum vertical cooling

It was shown in Section 7.8 that an optimum gain $g_{z,0}$ exists giving the highest cool-
ing rate. The total power $P_{z,\text{tot}}[\text{W}]$ on the two pick-up plates required for obtaining
$g_{z,0}$ is given for a large ratio of the noise power to the signal power by

$$P_{z,\text{tot}} = \frac{g_{z,0}^2}{8K_p N_S} \frac{\sigma_z(\text{P.U.}) \beta d pc}{\sqrt{\beta_z(\text{P.U.}) \beta_z(\text{P.K.}) le}},$$

$g_{z,0}$ is given in Section 7.8,
$N_S$ is defined in Section 7.4,
$K_p$ is the line impedance for the plates [\Omega],
$\sigma_z(\text{P.U.})$ is the r.m.s. value of the beam height (Section 1.6) at the pick-up [m],
$\beta_z(\text{P.U.})$ and $\beta_z(\text{K})$ are the vertical betatron amplitude functions at the pick-up and at the
kicker, respectively [m],
$d$ is the vertical spacing of the electrodes [m],
$p$ is the nominal beam momentum [eV/c],
$e$ is the length of the vertical kicker [m].
7.1) Noise to signal ratio in a vertical cooling system

This ratio is given by the electronic noise of the amplifier system to the difference signal current from the vertical pick-up

\[ \kappa_2 = 8 \pi \frac{I_0^2}{I_{sc}} . \]

The currents \( I_0 \) and \( I_{sc} \) are defined in Section 7.6.

8. FORMULAE IN CONNECTION WITH TUNE SHIFTS

Symbols frequently used in this chapter

- \( E_o \): rest energy of proton [GeV]
- \( a \): half-width of a small beam or single pulse [m]
- \( b \): half-height of a small beam or single pulse [m]
- \( w_{1/2} \): half-width of a large stacked beam [m]
- \( g \): half-height of a ferromagnetic gap or radius of a circular yoke [m]
- \( h \): half-height of the gap between conducting plates or of a pipe [m]
- \( v \): half-width of a conducting pipe [m]
- \( y_1 \): general transverse coordinate of the centre-of-mass of the whole beam [m]
- \( E_y \): transverse component of the electrostatic field [V m\(^{-1}\)]
- \( \varepsilon_0 \): permittivity of free space [V\(^{-1}\) m\(^{-1}\)]
- \( \lambda_c \): line density of electrical charges [A m\(^{-1}\)]
- \( B_f \): bunching factor (Section 3.12)
- \( \eta_0 \): neutralization factor (Section 4.8)
- \( x \): horizontal position of a particle which undergoes a given tune shift w.r.t. the centre of the perturbing beam [m]

8.1 Incoherent tune shift due to direct space charge effects

The tune shift considered here is that due to direct space charge fields of a single beam and experienced by a particle sitting in the beam itself

\[ \Delta Q_{y, \text{dir}} = - k \frac{R}{b^3} \frac{I}{\gamma} \frac{E_y}{B_f} \left( \frac{1}{y^2} - \eta_0 \right) \frac{\varepsilon_0 y}{b^2} , \]

where:

\[ k = \frac{2 \gamma P}{ec} = 6.39062 \times 10^{-6} \text{ [A}^-1\text{]} . \]

In the case of a bunched beam, this expression gives the maximum incoherent tune shift due to direct space charge effects, experienced by the particles set at the bunch centre.
The geometric coefficient \( \varepsilon_{0,Y} \) associated with direct space charge fields is defined by

\[
\varepsilon_{0,Y} = \frac{\pi \varepsilon_0 b^2}{2 \lambda_c} \frac{\partial E_Y}{\partial y},
\]

\( E_Y \) is the electrostatic field component due to the beam itself \([V m^{-1}]\),
\( \lambda_c \) is the line density of charges \([As m^{-1}]\) related to the averaged current \( I \) in a pulse via (Section 3.11)

\[
\lambda_c = \frac{I}{B_c}.
\]

For the ISR: this formula applies either to a single pulse for which \( B_f \) is of the order of 0.1 or to a stacked beam where \( B_f = 1 \). If the neutralization is negligible (bunched beam), the change of the betatron frequency due to direct space charge effects is rather small, since it is proportional to \( 1/\gamma^3 \). For large neutralization, the direct space charge effect can become of the same order of magnitude as the indirect effect (Section 8.2).

8.2 Tune shifts due to indirect space charge effects

The tune shifts considered here are those due to the presence of material boundaries around the beam excluding the direct space charge fields. It has become customary to separate the expressions for the tune shift associated with the oscillation of one single particle sitting in the perturbing beam and for the tune shifts associated with the oscillation of the whole beam. The first one is the so-called incoherent tune shift and the second one the coherent tune shift.

The incoherent tune shift (affecting, for instance, the oscillation frequency of a particle inside a stack) is

\[
\Delta Q^\text{inc}_Y = -k \frac{RI}{\beta_y} \left[ \sum_i \bar{y}^{(i)}_y c_1^{(i)} \frac{\xi_{1,Y}^{(i)}}{h^2} \left( 1 + \frac{1}{B_f \beta^2 \gamma^2} \right) (1 - \eta_e) + \sum_i \bar{y}^{(i)}_y c_2^{(i)} \frac{\xi_{2,Y}^{(i)}}{g^2} \right].
\]

The coherent tune shift valid for penetrating alternating magnetic fields at low frequency (affecting, for instance, the closed orbits) is

\[
\Delta Q^\text{coh}_Y = -k \frac{RI}{\beta_y} \left[ \sum_i \bar{y}^{(i)}_y c_1^{(i)} \frac{\xi_{1,Y}^{(i)}}{h^2} \left( 1 + \frac{1}{B_f \beta^2 \gamma^2} \right) (1 - \eta_e) + \sum_i \bar{y}^{(i)}_y c_2^{(i)} \frac{\xi_{2,Y}^{(i)}}{g^2} \right].
\]

The coherent tune shift valid for non-penetrating alternating magnetic fields at high frequency (affecting, for instance, the oscillation frequency of a kicked stack) is

\[
\Delta Q^\text{coh}_Y = -k \frac{RI}{\beta_y} \left[ \sum_i \bar{y}^{(i)}_y c_1^{(i)} \frac{\xi_{1,Y}^{(i)}}{h^2} \left( \varepsilon_{0,Y} + \frac{1}{B_f \beta^2 \gamma^2} \right) (1 - \eta_e) + \sum_i \bar{y}^{(i)}_y c_2^{(i)} \frac{\xi_{2,Y}^{(i)}}{g^2} \right],
\]

where \( k \) is defined in Section 8.1.
In the case of a bunched beam, these expressions give the maximum tune shifts due to indirect space charge effects, experienced by the particles set at the bunch centre.

The summations apply to the different possible geometries in the material boundaries, which may appear on the circumference of the machine. In each of these three expressions, the first summation concerns the electrostatic image effects (bottom index for \( C, \varepsilon \) and \( \xi \) equal to 1) and the second summation concerns the electromagnetic image effects (bottom index for \( C, \varepsilon \) and \( \xi \) equal to 2). The interdependence between the possible geometries considered in the literature, the material and the coefficients introduced here is summarized in the table below.

<table>
<thead>
<tr>
<th>Index ( i )</th>
<th>Geometry</th>
<th>Material</th>
<th>Coefficients</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>parallel plates</td>
<td>electrostatic</td>
<td>( C_1^{(1)} ); ( \varepsilon_{1,Y}^{(1)} ); ( \xi_{1,Y}^{(1)} )</td>
</tr>
<tr>
<td></td>
<td></td>
<td>ferromagnetic</td>
<td>( C_2^{(1)} ); ( \varepsilon_{2,Y}^{(1)} ); ( \xi_{2,Y}^{(1)} )</td>
</tr>
<tr>
<td>2</td>
<td>circular</td>
<td>electrostatic (vacuum pipe)</td>
<td>( C_1^{(2)} ); ( \varepsilon_{1,Y}^{(2)} ); ( \xi_{1,Y}^{(2)} )</td>
</tr>
<tr>
<td></td>
<td></td>
<td>ferromagnetic (magnet yoke)</td>
<td>( C_2^{(2)} ); ( \varepsilon_{2,Y}^{(2)} ); ( \xi_{2,Y}^{(2)} )</td>
</tr>
<tr>
<td>3</td>
<td>elliptical</td>
<td>electrostatic</td>
<td>( C_1^{(3)} ); ( \varepsilon_{1,Y}^{(3)} ); ( \xi_{1,Y}^{(3)} )</td>
</tr>
</tbody>
</table>

\( C_1^{(i)} \) and \( C_2^{(i)} \) are the fractions of the circumference occupied by an electrostatic and a ferromagnetic material in the \( i \)th geometry, respectively (see table above), \( \bar{E}_y^{(i)} \) is the average value of the betatron amplitude function in the fraction of the circumference defined by \( C^{(i)}[m] \).

The general definitions of the geometric coefficients \( \varepsilon_{1,Y} \) and \( \varepsilon_{2,Y} \), \( \xi_{1,Y} \) and \( \xi_{2,Y} \) associated with indirect space charge fields are

\[
\begin{align*}
\varepsilon_{1,Y} & = \frac{\pi \varepsilon_0 h^2}{\lambda_c} \frac{\partial E_y}{\partial y} \bigg|_{y_1 = 0} \\
\varepsilon_{2,Y} & = \frac{\pi \varepsilon_0 B_0^2}{\lambda_c \beta} \frac{\partial B_z}{\partial y} \bigg|_{y_1 = 0} \\
\xi_{1,Y} & = \frac{\pi \varepsilon_0 h^2}{\lambda_c} \left[ \frac{\partial E_y}{\partial y} + \frac{\partial E_y}{\partial y_1} \right] \bigg|_{y_1 = y_1} \\
\xi_{2,Y} & = \frac{\pi \varepsilon_0 B_0^2}{\lambda_c \beta} \left[ \frac{\partial B_z}{\partial y} + \frac{\partial B_z}{\partial y_1} \right] \bigg|_{y_1 = y_1}
\end{align*}
\]

\( E_y \) and \( B_z \) are the electromagnetic components due to the material surrounding the beam \([V \cdot m^{-1}]\) and \([T]\),

\( \lambda_c \) is the line density of charges given in Section 8.1 \([As \cdot m^{-1}]\).
For the ISR: the incoherent tune shift is the most important one and has to be compensated in order to avoid the disturbing resonances (Chapter 10). The direct space charge contribution can be neglected, since it appears that \(1/y^2 = \eta_c\) in the ISR at 26 GeV/c. The incoherent horizontal tune shifts are given in Fig. 8 for off-centred beams and the image space terms have been calculated with the following values:

\[
\begin{align*}
    a &= 0.0025 \text{ m} \\
    C_{Z}^{(1)} &= C_{Z}^{(3)} = 0.62 \\
    h_{\text{ell}} &= 0.026 \text{ m} \\
    \bar{\beta}_x &= 16.854 \text{ m} \\
    C_{l}^{(2)} &= 0.38 \\
    \nu &= 0.08 \text{ m} \\
    I &= 1 \text{ A} \\
    B_f &= 1 \\
    h_{\text{circ}} &= 0.08 \text{ m} \\
    \gamma &= 20 \\
    g &= 0.05 \text{ m}
\end{align*}
\]

For obtaining the horizontal tune shift for other values of the parameters \(\gamma\), \(\bar{\beta}_x\) and \(I\), it is sufficient to scale the curves of Fig. 8 in agreement with the formula. The ratio of the horizontal to the vertical tune shift is approximately given by

\[
\frac{\Delta \phi_x^{\text{inc}}}{\Delta \phi_z^{\text{inc}}} = -\frac{\bar{\beta}_x}{\bar{\beta}_z}.
\]

For a quick estimation of the incoherent tune shifts it is also possible to use the simple empirical formulae deduced from measurements performed at the ISR:

\[
\begin{align*}
    \Delta \phi_x^{\text{inc}} &= 0.04 \frac{I}{\gamma} \left[1 - \frac{0.2}{w_{1/2}} (x - x_1)^2\right] \\
    \Delta \phi_z^{\text{inc}} &= -1.25 \Delta \phi_x^{\text{inc}}.
\end{align*}
\]

### 8.3 Geometric coefficients for the tune shifts due to direct fields

This concerns the coefficient \(e_{0x,y}\) introduced in Section 8.1, and two cases can be considered:

a) Line charge or single pulse of elliptic cross-section and Gaussian distribution in both dimensions:

\[
\begin{align*}
    e_{0x} &= \frac{b^2}{\epsilon^2} \left\{ \frac{b}{a} \exp \left(-\frac{2x^2}{a^2}\right) - 1 + \sqrt{2\pi} \frac{x}{\epsilon} \left[ \text{erf} \left( \sqrt{2} \frac{bx}{a\epsilon} \right) - \text{erf} \left( \sqrt{2} \frac{x_0}{\epsilon} \right) \right] \exp \left(\frac{2x^2}{\epsilon^2}\right) \right\} \\
    e_{0z} &= -e_{0x} + \frac{b}{a} \exp \left(-\frac{2x^2}{a^2}\right),
\end{align*}
\]

where:

\[
\epsilon^2 = b^2 - a^2 > 0, \quad a \text{ and } b \text{ being } 2 \text{ times the r.m.s. beam dimensions.}
\]

At the pulse centre \((x = 0)\), this expression simply becomes

\[
\begin{align*}
    e_{0x} &= \frac{b^2}{a(a + b)} \\
    e_{0z} &= \frac{b}{a + b}.
\end{align*}
\]
Fig. 8 Incoherent tune shifts excited by image forces in the ISR.
(Calculated for $\gamma = 20$, $I = 1$ A, stack width 5 mm, elliptic chamber over 62% of ring with half-height 26 mm and half-width 80 mm, circular chamber over 38% of ring with radius 80 mm, parallel pole pieces over 62% of ring with half-separation 50 mm.)
b) Stacked beam with a uniform distribution and Gaussian tails in the horizontal direction and with a Gaussian profile in the vertical direction:

\[
\varepsilon_{o,x} = \frac{\sqrt{\pi}}{4^{1/2}} \frac{b^2}{\sqrt{w_{1/2}} \varepsilon} \left\{ \exp \left( \frac{x + w_{1/2}}{\varepsilon} \right)^2 \left[ \text{erf} \left( \frac{\sqrt{2} \frac{b}{\delta} \frac{x + w_{1/2}}{\varepsilon}}{\sqrt{2} \frac{x + w_{1/2}}{\varepsilon}} \right) - \text{erf} \left( \frac{\sqrt{2} \frac{b}{\delta} \frac{x - w_{1/2}}{\varepsilon}}{\sqrt{2} \frac{x - w_{1/2}}{\varepsilon}} \right) \right] - \exp \left( \frac{x - w_{1/2}}{\varepsilon} \right)^2 \left[ \text{erf} \left( \frac{\sqrt{2} \frac{b}{\delta} \frac{x - w_{1/2}}{\varepsilon}}{\sqrt{2} \frac{x - w_{1/2}}{\varepsilon}} \right) - \text{erf} \left( \frac{\sqrt{2} \frac{b}{\delta} \frac{x + w_{1/2}}{\varepsilon}}{\sqrt{2} \frac{x + w_{1/2}}{\varepsilon}} \right) \right] \right\}
\]

\[
\varepsilon_{o,z} = -\varepsilon_{o,x} + \frac{\sqrt{\pi}}{4^{1/2}} \frac{b}{\sqrt{w_{1/2}}} \left[ \text{erf} \left( \frac{\sqrt{2} \frac{x + w_{1/2}}{\delta}}{\sqrt{2} \frac{x - w_{1/2}}{\delta}} \right) \right],
\]

where:

\[\varepsilon^2 = b^2 - \delta^2,\]

\[\delta\] is the width of the horizontal tails [m], i.e. twice the standard deviation of the tails, and \(b\) is twice the r.m.s. vertical beam dimension.

At the beam centre, this expression becomes

\[
\varepsilon_{o,x} = \frac{\sqrt{\pi}}{2^{1/2}} \frac{b^2}{\sqrt{w_{1/2}} \varepsilon} \left[ \text{erf} \left( \frac{\sqrt{2} \frac{b \frac{w_{1/2}}{\varepsilon}}}{\sqrt{2} \frac{w_{1/2}}{\varepsilon}} \right) - \text{erf} \left( \frac{2\sqrt{2} \frac{w_{1/2}}{\varepsilon}}{\varepsilon} \right) \right]
\]

\[
\varepsilon_{o,z} = -\varepsilon_{o,x} + \frac{\sqrt{\pi}}{2^{1/2}} \frac{b}{\sqrt{w_{1/2}}} \text{erf} \left( \frac{\sqrt{2} \frac{w_{1/2}}{\delta}}{\varepsilon} \right),
\]

and at the beam edge \((x = w_{1/2})\)

\[
\varepsilon_{o,x} = \frac{\sqrt{\pi}}{4^{1/2}} \frac{b^2}{\sqrt{w_{1/2}} \varepsilon} \left[ \text{erf} \left( \frac{2\sqrt{2} \frac{w_{1/2}}{\varepsilon}}{\varepsilon} \right) - \text{erf} \left( \frac{2\sqrt{2} \frac{w_{1/2}}{\varepsilon}}{\varepsilon} \right) \right]
\]

\[
\varepsilon_{o,z} = -\varepsilon_{o,x} + \frac{\sqrt{\pi}}{4^{1/2}} \frac{b}{\sqrt{w_{1/2}}} \text{erf} \left( \frac{w_{1/2}}{\varepsilon} \right).
\]

Typical curves for \(\varepsilon_{o,x}\) and \(\varepsilon_{o,z}\) corresponding to stacks with \(b/w_{1/2} = 7/12\) are given in Fig. 9.

8.4 Geometric coefficients for the tune shifts due to indirect fields in a geometry with horizontal parallel plates

This concerns the coefficients \(\xi_{1,1}^{(1)}\), \(\xi_{2,1}^{(1)}\), \(\xi_{1,1}^{(1)}\) and \(\xi_{2,1}^{(1)}\) introduced in the table of Section 8.2.

For horizontal parallel plates, two coefficients vanish and some relations between the coefficients exist

\[
\xi_{1,1}^{(1)} = \xi_{2,1}^{(1)} = 0
\]

\[
\xi_{1,2}^{(1)} = \xi_{2,2}^{(1)}
\]

\[
\xi_{1,1}^{(1)} = -\xi_{1,1}^{(1)}
\]

\[
\xi_{1,2}^{(1)} = -\xi_{1,2}^{(1)}
\]

\[
\xi_{2,2}^{(1)} = -\xi_{2,2}^{(1)}
\]

\[
\xi_{2,2}^{(1)} = -\xi_{2,2}^{(1)}
\]
Fig. 9  Geometric coefficients for the tune shifts due to direct fields
(Calculated for stacks having the following distributions:
horizontal uniform \( w_{1/2} \) with tails \( \delta \),
vertical Gaussian \( b \), \( b/w_{1/2} = 7/12 \).)
Using the parameters

\[ \alpha_1 \equiv \frac{\pi}{2h} \quad \text{and} \quad \alpha_2 \equiv \frac{\pi}{2g} , \]

the expressions for the three coefficients, which are independent, are given below for two different cases with the beam vertically centred \( (z_1 = 0) \):

a) **Line-charge or single pulse at \( x_1 \):**

\[
\begin{align*}
\varepsilon_{1,z}^{(1)} &= \frac{\pi^2}{8} \frac{\text{ch}[\alpha_1(x-x_1)]}{\text{sh}^2[\alpha_1(x-x_1)]} - \frac{h^2}{2(x-x_1)^2} , \\
\varepsilon_{2,z}^{(1)} &= \frac{g^2}{2(x-x_1)^2} - \frac{\pi^2}{8 \text{sh}^2[\alpha_2(x-x_1)]} , \\
\varepsilon_{1,z}^{(1)} &= \frac{\pi^2}{16 \text{ch}^2[\alpha_1(x-x_1)]} .
\end{align*}
\]

At the pulse location \( (x = x_1) \), these expressions simply become

\[
\begin{align*}
\varepsilon_{1,z}^{(1)} &= \frac{\pi^2}{48} ; \\
\varepsilon_{2,z}^{(1)} &= \frac{\pi^2}{24} ; \\
\varepsilon_{1,z}^{(1)} &= \frac{\pi^2}{16} .
\end{align*}
\]

b) **Flat stacked beam centred on \( x_1 \):**

\[
\begin{align*}
\varepsilon_{1,z}^{(1)} &= \frac{\pi h}{4w_{1/2}} \frac{\text{ch}[\alpha_1(x-x_1)] \text{sh}(\alpha_1 w_{1/2})}{\text{sh}^2[\alpha_1(x-x_1)] - \text{sh}^2(\alpha_1 w_{1/2})} - \frac{1}{2} \frac{h^2}{(x-x_1)^2 - w_{1/2}^2} , \\
\varepsilon_{2,z}^{(1)} &= \frac{1}{2} \frac{g^2}{(x-x_1)^2 - w_{1/2}^2} - \frac{\pi g}{8w_{1/2} \text{sh}^2[\alpha_2(x-x_1)] - \text{sh}^2(\alpha_2 w_{1/2})} , \\
\varepsilon_{1,z}^{(1)} &= \frac{\pi h}{4w_{1/2}} \frac{\text{ch}(\alpha_1 w_{1/2})}{\text{ch}[\alpha_1(x-x_1)] + \text{ch}(\alpha_1 w_{1/2})} .
\end{align*}
\]

At the beam centre \( (x = x_1) \), these expressions become

\[
\begin{align*}
\varepsilon_{1,z}^{(1)} &= \frac{h^2}{2w_{1/2}^2} \left[ 1 - \frac{\alpha_1 w_{1/2}}{\text{sh}(\alpha_1 w_{1/2})} \right] , \\
\varepsilon_{2,z}^{(1)} &= \frac{g^2}{2w_{1/2}^2} \left[ \alpha_2 w_{1/2} \coth(\alpha_2 w_{1/2}) - 1 \right] , \\
\varepsilon_{1,z}^{(1)} &= \frac{\pi^2}{8 \alpha_1 w_{1/2}} \text{th} \left( \frac{\alpha_1 w_{1/2}}{2} \right) ,
\end{align*}
\]

and at the beam edges \( (x = x_1 \pm w_{1/2}) \)

\[
\begin{align*}
\varepsilon_{1,z}^{(1)} &= \frac{h^2}{8w_{1/2}^2} \left[ 1 - \frac{2\alpha_1 w_{1/2}}{\text{sh}(2\alpha_1 w_{1/2})} \right] , \\
\varepsilon_{2,z}^{(1)} &= \frac{g^2}{8w_{1/2}^2} \left[ 2\alpha_2 w_{1/2} \coth(2\alpha_2 w_{1/2}) - 1 \right] , \\
\varepsilon_{1,z}^{(1)} &= \frac{\pi^2}{16 \alpha_1 w_{1/2}} \text{th} \left( \alpha_1 w_{1/2} \right) .
\end{align*}
\]
In all these expressions, \( h \) and \( g \) are the half-distances between the two conducting and ferromagnetic plates, respectively [m].

Typical curves for \( \varepsilon_{1,z}^{(1)} \) and \( \varepsilon_{2,z}^{(1)} \) with different ratios \( w_{1/2}/h \) and \( w_{1/2}/g \) are given in Fig. 10.

For the ISR: since the main magnets are not pure dipoles but are gradient magnets, the above formulae for \( \varepsilon_{1,z}^{(1)} \) do not apply exactly. However, a correction factor has been estimated\(^{23} \) for a wedge-shaped gap and a single pulse:

\[
\varepsilon_{1,z}^{(1)} \text{ (wedge gap) } = \varepsilon_{2,z}^{(1)} \left[ 1 + \frac{6}{\pi^2} \frac{gB}{G} + \left( \frac{23}{3} + \frac{5}{\pi^2} \right) \left( \frac{gB}{G} \right)^2 \right].
\]

The negative sign applies to magnets which have a gap open towards the outside of the machine and a return yoke on the inside (so-called D-units).

The positive sign applies to magnets which have a gap open towards the inside of the machine and a return yoke on the outside (so-called F-units). Since in the ISR there are an equal number of both types of magnets, the linear correction term cancels and only the quadratic correction term, which is much smaller, has to be considered.

\( g \) is the half gap [m] at the pulse position and \( G \) is the magnet gradient [Tm\(^{-1}\)].

The values of \( G/B = -n/p \) for the ISR are given in Table 2.

8.5 Geometric coefficients for the tune shifts due to indirect fields in a circular geometry

This concerns the coefficients \( \varepsilon_{1,y}^{(z)} \), \( \varepsilon_{2,y}^{(z)} \) and \( \varepsilon_{3,y}^{(z)} \) introduced in the table of Section 8.2 (\( \varepsilon_{2,y}^{(z)} \) is not given in the literature).

In this geometry, there are two relations between the coefficients:

\[
\varepsilon_{1,z}^{(z)} = \varepsilon_{1,x}^{(z)},
\]

\[
\varepsilon_{2,z}^{(z)} = \frac{\mu-1}{\mu+1} \varepsilon_{1,z}^{(z)}(g),
\]

\( \mu \) is the constant relative permeability of a circular magnetic yoke,

\( \varepsilon_{1,z}^{(1)}(g) \) means that \( g \) must replace \( h \) in the expression of \( \varepsilon_{1,z}^{(z)} \).

The expressions of the three independent coefficients are given below for two different cases, the beam being vertically centred \( (z_1 = 0) \):

a) Line-charge or single pulse at \( x_1 \):

\[
\varepsilon_{1,z}^{(z)} = \frac{-h^2}{2} \frac{x_1^2}{(h^2 - x_1^2)^2},
\]

\[
\varepsilon_{1,x}^{(z)} = \frac{h^2}{2} \frac{h^2 - x_1^2}{(h^2 - x_1^2)^2},
\]

\[
\varepsilon_{1,z}^{(z)} = \frac{h^2}{2} \frac{h^2 + x_1^2}{(h^2 - x_1^2)^2}.
\]
Fig. 10  Geometric coefficients $\varepsilon_{1,z}^{(1)}$ and $\varepsilon_{2,z}^{(1)}$ in a geometry with horizontal parallel plates.
At the pulse location \((x = x_1)\), these expressions become

\[
\xi_{1,z}^{(z)} = -\frac{h^2}{2} \frac{x_1^2}{(h^2 - x_1^2)^2},
\]

\[
\xi_{1,x}^{(z)} = \frac{h^2}{2} \frac{1}{h^2 - x_1^2},
\]

\[
\xi_{1,x}^{(z)} = \frac{h^2}{2} \frac{h^2 + x_1^2}{(h^2 - x_1^2)^2},
\]

and for the pulse at the chamber centre \((x = x_1 = 0)\)

\[
\xi_{1,z}^{(z)} = \xi_{1,x}^{(z)} = 0; \quad \xi_{1,z} = \xi_{1,x} = \frac{1}{2}.
\]

b) **Flat stacked beam centred on** \(x_1:\)

\[
\xi_{1,z}^{(z)} = -\frac{h^2}{2x^2} \left( 1 + \frac{h^2}{C} + \frac{h^2}{w_{1/2}^2} \ln D \right),
\]

\[
\xi_{1,z}^{(z)} = -\frac{h^2}{2x^2} \left( 1 + \frac{h^2 - x^2}{C} + \frac{h^2}{w_{1/2}^2} \ln D \right),
\]

\[
\xi_{1,x}^{(z)} = -\frac{h^2}{2x^2} \left( 1 + \frac{h^2 + x^2}{C} + \frac{h^2}{w_{1/2}^2} \ln D \right),
\]

where:

\[
C = \frac{1}{h^2} \left[ \left( h^2 - xx_1 \right)^2 - w_{1/2}^2 x^2 \right],
\]

\[
D = \frac{h^2 - xx_1 - w_{1/2} x}{h^2 - xx_1 + w_{1/2} x}.
\]

At the chamber centre \((x = 0)\), these expressions become

\[
\xi_{1,z}^{(z)} = -\frac{3x_1^2 + w_{1/2}^2}{6h^2},
\]

\[
\xi_{1,z}^{(z)} = \frac{1}{2} \left( 1 - \frac{3x_1^2 + w_{1/2}^2}{3h^2} \right),
\]

\[
\xi_{1,x}^{(z)} = \frac{1}{2} \left( 1 + \frac{3x_1^2 + w_{1/2}^2}{3h^2} \right).
\]

In all these expressions, \(h\) and \(g\) are the radii of the electrostatic and ferromagnetic boundaries, respectively \([m]\).

The parameter \(\xi_{1,z}^{(z)}\) gives all the other parameters \(\xi\) by virtue of the above relations. For stacks in a circular pipe, typical curves for \(\xi_{1,z}^{(z)}\) with different ratios \(w_{1/2}/h\) and two positions of the stack \((x_1 = 0\) and \(x_1 = 0.2 \ h\)) are given in Figs. 11 and 12.

For the ISR: the curves of Figs. 11 and 12 can directly be used when the stacked beam is either centred by decreasing the magnetic field or left at the position obtained after stacking, i.e. \(x_1/h \approx 0.2\).
Fig. 11 Geometric coefficient $\varepsilon_{1,x}^{(2)}$ in a circular geometry for centred beams

Fig. 12 Geometric coefficient $\varepsilon_{1,z}^{(2)}$ in a circular geometry for off-centred beams
8.6 Geometric coefficients for the tune shifts due to indirect fields in an elliptical geometry

This concerns the coefficients $\varepsilon_1^{(3)}$ and $\xi_1^{(3)}$ introduced in the table of Section 8.2 (a magnetic yoke of elliptical section having never been considered in the literature).

In this geometry, the following relation between the coefficients exists:

$$\varepsilon_1^{(3)}_{i, z} = - \varepsilon_1^{(3)}_{i, x}$$

The expressions of the three remaining coefficients are very complex and are given here only for a line-charge or single pulse, at $x_1$ on the major axis inside focal points ($|x_1| < d$, $z_1 = 0$):

$$\varepsilon_1^{(3)}_{i, z} = \frac{h^2}{2} \left[ \left( \frac{2K}{\pi} \right)^2 \frac{DD_1}{F_1} \frac{1}{(S-S_1)^2} \frac{LSS_1}{SS_1^1} - \frac{2xK}{\pi F_3^1} \frac{CD_1}{S-S_1} - \frac{1}{(x-x_1)^2} \right] ,$$

$$\varepsilon_1^{(3)}_{i, x} = \frac{h^2}{2} \left[ \left( \frac{2K}{\pi} \right)^2 \frac{DD_1}{(S-S_1)^2} \frac{1}{F_1} \frac{CC_1}{SS_1} - \frac{2xK}{\pi F_3^1} \frac{CD_1}{S-S_1} \right] ,$$

$$\xi_1^{(3)}_{i, x} = \frac{h^2}{2} \left[ \left( \frac{2K}{\pi} \right)^2 \frac{1}{(S-S_1)^2} \left( CC_1 \frac{1-k^2SS_1}{SS_1^1} - DD_1 \frac{1-SS_1}{F_1^2} \right) + \frac{2xK}{\pi F_3^1} \frac{CD_1}{S-S_1} \right] ,$$

where:

$S$, $C$ and $D$ are the Jacobian elliptic functions $^7,^{16}$ sn, cn and dn of argument $u$ (respectively $u_1$ for subscript 1) with

$$u_1 = \frac{2K}{\pi} \arcsin \frac{X(1)}{d} ,$$

and $d = \sqrt{v^2 - h^2}$ ,

$K(k)$ is the complete elliptic integral of the first kind $^7,^{16}$ and the modulus $k$ is determined by the algebraic equation

$$q = \frac{\sqrt{v} - h}{\sqrt{h}} ,$$

where the "nome" $q$ is defined by

$$q = \exp \left(-\pi K'/K\right) ,$$

$K' = K(k')$ is the complete elliptic integral of the complementary modulus $k' = \sqrt{1-k^2}$ ,

$F$ and $F_1$ are defined according to

$$F_1 = \sqrt{d^2 - x_1^2} .$$

At the pulse location coinciding with the chamber centre ($x = x_1 = 0$), these expressions become
\[ \varepsilon_{1,z}^{(3)} = \frac{h^2}{12d^2} \left[ (1 + k^{1/2}) \left( \frac{2K}{\pi} \right)^2 - 2 \right] \]

\[ \varepsilon_{1,z}^{(3)} = \frac{h^2}{4d^2} \left[ \left( \frac{2K}{\pi} \right)^2 - 1 \right] \]

\[ \varepsilon_{1,x}^{(3)} = \frac{h^2}{4d^2} \left[ 1 - \left( \frac{2K}{\pi} \right)^2 \right] \]

with the same definitions as above.

For stacked beams of finite width, it is not possible to integrate analytically the expressions valid for a line-charge. However, the integrals have been evaluated numerically by computer for the parameter \( \varepsilon_{1,z}^{(3)} \). For stacks in an elliptical pipe (\( v/h = 3 \)), typical curves for \( \varepsilon_{1,z}^{(3)} \) with different ratios \( w_{1/2}/v \) and two positions of the stack (\( x_1 = 0 \) and \( x_2 = 0.2 \) \( v \)) are given in Figs. 13 and 14.

For the ISR: the curves of Figs. 13 and 14 can be used directly, since the aspect ratio \( v/h \) of the elliptical chamber is very close to 3 and since the stacked beam is either centred by decreasing the magnetic field or left at the position obtained after stacking, i.e. \( x_1/v \approx 0.2 \).

8.7 Tune shift of a particle crossing a cylindrical coasting beam

A particle crossing with a horizontal angle \( \psi \), an azimuthally uniform, Gaussian beam of circular cross-section undergoes a linear tune shift per intersection in the plane \((\gamma, s)\) of (28)

\[ \Delta \psi_{bb} \left( Y_1 \right) = k_{bb} \frac{1 + \beta_1 \beta_2}{\beta_1 \beta_2 \sigma^2} \frac{I_2 \beta_{y,1}}{\sigma_{y,z,1}} \frac{d}{y_1} \int_{-\infty}^{\infty} \left( \frac{d}{y_1} \right) \frac{\beta_{y,1} \sin \psi}{\sigma_{y,z,1}} \left( \frac{d}{y_1} \right) \right) \]

where:

\[ k_{bb} = \frac{r_P E_0}{4\pi e c^2} = 2.385611 \times 10^{-8} \left[ \text{A}^{-1} \text{GeV/c} \right] \]

\[ J_1 (\xi, \omega) = \cos^2 \psi \int_{-\infty}^{\infty} \frac{\xi}{t^2 + \frac{1}{t^2}} \left\{ 1 + \frac{\omega^2 t^2}{1 + t^2 \cos^2 \psi} \right\} \exp \left[ -\frac{1}{2} \frac{\omega^2 t^2}{1 + t^2 \cos^2 \psi} \right] - 1 \right\} dt \]

\[ J_2 (\xi, \omega) = \int_{-\infty}^{\infty} \frac{\xi}{t^2 + \frac{1}{t^2}} \left\{ 1 - \exp \left[ -\frac{1}{2} \frac{\omega^2 t^2}{1 + t^2 \cos^2 \psi} \right] \right\} dt \]

Index 1 refers to the perturbed particle,
Index 2 refers to the perturbing beam,
\( \beta_{y,1} \) is the minimum of the betatron amplitude function which is assumed to be at the point where the particle crosses the beam centre [m]. The variation of \( \beta_x \) is quadratic with the longitudinal coordinate and \( \beta_x = \beta_z \) since the beam is circular.
Furthermore, the functions \( \beta_x \) associated with the particle and with the beam are assumed to be equal,
\( \sigma_{y,z,1} \) is the r.m.s. value of the beam 2 radius, taken at the point where \( \beta_y \) is minimum [m],
\( d \) is the interaction length [m],
\( \psi \) is the horizontal colliding angle [rad],

\[ \]
Fig. 13 Geometric coefficient $c_{1,z}^{(3)}$ in an elliptical geometry for centred beams.
Fig. 14  Geometric coefficient $e^{(3)}_{1,z}$ in an elliptical geometry for off-centred beams
$J_x$ and $J_z$ are integrals, which are given in Figs. 15 and 16 for different values of the parameters $\xi$ and $\omega$, and for $\cos \psi \equiv 1$.

If the crossing angle is vertical, the expression for $\Delta_{\gamma}^{bb}$ remains valid, provided that the definitions of $J_x$ and $J_z$ are exchanged.

![Graph showing $J_x$ as a function of $\xi$ for different values of $\omega$.](image)

**Fig. 15** Integral $J_x (\xi, \omega)$ for estimation of particle-beam tune shifts (formula in Section 8.7)
8.6 Tune shift of a particle crossing a coasting or bunched beam at zero angle

A particle crossing with zero angle a strong beam of any cross-section undergoes a linear tune shift per intersection in the plane \((y,s)\) of

\[
\Delta \nu^b_y = 2 k^{bb} \frac{1 + \beta_1 \beta_2}{\beta_1 \beta_2 \beta_1} \int_0^d \beta_y I_s \frac{e_{0,y}}{\sigma_{z_2}^2} \, ds ,
\]

\(k^{bb}\) is given in Section 8.7, as well as the meaning of the indices 1 and 2,
\(d\) is the interaction length \([m]\),
\(\beta_{y_1}\) is the betatron amplitude function associated with the particle inside the interaction region \([m]\),
\(\sigma_{z_2}\) is the r.m.s. value of the beam 2 height \([m]\),
e\(e_{0,y}\) is the geometric coefficient for direct fields. It is explicitly given in Section 8.3 for a Gaussian pulse of elliptic cross-section and for a stacked beam of uniform distribution with tails in the horizontal direction and of Gaussian profile in the vertical direction. The variable \(x\) used in Section 8.3 represents in this case the radial distance of the particle from the strong beam centre. All the parameters \(a, b, x, \omega, k, \delta\) appearing in Section 8.3 can be functions of the coordinate \(s\).
In the special case where the functions \( \beta_{y_1}, \beta_{y_2} \) and the radial position \( x \) of the particle are constant through the interaction region, the linear tune shift per intersection for an either coasting or bunched beam becomes

\[
\Delta Q_{y_1}^{bb} = 2k_{bb} \left( \frac{1 + \beta_1 \beta_2}{\beta_1 \beta_2 p_1} \frac{\beta_{y_1,1} I_2}{\sigma_{z_2,1}^2} \right) \left\{ \begin{array}{cc}
\frac{2\beta_1}{\beta_1 + \beta_2} & \text{coasting beam} \\
\frac{\eta R}{M} & \text{bunched beam}
\end{array} \right.
\]

\( \beta_{y_1,1} \) is the betatron amplitude function associated with the particle in the crossing region \([m]\),

\( \sigma_{z_2,1} \) is the r.m.s. value of the strong beam height in the crossing region \([m]\),

\( M \) is the number of bunches in the strong beam.

For a strong bunched beam, it has been assumed that the perturbed particle was synchronous with the bunches.

If the strong beam has an elliptic cross-section and the particle remains radially centred w.r.t. the perturbing beam, the expressions given in Section 8.3.a) for \( x = 0 \) apply (with the convention \( a = 2\sigma_{x_1,1} \) and \( b = 2\sigma_{z_1,1} \)) and the relation for \( \Delta Q_{y_1}^{bb} \) takes the more usual form

\[
\Delta Q_{y_1}^{bb} = 2k_{bb} \left( \frac{1 + \beta_1 \beta_2}{\beta_1 \beta_2 p_1} \frac{\beta_{y_1,1} I_2}{\sigma_{y_2,1} (\sigma_{x_2,1} + \sigma_{z_2,1})} \right) \left\{ \begin{array}{cc}
\frac{2\beta_1}{\beta_1 + \beta_2} & \text{coasting beam} \\
\frac{\eta R}{\beta_1 + \beta_2} & \text{bunched beam}
\end{array} \right.
\]

\( \sigma_{x_2,1} \) is the r.m.s. value of the strong beam width in the crossing region \([m]\).

8.9 Tune shift of a particle crossing a coasting beam at a large angle

For a particle crossing with a relatively large horizontal angle \( \psi \) a circular pulse of Gaussian transverse distribution, the following condition and approximation apply to the relations given in Section 8.7:

\[
\omega(z) \gg 1 \ ; \ \cos \psi \approx 1 ,
\]

\[
J_z(\zeta, \omega >> 1) \approx \frac{2}{\omega^2} - \frac{2}{\zeta^2 \omega^2} \left[ 1 - \exp \left( -\frac{1}{2} \zeta^2 \omega^2 \right) \right] + \frac{\sqrt{2} \pi}{\zeta \omega} \text{erf} \left( \frac{1}{\sqrt{2} \xi \omega} \right)
\]

\( \omega(z) \) means that \( \omega, \) defined in Section 8.7, has to be taken for \( y \equiv z \).

If, furthermore, the conditions \( \omega(z) \zeta(z) \gg 1 \) and \( \omega(z) \gg \zeta(z) \) are verified, \( J_z \) tends towards

\[
J_z \approx \frac{\sqrt{2} \pi}{\zeta \omega} ,
\]

\( \zeta(z) \) means that \( \zeta, \) defined in Section 8.7, has to be taken for \( y \equiv z \).
Using the last approximation for \( J_z \), the linear tune shift per intersection of a particle crossing a coasting beam at a large horizontal angle is

\[
\Delta Q_{x_1}^{bb} = 0 ,
\]

\[
\Delta Q_{z_1}^{bb} = 2z_2 \pi k^{bb} \frac{I_z}{p_1} \frac{\beta_{z_1} \beta_{z_2}}{\sigma_{z_2,I} \sin \psi} ,
\]

\( k^{bb} \) is given in Section 8.7 as well as the meaning of the indices 1 and 2, \( \beta_{z_1,I} \) and \( \sigma_{z_2,I} \) are defined in Section 8.8 and the angle \( \psi \) in Section 8.7.

For a stacked beam consisting of a radial succession of circular pulses with Gaussian transverse distribution, \( I_z \) represents the total current and \( \sigma_{z_2,I} \) is the r.m.s. value of the thin dimension of the beam.

It is interesting to note that the octupole terms in the tune shift, seen by an off-centred particle, are opposed to the linear one given above, so that the total tune shift seen by the particles of a wide beam is distributed between 0 and \( \Delta Q_{z_1}^{bb} \) around an average value which is\(^{29}\)

\[
\overline{\Delta Q_{z_1,\text{tot}}}^{bb} \approx 0.7 \Delta Q_{z_1}^{bb} .
\]

If the crossing angle \( \psi \) is vertical, the above expression of \( \Delta Q_{z_1}^{bb} \) applies to \( \Delta Q_{x_1}^{bb} \), provided that the parameters \( \beta_{x_1,I} \) and \( \sigma_{x_1,I} \) are used instead of \( \beta_{z_1,I} \) and \( \sigma_{z_2,I} \). \( \Delta Q_{z_1}^{bb} \) becomes equal to zero.

For the ISR: the crossing plane is horizontal and the nominal angle \( \psi \) is equal to 0.2578 rad. In agreement with the above formulae, the total vertical averaged tune shift of a wide beam can be estimated from

\[
\overline{\Delta Q_{z_1,\text{tot}}}^{bb} = 6.567 \times 10^{-8} \frac{I_z \beta_{z_1,I}}{p_1 \sigma_{z_2,I}} ,
\]

while the horizontal tune shift is zero. The measurements made in the ISR with pulses give values of \( \overline{\Delta Q_{z_1,\text{tot}}}^{bb} \) which are close to this theoretical value (remembering that the linear tune shift \( \Delta Q_{z_1}^{bb} \) is \( \sim 1.43 \) larger).

When the Split Field Magnet is set, the crossing angle \( \psi \) changes in intersection 4. The values of \( \psi \) and the orbit characteristics in intersection 4 are given in Table 10 for different field levels.

The values of \( \beta_{z,I} \) are given in Table 3 and the relativistic parameters in Table 5.

8.10 Tune shift of a particle crossing a bunched beam at a large angle

The conditions and approximations given in Section 8.9 remain valid for a bunched beam, the sole difference coming from the non-uniform azimuthal distribution of the current \( I_z \). If the perturbed particle is synchronous with the bunches of the strong beam, the expressions for the linear tune shifts per intersection can be written as follows:
\[\Delta Q_{X1}^{bb} = 0,\]

\[\Delta Q_{Z1}^{bb} = 2\sqrt{\pi} k^{bb} \frac{1}{\beta_1 \beta_2 p_1} \frac{\beta_{z1}}{\sigma_{z1} \sin \psi} \int_0^{\theta+(1+\beta_2) \frac{w_2}{R \sin \psi}} I_{\theta,2} d\theta,\]

\(k^{bb}\) is given in Section 8.7 as well as the meaning of the indices 1 and 2, \(\beta_{z1}, I\) and \(\sigma_{z1}, I\) are defined in Section 8.8 and the angle \(\psi\) in Section 8.7, \(\theta\) is the angular shift of the perturbed particle w.r.t. the bunch centre [rad], \(w_2\) is the total width of the strong beam [m], \(I_{\theta,2}\) is the azimuthal current distribution inside a bunch of the perturbing beam [A] (Section 3.10).

If the interaction length is small compared with the bunch length, \(I_{\theta,2}\) can be taken as constant in the integration interval \(\Delta \theta\), so that the integral can be replaced by \(I_{\theta,2} \Delta \theta\), where

\[\Delta \theta = \left(1 + \frac{\beta_2}{\beta_1}\right) \frac{w_2}{R \sin \psi}.\]

9. **FORMULAE IN CONNECTION WITH LINEAR COUPLING\(^{30,31,12}\)**

**Symbols frequently used in this chapter**

- \(C\) complex coefficient of linear coupling
- \(\Delta\) distance from a linear coupling resonance
  
  \((=Q_x - nQ_y - p, n \text{ and } p \text{ being integers, } n = \pm 1)\)

9.1 **Complex coefficients of linear coupling**

For both sum and difference resonances \(Q_x + nQ_y = p, (n = \pm 1)\), which are associated with a linear coupling of transverse betatron oscillations, a general complex coefficient can be defined:

\[C = \frac{1}{4\pi R} \int_0^{2\pi} \sqrt{\beta_x \beta_2} \left[ K + \frac{1}{2} MR \left( \frac{\alpha_x}{\beta_x} - \frac{\alpha_2}{\beta_2} \right) - \frac{i}{2} MR \left( \frac{1}{\beta_x} - \frac{n}{\beta_2} \right) \right] \exp \left\{ i \left[ (\nu_x - Q_x \theta) + n(\nu_x - Q_x \theta) + p\theta \right] \right\} d\theta,\]

with the following definitions:

\[K(\theta) = \frac{1}{2} \frac{R^2}{B_0} \left( -\beta_2 \frac{\alpha_2}{\beta_x} + \frac{\alpha_2}{\beta_2} \right),\]

\[M(\theta) = \frac{R}{B_0} B_0^{\theta},\]

\(n = \pm 1 \) (depending on the resonance),
\( \alpha_y \) is the Twiss parameter (Section 1.1),
\( \nu_y \) is the phase of the betatron oscillation (Section 1.1),
\( \theta \) is the azimuthal angle (Fig. 1),
\( \partial B_y/\partial y \) is the skew gradient in the horizontal or vertical direction [Tm\(^{-1}\)].

For the ISR: using the above formula and taking, for example, the value of \( \alpha_y \), \( \theta_y \), \( \nu_y \) and \( Q_y \) for the 8C working line, it is possible to estimate the effect on the coupling due to random main magnet tilts

\[
|C| = 10 \, \phi_{r.m.s.},
\]

\( \phi_{r.m.s.} \) is the root mean square value of the distributed tilts [rad].

9.2 Fraction of energy interchanged in a kicked beam

The energy associated with betatron oscillations in a transverse plane is defined here as a square of the amplitude of the oscillation envelope. For the difference resonance \( Q_x - Q_z = p \), there is an interchange of energy between horizontal and vertical oscillations following a kick in a transverse plane and the fraction of interchanged energy is

\[
F = \frac{|C|^2}{\Delta^2 + |C|^2},
\]

where \( |C| \) indicates the modulus of \( C \).

9.3 Parameters of the coherent oscillations of a kicked beam

When close to the difference resonance \( Q_x - Q_z = p \), the coupling provokes a modulation of the amplitude of the oscillations in the same plane as an angular kick is given.

\[
S = \frac{|\Delta|}{\sqrt{\Delta^2 + |C|^2}},
\]

\[
T = \frac{1}{f_{rev} \sqrt{\Delta^2 + |C|^2}},
\]

\( S \) is the ratio of the minimum to the maximum of the amplitude modulation,
\( T \) is the period of the modulation [s],
\( f_{rev} \) is the revolution frequency [s\(^{-1}\)] (Section 3.3).

The above formulae provide a simple method for measuring \( |C| \) and \( |\Delta| \) (see also Section 9.5).

For the ISR: \( f_{rev} \) is given in Table 2.

9.4 Time shift of the coherent oscillations of a kicked beam

When kicking a pulse at an angle \( d\theta/\theta_x = \sqrt{\beta_x/\beta_z} \) while being close to the resonance \( Q_x - Q_z = p \), the amplitude modulation observed in the horizontal or vertical direction is shifted, compared with the modulation following a simple horizontal or vertical kick.

\[
\delta_t = \frac{T}{2\pi} \arctg \sqrt{\frac{\Delta^2 + |C|^2}{\Delta \, (\text{Im.C})}}
\]
(Im.C) and (Re.C) are the imaginary and real parts of C (Section 9.1),

T is given in Section 9.3,

δ_t is the time shift of the amplitude modulation [s].

9.5 Measurement of the coupling coefficient

By combining the formulae of Sections 9.3 and 9.4, |Δ|, Re.C, Im.C and |C| can be expressed in terms of measurable quantities

\[
|Δ| = \frac{S}{T f_{\text{rev}}} \quad \text{kicking in a transverse plane,}
\]

\[
|C| = \frac{1}{T f_{\text{rev}}} \sqrt{1 - S^2}
\]

\[
\frac{\text{Im.C}}{\text{Re.C}} = \frac{SΔ}{|Δ|} \tan \left(2\pi \frac{δ_t}{T}\right) \quad \text{kicking at an angle } \frac{dz}{dx} = \sqrt{β_x/β_z}.
\]

S, T and f_{rev} are defined in Section 9.3 and δ_t is defined in Section 9.4

9.6 Changes in beam dimensions due to linear coupling

Averaging on time the modulation of the amplitude of the betatron oscillations due to linear coupling with n = -1 (Section 9.1) gives the apparent beam sizes. Without shaving, the sizes are\(^{12}\)

\[
\frac{\sigma_x}{\sigma_{x,0}} = \sqrt{\frac{β_x}{β_{x,0}}} \sqrt{1 + \frac{1}{2} \left[ \left(\frac{\sigma_{x,0}}{\sigma_{x,0}}\right)^2 - 1 \right] \frac{|C|^2}{A^2 + |C|^2}},
\]

\[
\frac{\sigma_z}{\sigma_{z,0}} = \sqrt{\frac{β_z}{β_{z,0}}} \sqrt{1 + \frac{1}{2} \left[ \left(\frac{\sigma_{z,0}}{\sigma_{z,0}}\right)^2 - 1 \right] \frac{|C|^2}{A^2 + |C|^2}},
\]

σ_x and σ_z are the r.m.s. values of the horizontal and vertical distributions of the amplitudes, respectively, at the azimuthal position θ [m],

β_x and β_z are the betatron amplitude functions at θ [m],

σ_{x,0} and σ_{z,0} are the initial values of σ_x and σ_z at the position θ_0 [m],

β_{x,0} and β_{z,0} are the initial values of β_x and β_z at θ_0 [m],

C is explicitly given in Section 9.1.

Considering a vertical shaving of the beam, the remaining particles are initially in a 4-dimensional phase space volume, which is given by

\[
v_o^2 + v_o'^2 + (A + \sqrt{A^2 + B^2}) \leq \frac{z_s^2}{β_z s},
\]

where:

\[
A = \frac{|C|^2}{2\eta_C}(u_o^2 + u_o'^2 - v_o^2 - v_o'^2) - \frac{(\text{Re.C})A}{\eta_C}(u_o v_o + u_o' v_o') + \frac{(\text{Im.C})A}{\eta_C}(u_o v_o - u_o' v_o'),
\]

\[
B = - \frac{(\text{Im.C})}{\eta_C}(u_o v_o + u_o' v_o') - \frac{(\text{Re.C})}{\eta_C}(u_o v_o' - u_o' v_o'),
\]
with

\[ u_o = \frac{x_o}{\sqrt{\beta_{x,o}}} \]

\[ \gamma_o = \frac{\gamma_o}{\sqrt{\beta_{z,o}}} \]

\[ u_o^* = \frac{\beta_{x,o} x_o' + \alpha_{x,o} x_o}{\sqrt{\beta_{x,o}}} \]

\[ \gamma_o^* = \frac{\beta_{z,o} z_o' + \alpha_{z,o} z_o}{\sqrt{\beta_{z,o}}} \]

\[ \eta_c = \sqrt{\Delta^2 + |C|^2} \]

(Re.C) and (Im.C) are the real and imaginary parts of C (Section 9.1),
\( x_o, x_o', z_o \) and \( z_o' \) are the initial coordinates of the particles,
\( z_{s} \) is the position of the scraper blade [m],
\( \beta_{z,s} \) is the betatron amplitude function at the scraper [m],
\( \alpha_{x,o} \) and \( \alpha_{z,o} \) are the initial values of \( \alpha_x \) and \( \alpha_z \) at \( \theta_o \).

Averaging on time the amplitude of the oscillations inside the phase-space volume defined above gives the change of the vertical beam size as a function of the shaver position or of the fraction of particles left. Typical curves giving the remaining fraction of a shaved beam \( F_{s,z} \) as a function of the ratio of the final to the initial beam height \( \sigma_z/\sigma_{z,o} \) are represented in Figs. 17 and 18 for different emittance ratios and different values of F (Section 9.2). These results can be used for horizontally shaved beams, provided the corresponding variables are used.

10. FORMULAE IN CONNECTION WITH SUM AND DIFFERENCE RESONANCES\(^{5,30,33,34,35,36}\)

Symbols frequently used in this chapter

\( \alpha_y \)  
Twiss parameter (Section 1.1)

\( \gamma_y \)  
phase of betatron oscillation (Section 1.1)

\( E_{yo} \)  
ninitial transverse emittances associated with one particle

and assumed different from zero [m rad]

\( n_x, n_z \)  
integers defining the resonance of interest

\( N \)  
order of the resonance: \( N = |n_x| + |n_z| \)

\( \delta_y = \sqrt{\frac{E_y}{E_{yo}}} \)  
amplitude growth in a resonance

\( e \)  
distance from the resonance

\( B_0 \)  
magnetic rigidity [Tm] (Section 2.1)
Fig. 17  Vertical blow-up due to linear coupling for different rates of vertical shaving and low emittance ratios
Fig. 18 Vertical blow-up due to linear coupling for different rates of vertical shaving and high emittance ratios
10.1 Resonances in a coasting beam due to the magnetic fields of the machine magnets

The presence of a three-dimensional magnetic field can provoke instabilities and resonances in the transverse motions of the particles in a circulating coasting beam. The condition that a resonance may occur in the two-dimensional oscillations of the particles is

\[ e = n_x Q_x + n_z Q_z - p = 0. \]

The excitation terms, which characterize the perturbation, have the following forms in the case of a three-dimensional magnetic field verifying \( \mathbf{\nabla} \times \mathbf{B} = 0 \):

\[
\begin{align*}
\frac{d}{d\phi} &= \frac{1}{2\pi} \int_0^{2\pi} R^2 \left( |n_x|/2 \right) \beta_x(|n_x|/2) \frac{B_0}{R^2} \left[ (-1)^{(|n_x|+2)/2} \left( \frac{n_x}{\beta_x} \right) \right] \\
&+ \frac{RF(N-1)}{X} \left( |n_x| \frac{\alpha_x}{\beta_x} - |n_z| \frac{\alpha_z}{\beta_z} \right) - \frac{iRF(N-1)}{X} \left( \frac{n_x}{\beta_x} - \frac{n_z}{\beta_z} \right) \\
&\exp \left\{ i \left[ n_x \frac{\alpha_x}{\beta_x} + n_z \frac{\alpha_z}{\beta_z} - (n_x Q_x + n_z Q_z - p) \theta \right] \right\} d\theta,
\end{align*}
\]

the indices \( z \) and \( x \) of \( K_x^{(N-1)} \) and \( F_x^{(N-1)} \) being associated with \( d_p \) and \( f_p \), respectively, with, for \( N = 2 \):

\[
\begin{align*}
k_x^{(1)} &= \frac{R^2}{2B_0} \left( \frac{\partial B_x}{\partial x} - \frac{\partial B_z}{\partial z} \right)_{x=z=0} \\
k_z^{(1)} &= \frac{R^2}{2B_0} \beta_z (x=z=0) \\
F_x^{(1)} &= 0
\end{align*}
\]

and, for \( N \gg 3 \):

\[
\begin{align*}
k_x^{(N-1)} &= \begin{cases} \frac{(-1)(k_x+2)/2}{(k_x+1)! (N-k_z-2)!} \left( \frac{k_x+1}{N-1} \right) \left| \frac{\beta_z (N-k_z-2) - \beta_z (k_x+1)}{\alpha_x (N-k_z-2) - \alpha_x (N-k_x-2)} \right|_{x=z=0} \\
&+ \frac{\beta_z (k_x+1)}{\alpha_x (N-k_x-2)} \left( \frac{\partial B_z}{\partial z} \right)_{x=z=0} \\
k_z^{(N-1)} &= \begin{cases} (-1)(k_z+1)/2 \end{cases} \left( \frac{k_z+1}{N-1} \right) \left| \frac{\beta_x (N-k_x-2) - \beta_x (k_z+1)}{\alpha_z (N-k_x-2) - \alpha_z (N-k_z-2)} \right|_{x=z=0} \\
&+ \frac{\beta_x (k_z+1)}{\alpha_z (N-k_z-2)} \left( \frac{\partial B_x}{\partial x} \right)_{x=z=0}
\end{cases}
\end{align*}
\]

where the integer \( k_z \) is free within the limits given by \( 0 \leq k_z \leq (N-2) \).
For a two-dimensional magnetic field, the parameters $K$ and $F$ simply become:

$$
K^{(N-1)}_x = F^{(N-1)}_z = 0,
$$

$$
K^{(N-1)}_x = \frac{R^2}{B_0} \frac{\partial^{(N-1)} B_x}{\partial x^{(N-1)}},
$$

$$
K^{(N-1)}_z = \frac{R^2}{B_0} \frac{\partial^{(N-1)} B_z}{\partial x^{(N-1)}}.
$$

The magnetic resonances can be damped by some zero harmonics of the field. These damping terms associated with the field of the machine magnets are

$$
k_{qs}^{(2\nu)} = \frac{1}{2\pi (2\nu)! (q! s!)^2} \int_0^{2\pi} \beta_x^q \beta_z^s \left[ (-1)^{s+1} k_z^{(2\nu-1)} f_z^{(2\nu-1)} \right] \frac{\partial}{\partial \theta} \left( \frac{\alpha_x}{\beta_x^s} - s \frac{\alpha_z}{\beta_z^s} \right) d\theta,
$$

with $\nu = q + s$; $\nu$, $q$, $s$ are integers; $\nu \leq N/2$ ($N/2$ is the smallest integer $\geq N/2$).

$k_z^{(2\nu-1)}$ and $f_z^{(2\nu-1)}$ are defined as above using $2\nu$ instead of $N$.

**10.2 Betatron-synchrotron resonances due to the magnetic fields of the machine magnets**

The presence of a magnetic field can provoke instabilities and resonances in the transverse motions of the particles in a bunched beam. These resonances are called betatron-synchrotron resonances, since the synchrotron oscillations of the particles inside the bunches are coupled with the betatron oscillations to induce transverse instabilities. In this mechanism, the synchrotron motion manifests itself by an oscillation of the momentum of nonsynchronous particles given in Section 3.2. Analysing this oscillation in harmonics of the revolution frequency, it is possible to write

$$
\frac{\Delta p}{p} = \frac{p - p_o}{p} = \sum_{k=0}^{\infty} u_k \exp \left( i k Q_s \theta \right),
$$

$k$ is an integer,

$p$ and $p_o$ are the momentum of the nonsynchronous and synchronous particles, respectively [eV/c],

$Q_s$ is the number of synchrotron oscillations per revolution. $Q_s = f_p / f_{rev}$, these frequencies being defined in Section 3.3.

The horizontal position $x_p$ of an off-momentum particle is related to $\alpha_{px}$ (Section 1.1), but this is no longer an oscillation of period $2\pi$ in $\theta$. Using the form of $\Delta p/p$ given above, the position $x_p$ for a nonsynchronous particle sitting in a bunch can be written as follows:

$$
x_p = \sum_k \frac{\beta_x \beta_x^s}{R} \frac{u_k \delta \xi}{\rho_o \delta_x} \exp \left[ i \left( \mu_k - \Delta_k \xi \right) \right].
$$
where:

\[ c_b = \frac{1}{2\pi} \int_{b}^{2\pi} e^{\frac{\beta x}{2R}} \exp \left\{ i[(Q_x - b)\theta - \nu_x] \right\} d\theta, \]

\[ u_k = \frac{Q_s}{2\pi} \int_{0}^{2\pi/Q_s} \frac{\Delta p}{p} \exp (-i k Q_s \theta) d\theta, \]

\[ \Delta = Q_x - b - k Q_s. \]

The index \( \ell \) (integer) has been introduced for numbering the two integers \( k \) and \( b \), in order to replace the double summation on \( k \) and \( b \) by a single summation on \( \ell \). With every \( \ell \), a pair of values \((b_\ell, k_\ell)\) is associated. \( \Delta_\ell \) means that \( \Delta \) is taken for this pair of values, while \( c_\ell \) and \( u_\ell \) mean that \( c_b \) and \( u_k \) are taken for the values \( b_\ell \) and \( k_\ell \) of the pair, respectively.

\( \rho_o \) is the radius of curvature of the synchronous particle \([m]\).

This expression of \( x_p \) obviously shows the existence of dipole resonances in the absence of perturbing fields. These resonances are given by

\[ \Delta = Q_x - b - k Q_s = 0, \]

so that \( x_p \) goes to infinity.

Considering now two-dimensional perturbing fields of the machine magnets, the condition that a resonance may occur in the transverse oscillations of the particles will be

\[ e = n Q_x + n Q_z - p - m Q_s - \sum_{\ell} (b_{\ell} + k_{\ell} Q_s) j_{\ell} = 0, \]

\( p, m, j_\ell \) being also integers and \( \sum_{\ell} j_\ell = n - N + 1 \), with \( n \geq N-1 \).

In this case, the excitation terms characterizing the perturbation are

\[ \begin{align*}
\left. \frac{d}{p} \right|_{f_{p}} &= \frac{(n-N+1)!(N+1)!}{n!} \left( \frac{\beta x}{R} \right)^{n-N+1} (-u_m)^{\frac{1}{2}} \prod_{\ell} \left( \frac{u_{\ell} c_{\ell}}{\Delta_{\ell}} \right)^{1/2} \int_{0}^{2\pi} \frac{\Delta p}{p} \frac{\beta_{x}}{\beta_{z}} \exp \left\{ i[(n_x + n - N + 1)(\nu_x - Q_x \theta) + n_z(\nu_z - Q_z \theta) + p\theta] \right\},
\end{align*} \]

the indices \( z \) and \( x \) of \( B \) are associated with \( d_{p} \) and \( f_{p} \), respectively, the symbol \( \prod_{\ell} \) indicates a product of terms numbered by the index \( \ell \),

\( n \) is any integer \( \geq N-1 \),

\( j_\ell \) are integers verifying \( \sum_{\ell} j_\ell = n-N+1 \),

\( p, m \) are any integers.

The coefficients \( \Delta_\ell, u_{\ell} k_\ell \) and \( c_{\ell} \) are those appearing in the expression for \( x_p \) given above. The coefficient \( u_m \) is any harmonics amplitude of \( \Delta p/p \).
The two products appearing in this expression should contain exactly the terms appearing in $x_p$, and no others. Nevertheless, only the dominant terms corresponding to small $\Delta_k$ can be considered for $x_p$, $d_p$ and $f_p$ in order to limit the number of terms in a practical case.

The resonances can be damped as mentioned in Section 10.1. The damping terms are

$$h_{q^{(2\nu)}} = \frac{R}{B_{p_{0}(2R)i(q!)}^2} \frac{(2\nu-1)!(n-2\nu+1)!}{(n!)^2} \left( \frac{R}{Z} \frac{R}{p_{0}} \right)^{n-2\nu+1} \left( \frac{u_m}{\Delta_k} \right)^{j_k} \left( \frac{\eta_k c_{b_k}}{\Delta_k} \right)^{j_k} \left( \frac{\eta_k c_{b_k}}{\Delta_k} \right)^{j_k} \int_0^{2\pi} d\theta \frac{\beta_{x}}{\Delta_k^n} \frac{\beta_{z}}{\Delta_k^n} \beta_{z}^s \exp \left\{ i \left[ (n - 2\nu + 1)(\mu_x - Q_x \theta) + p\theta \right] \right\},$$

with $\nu = q + s$; $\nu$, $q$, $s$ are integers,

$n$ is any integer $\geq 2\nu - 1$,

$j_k$ are integers verifying $\sum_k j_k = n - 2\nu + 1$,

$m$, $p$ are integers verifying

$$p + m Q_S = -\sum_k (b_k + k Q_S) j_k.$$

### 10.3 Resonances due to coasting beams crossing each other

When two coasting beams are crossing at a small angle $\psi$, a particle of the beam 1 sees the electric and magnetic fields of beam 2. As for the case mentioned in Section 10.1, these fields can provoke instabilities and resonances in the transverse motions of the particles of beam 1. The condition that such a resonance may occur in beam 1 is again (Section 10.1)

$$e = n_x Q_{x_1} + n_z Q_{z_1} - p = 0.$$

The index 1 means that the tunes are those of beam 1 particles.

The excitation terms have the following forms using the coordinate system of Fig. 19:

$$d_p = \frac{1}{2\pi} \frac{1}{\beta_1} \int_0^{2\pi} \beta_{x_1} \beta_{z_1} \beta_{z_1} \exp \left\{ i \left[ n_x u_{x_1} + n_z u_{z_1} - (n_x Q_{x_1} + n_z Q_{z_1} - p) \theta_1 \right] \right\} d\theta_1.$$

$V$ is a force potential which does not necessarily obey Laplace's equation $[V]$.

The index 1 in the coordinates $x$, $z$, $\theta$ and in the parameter $\beta$ means that the functions such as $\beta_x$, $V$, $u_x$, and $Q_x$ and the longitudinal speed have to be taken in the coordinate system of the beam 1 particles (Fig. 19).

Since the fields are due to the particles of beam 2, the potential $V$ is given by the following relations:
Fig. 19 Coordinate systems used at the crossing point of two beams

\[
\left\{ \begin{array}{l}
\frac{\partial^2}{\partial x_1^2} \left| \begin{array}{c}
\beta_2 \left( \frac{a^{(N-1)}}{|n_x|} \right) \frac{E_{x_1}}{\partial x_1} \\
\beta_2 \left( \frac{a^{(N-1)}}{|n_z|} \right) \frac{E_{z_1}}{\partial z_1}
\end{array} \right| \geq 1 \\
(1 + \beta_1 \beta_2) \left( \frac{a^{(N-1)}}{|n_x|} \left( \frac{a^{(N-1)}}{|n_z|} \right) \right) \geq 1
\end{array} \right.
\]

\[\beta_2\] is the relative longitudinal speed of beam 2.

The electric field components \( [V \ m^{-1}] \) in the coordinate system of beam 2 (Fig. 19) for Gaussian transverse distributions of particles are given by

\[
E_{y_2} = \frac{I_2}{\pi \epsilon_0 \beta_2 c} \int_0^\infty y_2 \exp \left[ -\left( \frac{2x_2^2}{a_2^2 + t} + \frac{2z_2^2}{b_2^2 + t} \right) \right] \frac{dt}{(a_2^2 + t) \sqrt{(a_2^2 + t)(b_2^2 + t)}}
\]

where index 2 means that all the functions are taken for beam 2,

\( \epsilon_0 \) is the free-space permittivity \([As \ \ V^{-1} \ m^{-1}]\) (Table 1),

\( a_2 \) is the half-beam 2 width (twice the r.m.s. of the radial dimension) \([m]\),

\( b_2 \) is the half-beam 2 height (twice the r.m.s. of the vertical dimensions) \([m]\).

In general, \( a_2 \) and \( b_2 \) are quadratically varying with \( \theta_2 \).

The passage from the field components \( E_{y_2} \) to the components \( E_{y_1} \) depends on the colliding angle \( \psi \), since the coordinates transformation from beam 2 to beam 1 only depends on this angle (Fig. 19).

For beam-beam resonances, the damping terms mentioned in Section 10.1 are

\[
h^{(2)}_{qs} = \frac{1}{2\pi \ eta_1 c (2R)^{(q!s!)^2}} \int_0^{2\pi} g_x q_1 g_x s_1 \ r^2 \ \
\rho \left( \frac{2}{a_1 (2q)} \ d\theta_1 \right) \frac{2}{a_2 (2s)} \ d\theta_1,
\]
\( \nu, q, s \) are integers satisfying \( \nu = q + s \),
\( V \) is the potential given above \([V]\).

In actual fact, the integrals for \( d_p, f_p \) and \( f_{qs}^{(2\nu)} \) are different from zero only in the interval \( (\theta_1 - d/2R, \theta_1 + d/2R) \), if \( d \) is the length \([m]\) of the crossing region in which the two beams interact and \( \theta_1 \) is the azimuth of the considered crossing point.

10.4 Resonances due to overlap knock-out

The interaction between a coasting beam (beam 1) and a bunched beam (beam 2) can provoke instabilities and resonances in the transverse motions of the particles. The condition for a resonance to occur in the coasting beam is

\[
e = n_1 Q_{x_1} + n_2 Q_{z_1} - p - q \frac{f_{rev_2}}{f_{rev_1}} = 0.
\]

The indices 1 and 2 refer to the coasting and bunched beams, respectively.

\( q \) is the harmonic of the bunch frequency exciting the resonance (Section 3.15),
\( f_{rev_2} \) is the revolution frequency of the bunches \([s^{-1}]\) (Section 3.3),
\( f_{rev_1} \) is the revolution frequency of the particle at resonance \([s^{-1}]\) (Section 3.3).

The excitation terms characterizing the perturbation have the following forms in the case of the fields induced by the bunches:

\[
d_p f_p = \frac{1}{2\pi \beta_1} \int_0^{2\pi} \beta_{x_1} \beta_{z_1} \frac{\partial V}{\partial x_1} \frac{\partial V}{\partial z_1} d\theta_1 \exp \left\{ i \left[ n_1 Q_{x_1} + n_2 Q_{z_1} - (n_1 Q_{x_1} + n_2 Q_{z_1} - p) \theta_1 + q \theta_2 \right] \right\}
\]

\[
= \frac{1}{2\pi} \int_0^{2\pi} I_{\theta_2} \exp (i q \theta) d\theta
\]

\( \theta_1 \) is the azimuthal angle of the particle at resonance \([rad]\),
\( \theta_2 \) is the angular position taken in the bunch frame and corresponding to the particle position \( \theta_1 \) \([rad]\) \((\text{Fig. 20})\),
\( \theta \) is the azimuthal variable of integration taken in the bunch frame \([rad]\),
\( \beta_1 \) is the relativistic parameter of the coasting beam,
\( I_{\theta_2} \) is the azimuthal current distribution inside one bunch of beam 2 \([A]\) \((\text{Section 3.10})\),
\( I_2 \) is the peak current inside a bunch of beam 2 \([A]\) \((\text{Section 3.10})\).

The last integral on \( \theta \) gives exactly the coefficients \( c_q \) defined in Section 3.13.

\( V \) is the potential \([V]\) of the electromagnetic forces due to the peak current of the bunched beam and \( V \) does not obey Laplace's equation. Looking at these excitation forces, two different cases of overlap knock-out can be considered:

i) Single-beam overlap knock-out:

The bunched beam and the coasting beam are two parts of the same beam \((\text{Fig. 20})\). Then, the potential \( V \) comes from the electromagnetic forces of the bunched beam and also from the coupling forces induced by the bunches via the material close to the beam.
Angle definition for two-beam overlap knock-out, depending on the directions of propagation

Angle definition for single beam overlap knock-out

Fig. 20 Definition of the azimuthal variables used in Section 10.4

ii) Two-beam overlap knock-out:

The bunched beam is different from the coasting beam and crosses it at a finite number of intersections (Fig. 20). In this case, the effect of the surrounding material is less important and the potential $V$ is essentially due to the direct electromagnetic forces.

For the direct forces, the potential $V$ is given by the following relations:
\[
\frac{\sum_{N} V}{n_{x} n_{z}} = \left\{ \begin{array}{l}
(1 + \text{sign } \beta_{1} \beta_{2}) \frac{a^{(N-1)} E_{x_{1}}}{n_{x} n_{z}} \left| n_{x} \right| \geq 1 \\
(1 + \text{sign } \beta_{1} \beta_{2}) \frac{a^{(N-1)} E_{z_{1}}}{n_{x} n_{z}} \left| n_{z} \right| \geq 1
\end{array} \right.
\]

where:

- $E_{x_{1}}$ and $E_{z_{1}}$ are the electric field components in the coordinate system of beam 1.
- The corresponding components $E_{x_{2}}$ in the coordinate system of beam 2 are given in Section 10.3 for a beam of elliptic cross-section,
- $\beta_{1}$ is the relativistic parameter of the coasting beam,
- $\beta_{2}$ is the relativistic parameter of the bunched beam,
- sign is +1 if the two beams go in opposed directions and is -1 in the other case.

10.5 Bandwidth for a sum resonance

Given a single resonance $\varepsilon = 0$ (Sections 10.1 to 10.4) and a point $(Q_{x}, Q_{z})$ in the tune diagram, the distance of this point from the resonance is defined as the value of $\varepsilon$ with these tunes. Knowing the limits $\varepsilon_{1}$ and $\varepsilon_{2}$ on each side of the resonance beyond which the motion becomes stable, the bandwidth is defined as $\Delta \varepsilon = |\varepsilon_{1}| + |\varepsilon_{2}|$ (Fig. 21)

\[
\Delta \varepsilon = \frac{R}{2(N-2)} \frac{R_{p}}{|n_{x}|! |n_{z}|!} \left( \frac{n_{x}^{2} E_{x_{0}}}{n_{z}^{2} E_{z_{0}}} \right)^{2} \left( \frac{n_{x}^{2} E_{x_{0}}}{n_{z}^{2} E_{z_{0}}} \right)^{-2}\left( \frac{n_{x}^{2} E_{x_{0}}}{n_{z}^{2} E_{z_{0}}} \right)^{1} \left( \frac{n_{x}^{2} E_{x_{0}}}{n_{z}^{2} E_{z_{0}}} \right)^{-1}
\]

for $n_{z}$ even

\[
\Delta \varepsilon = \frac{R}{2(N-2)} \frac{R_{p}}{|n_{x}|! |n_{z}|!} \left( \frac{n_{y}^{2} E_{y_{0}}}{n_{z}^{2} E_{z_{0}}} \right)^{2} \left( \frac{n_{y}^{2} E_{y_{0}}}{n_{z}^{2} E_{z_{0}}} \right)^{-2}\left( \frac{n_{y}^{2} E_{y_{0}}}{n_{z}^{2} E_{z_{0}}} \right)^{1} \left( \frac{n_{y}^{2} E_{y_{0}}}{n_{z}^{2} E_{z_{0}}} \right)^{-1}
\]

for $n_{z}$ odd

where the excitation terms $d_{p}$ and $f_{p}$ are given in Sections 10.1 to 10.4 depending on the type of instability considered.

10.6 Trapping condition for the particles crossing a sum resonance

When the tunes, the synchrotron frequency and/or the revolution frequencies are changing with the time in such a way that the particles are crossing a sum resonance, these particles can remain trapped in the resonance if the rate of change of the distance from the resonance is sufficiently small. The condition for the trapping to occur is given by:

\[
\frac{d\varepsilon}{dt} \leq \frac{\Delta \varepsilon}{\varepsilon} \frac{1}{n_{x}^{2} E_{x_{0}} + n_{z}^{2} E_{z_{0}}} \sum_{k=2}^{N/2} \varepsilon(k-1) K_{x,k} \quad n_{x} \neq 0
\]

\[
\frac{d\varepsilon}{dt} \leq \frac{\Delta \varepsilon}{\varepsilon} \frac{1}{n_{x}^{2} E_{x_{0}} + n_{z}^{2} E_{z_{0}}} \sum_{k=2}^{N/2} \varepsilon(k-1) K_{z,k} \quad n_{z} \neq 0
\]

$\Delta \varepsilon$ is the bandwidth given in Section 10.5,
$N/2$ is the smallest integer larger or equal to $N/2$. 
\[ e = n_x Q_x + n_z Q_z - p \]

\[ h_{qs}^{(2\nu)} \text{ are zero} \]
\[ n_x \text{ and } n_z \text{ have same sign} \]

\[ g_y \geq 1 \]

Fig. 21 Resonance curves for sum resonances
The coefficients $K_{k,1}$ and $K_{k,2}$ are given by

$$K_{k,1} = E_{x_0}^{(k-1)} \left( \sum_{v=0}^{N/2} A_{x,v}^{(v-k)} \sum_{p=0}^{(v-k+1)} \frac{(v-k+p)(v-k+p-1)\ldots(v-k+1)}{p!} \left( \frac{n_z}{n_x} \right)^p h_{x,v}^{(2v)} \right)$$

$$K_{k,2} = E_{z_0}^{(k-1)} \left( \sum_{v=0}^{N/2} A_{z,v}^{(v-k)} \sum_{p=0}^{(v-k+1)} \frac{(v-k+p)(v-k+p-1)\ldots(v-k+1)}{p!} \left( \frac{n_x}{n_z} \right)^p h_{z,v}^{(2v)} \right)$$

in which:

$$A_{x} = \frac{R}{n_x} (n_x E_{x_0} - n_z E_{x_0})$$

$$A_{z} = \frac{R}{n_z} (n_z E_{z_0} - n_x E_{z_0})$$

and the coefficients $h_{x,v}^{(2v)}$ are given in Sections 10.1 to 10.4 depending on the type of instability considered.

The form of the rate of change $\frac{d\theta}{dt}$ appearing on the left hand side of the trapping condition depends on the resonances considered:

i) **Resonances in a coasting beam** due to the fields of the machine magnets or of an other coasting beam (Sections 10.1 and 10.3)

$$\frac{d\theta}{dt} = n_x \frac{d\theta_i}{dt} + n_z \frac{d\theta_j}{dt} - \frac{p}{f_{rev}} \frac{df_{rev}}{dt}$$

In the case given in Section 10.3, the same expression applies, but $f_{rev}$, $Q_x$ and $Q_z$ represent the revolution frequency and the tunes of a particle in beam 1.

ii) **Betatron-synchrotron resonances** (Section 10.2)

$$\frac{d\theta}{dt} = n_x \frac{d\theta_i}{dt} + n_z \frac{d\theta_j}{dt} - \frac{(p+\sum_k b_{k,k})}{f_{rev}} \frac{df_{rev}}{dt} - \left( m + \sum_k b_{k,k} \right) \frac{d\theta_S}{dt},$$

the definitions being given in Section 10.2.

iii) **Resonances due to overlap knock-out** (Section 10.4)

$$\frac{d\theta}{dt} = n_x \frac{d\theta_{i1}}{dt} + n_z \frac{d\theta_{j1}}{dt} - \frac{p}{f_{rev_1}} \frac{df_{rev_1}}{dt} - \frac{q}{f_{rev_2}} \frac{df_{rev_2}}{dt},$$

the definitions being given in Section 10.4.

10.7 **Maximum growth of amplitude for particles crossing a sum resonance**

When the particles cross a sum resonance with a rate of change of $\theta$ which is sufficient to avoid being trapped (see Section 10.6), there is still an amplitude growth. The maximum possible growth of the amplitude which may occur can be calculated for a single crossing as follows:
\[
\left(\frac{n_x E_{x0}}{n_z E_{z0}}\right)^{n_z/2} \int \frac{d g_x}{\left(\frac{n_x E_{x0}}{n_z E_{z0}} - 1 + g_x^2\right)^{n_x/2}} = \frac{\Delta e}{n_x E_{x0} + n_z E_{z0}} \frac{\sqrt{n_{bc}}}{2 R \sqrt{\left|\frac{de}{dt}\right|}} \text{ for } n_x \neq 0 \text{ only},
\]

\(\Delta e\) is the bandwidth defined in Section 10.5.

The relation for \(g_x\), valid if \(n_z \neq 0\) only, is simply obtained by exchanging the indices \(x\) and \(z\) in the preceding relation.

If the particle is crossing the resonance \(n_x\) times, the average growth of amplitude will be \(\sqrt{n_x/2}\) times larger than for a single crossing.

The rate of change of \(e\) is given in Section 10.6 for different types of resonances.

For monodimensional resonances \((n_x = N, n_z = 0)\) or \((n_x = 0, n_z = N)\), this relation becomes

\[
N = 2 \quad \Delta n \frac{g_y}{1 - g_y^{(2-N)}} = \frac{\Delta e}{N} \frac{\sqrt{n_{bc}}}{\sqrt{2R} \sqrt{\left|\frac{de}{dt}\right|}}.
\]

For bidimensional resonances \((n_x \neq 0, n_z \neq 0)\) of order \(N \leq 5\), the analytic form of the above integral is given in the Appendix.

10.8 Criterion concerning the distance of the working point from a sum resonance line

The perturbation of the oscillation amplitude of a single particle in a sum resonance decreases as the distance from the resonance increases. Assuming that the maximum tolerable blow-up for a single particle associated with the initial emittances \(E_{x0}\) and \(E_{z0}\) is \(A\) (Fig. 21), then the distance \(\delta e\) (Fig. 21) of the working point from the resonance must be in agreement with

\[
\delta e \geq \frac{\Delta e}{2} \left(1 + \frac{1}{A} \frac{|n_x|E_{z0} + |n_z|E_{x0}}{n_x E_{x0} + n_z E_{z0}}\right),
\]

\(\Delta e\) is the bandwidth defined in Section 10.5.

10.9 Bandwidth for a difference resonance

As in Section 10.5, \(e\) represents the distance from the resonance. Knowing the limits \(e_{1,1}\) and \(e_{2,1}\) on each side of the resonance between which the amplitude can become zero and the limits \(e_{1,2}\) and \(e_{2,2}\) on each side of the resonance between which the amplitude can become maximum, the bandwidth is defined as \(\Delta e = \frac{1}{2} (|e_{1,1}| + |e_{2,1}| + |e_{1,2}| + |e_{2,2}|)\) (Fig. 22)

\[
\Delta e = \frac{R}{2^{(N-2)}} \frac{B}{|n_x||n_z|} \left(\frac{|n_x|-2}{2} E_{x0} \frac{|n_z|-2}{2} E_{z0} \left(\frac{|n_x|E_{z0} + |n_z|E_{x0}}{E_{x0} + E_{z0}}\right) \left|\frac{dp}{d\epsilon}\right| \text{ for } n_z \text{ even}
\]

\[
\text{or } n_x \geq 0, n_z \leq 0 \quad \text{or} \quad n_x \leq 0, n_z \geq 0.
\]
\[ e = n_x Q_x + n_z Q_z - p \]

\[ h_{0s} \text{ are zero} \]

\[ n_x \text{ and } n_z \text{ have opposed signs} \]

\[ \delta e \]

\[ e_{1,1} \]

\[ e_{1,2} \]

\[ e_{2,1} \]

\[ e_{2,2} \]

\[ 0 \leq g_y \leq \delta_y \]

Fig. 22  Resonance curves for difference resonances
The parameters $d_p$ and $f_p$ are given in Sections 10.1 to 10.4, depending on the type of instability considered.

### 10.10 Maximum amplitude for particles on a difference resonance

Such a resonance is always stable and a particle remaining inside the band defined in Section 10.9 can oscillate between two limits (Fig. 22) given by

$$0 \leq g_x \leq \sqrt{1 + \frac{n_x}{n_z} \frac{E_{x_0}}{E_{x_0}}},$$

$$0 \leq g_z \leq \sqrt{1 + \frac{n_z}{n_x} \frac{E_{z_0}}{E_{z_0}}}.$$

### 10.11 Criterion concerning the distance of the working point from a difference resonance line

Since a difference resonance is stable, the amplitude of the single particle motion oscillates between two limits, but with increasing distance from the resonance these limits converge. Assuming that the maximum tolerable beating of the amplitude for a single particle associated with the initial emittances $E_{x_0}$ and $E_{z_0}$ is $\Lambda$ (Fig. 22), then the distance $\delta e$ (Fig. 22) of the working point from the resonance must be in agreement with

$$\delta e \geq \frac{1}{2} \frac{\Delta e}{\Lambda}.$$

$\Delta e$ is the bandwidth defined in Section 10.9.

### 10.12 Monodimensional resonances for coasting beams crossing at large angle

This is a particular case of the resonances mentioned in Section 10.3. The crossing angle can be either in the horizontal or in the vertical plane and it is assumed to be small as in Section 10.3, but large enough that only the monodimensional resonances $e = N y_1 - p = 0$ are excited in the perturbed beam.

It is assumed in what follows that the perturbing beam 2 has a transverse, uniform distribution in the colliding plane and a Gaussian distribution in the direction perpendicular to the colliding plane. Furthermore, the distance $y_1$ of the excited particle in beam 1 w.r.t. the beam 2 centre is taken as small. Under these conditions, the bandwidths of these monodimensional resonances become, using the formulae of Sections 10.3, 10.5 and 8.9

$$\Delta e = \frac{N}{2(N/2-2)} \frac{N}{(N-1)(N/2-1)!} y_1 \sigma_{y_1}^{bb} \quad \text{for N even},$$

$$\Delta e = \frac{N}{2(N-1)/2} \frac{N}{(N-1)!} y_1 \sigma_{y_2}^{bb} \quad \text{for N odd, } \frac{y_1}{\sigma_{y_2}} \ll 1,$$

$y_1$ is the distance of the particle from the beam 2 centre in the direction perpendicular to the colliding plane [m],

$\sigma_{y_2}$ is the r.m.s. value of the beam 2 size in the direction perpendicular to the colliding plane [m],

$\Delta e^{bb}$ is the linear beam-beam tune shift given in Section 8.9, in the direction perpendicular to the colliding plane.
Using the formula in Section 10.7 for monodimensional resonances, the transverse increase of the initial amplitude of particle 1 is given by the following equation when crossing the resonance by changing the tune and/or the revolution frequency:

\[
\frac{\Delta \sigma_{Y_1}}{\sigma_{Y_1}} = \frac{\sqrt{n \beta_1 c}}{\sqrt{2 R N}} \frac{\Delta \epsilon}{\int_{N}^{\frac{dQ_{Y_1}}{dt} - \frac{p}{f_{rev_1}} \frac{df_{rev_1}}{dt}}}
\]

\(\sigma_{Y_1}\) is the initial r.m.s. value of the transverse amplitude of particle 1 (Section 1.6) in the direction perpendicular to the colliding plane \([m]\),
\(\beta_1\) is the relative longitudinal speed of particle 1,
\(Q_{Y_1}\) is the particle tune in the direction perpendicular to the colliding plane,
\(f_{rev_1}\) is the revolution frequency of particle 1 \([s^{-1}]\),
\(\Delta \epsilon\) is the bandwidth given above.

Even if the mathematical definition used for the bandwidth does not apply to the dipole resonances, the final result concerning the transverse blow-up is still valid, using a virtual bandwidth \(\Delta \epsilon\) with \(N = 1\)

\[
\frac{\Delta Y_1}{Y_1} = \frac{\sqrt{n \beta_1 c}}{\sqrt{2 R \sigma_{Y_2}}} \frac{\Delta Q_{bb}^{bb} \sigma_{Y_1}}{\int_{N}^{\frac{dQ_{Y_1}}{dt} - \frac{p}{f_{rev_1}} \frac{df_{rev_1}}{dt}}}
\]

\(\Delta Y_1\) is the dipole shift of particle 1 at resonance in the direction perpendicular to the colliding plane \([m]\).

All other parameters are defined above.

For the ISR: The crossing plane is horizontal so that \(y = z\). The expression for \(\Delta Q_{bb}^{bb}\) appears in Section 8.9.

### 10.13 Monodimensional resonances due to overlap knock-out at large colliding angle

This is a particular case of the resonances mentioned in Section 10.4. All assumptions made in Section 10.12 are also applied in the present case and \(y\) again represents the direction perpendicular to the crossing plane. The bandwidth of the monodimensional resonances (Section 10.4)

\[
e = NQ_{Y_1} - p - q \frac{f_{rev_1}}{f_{rev_1}} = 0
\]

becomes, using the expressions in Sections 10.4, 10.5, 8.9 and 3.13,

\[
\Delta \epsilon = \frac{N}{2(N-2)} \frac{N}{(N-1)(N/2-1)!} \frac{c_q \Delta Q_{bb}^{bb}}{\sigma_{Y_1}} \text{ for } N \text{ even}
\]

\[
\Delta \epsilon = \frac{N}{2(N-1)/2} \frac{N}{(N-1)/2} \frac{\frac{y_1}{\sigma_{Y_2}}}{c_q \Delta Q_{bb}^{bb}} \text{ for } N \text{ odd, } \frac{y_1}{\sigma_{Y_2}} \ll 1,
\]

\(y_1, \sigma_{Y_2}, \Delta Q_{bb}^{bb}\) (Section 8.9) are defined in Section 10.12, \(c_q\) are the harmonics amplitudes defined in Section 3.13.
Using the formula in Section 10.7 for monodimensional resonances, the transverse increase of the initial amplitude of particle 1 when crossing the resonance by tune and/or frequency changes is given by

\[
\frac{\Delta \alpha}{\alpha_{y_1}} = \sqrt{\frac{\pi \beta_c}{2 R N}} \frac{\Delta \varepsilon}{\sqrt{\int N \frac{dQ_1}{dt} - \frac{p}{f_{rev_1}} \frac{df_{rev_1}}{dt} - \frac{q}{f_{rev_1}} \frac{df_{rev_2}}{dt}}},
\]

\(\sigma_{y_1}, \beta_1, \) and \(Q_{y_1}\) are defined in Section 10.12,
\(f_{rev_1}\) and \(f_{rev_2}\) are given in Section 10.4,
\(\Delta \varepsilon\) is the bandwidth given above.

Even if the mathematical definition used for the bandwidth does not apply to the dipole resonances, the final result concerning the transverse blow-up is still valid, using a virtual bandwidth \(\Delta \varepsilon\) with \(N = 1\)

\[
\frac{\Delta y_1}{y_1} = \frac{\sqrt{\pi \beta_c}}{\sqrt{2 R \sigma_{y_2}}} \frac{c_q \Delta Q_{y_1}}{\sigma_{y_1}} \frac{\Delta y_1}{y_1} \sqrt{\int N \frac{dQ_1}{dt} - \frac{p}{f_{rev_1}} \frac{df_{rev_1}}{dt} - \frac{q}{f_{rev_1}} \frac{df_{rev_2}}{dt}}},
\]

\(\Delta y_1\) is the dipole shift of particle 1 at resonance in the direction perpendicular to the colliding plane \([m]\).

All other parameters are defined above.

When the tune and the revolution frequency of particle 1 vary with momentum as given below,

\[
Q_{y_1} = Q_{y_1,0} + Q_1 \frac{\Delta p}{p_1},
\]

\[
f_{rev_1} = f_{rev_1,0} \left[1 + \eta \frac{\Delta p}{p_1}\right],
\]

the momenta at which the resonances appear in beam 1 are given by

\[
\frac{\Delta p}{p_1} = -Q_{y_1,0} + p + q f_{rev_2}/f_{rev_1,0} + NQ_{y_1},
\]

\(Q_{y_1,0}\) is the tune of beam 1 at centre line,
\(f_{rev_1,0}\) is the revolution frequency of beam 1 at centre line \([s^{-1}]\),
\(Q_1\) is the chromaticity of beam 1,
\(f_{rev_2}\) is the revolution frequency of the bunches \([s^{-1}]\),
\(\Delta p/p_1\) is the relative momentum difference between the particle at resonance and the particle at the centre line.

For the ISR: The crossing plane is horizontal so that \(y = z\). The expression for \(\Delta Q_{bb}\) appears in Section 8.9.
11. FORMULAE IN CONNECTION WITH COASTING BEAM INSTABILITIES\(^{28,39,40,41,42,43,44,45}\)

Symbols frequently used in this chapter

- \(\Delta p/p\) momentum spread for coasting beam
- \(Z_0 = \mu_0 c\) impedance of free space (Table 1) [\(\Omega\)]
- \(\sigma\) conductivity of the chamber material \([\Omega^{-1} \text{m}^{-1}]\)
- \(E_0\) rest energy of proton [eV]
- \(n\) is an integer.

11.1 Stability criterion for transverse resistive wall instabilities of coasting beams

The following stability criterion applies to a coasting beam\(^{49}\):

\[
|Z_\perp| \leq kF \frac{\beta \gamma}{B_y I} \left| n - Q_y \right| n - \frac{3Q_y}{\beta (\Delta p/p)} \left| \frac{\Delta p}{p} \right|
\]

where:

- \(k = \frac{E_0}{\epsilon} = 2.947618 \times 10^9 [V]\),
- \(Z_\perp\) is the total transverse impedance in the machine \([\Omega \text{m}^{-1}]\),
- \(n\) is every integer > \(Q_y\),
- \(3Q_y / \beta (\Delta p/p) = Q_y'\) is the chromaticity (Section 1.1) multiplied by the tune,
- \(\Delta p/p\) is the full relative momentum spread at half height,
- \(F\) is a form factor depending on the aspect ratio of the coasting beam.

Considering symmetric particle distributions with a central plateau and \(\sin^2\) edges, the aspect ratio of the beam is defined as plateau width over sum of edge widths at half-height. The minimum form factors associated with the worst possible phase of \(Z_\perp\) are given for different aspect ratios in the table below\(^{46}\):

<table>
<thead>
<tr>
<th>Aspect ratio of beam</th>
<th>0</th>
<th>1</th>
<th>10</th>
<th>100</th>
</tr>
</thead>
<tbody>
<tr>
<td>Minimum form factor F</td>
<td>1</td>
<td>0.89</td>
<td>0.65</td>
<td>0.45</td>
</tr>
</tbody>
</table>

This criterion can also be applied to a bunched beam if the rise time of the transverse instability is short compared with the period of the radio-frequency system. In this case, the current to be considered is the peak current of one bunch (Section 3.10).

For the ISR: The resistive wall instability is known under the name of "brickwall instability". Tables 3, 4, 5 and 6 give the values of \(B_y\), \(Q_y\) and \(Q_y'\), \(\beta\) and \(\gamma\), and \(n\), respectively.

11.2 Expressions for the transverse impedance

In the case of an elliptical vacuum chamber, the following expression applies\(^{46}\):

\[
Z_\perp = i R \left[ \frac{Z_0}{\beta^2 \gamma^2 \left( \frac{1}{a^2} - \frac{1}{b^2} \right)} - (1 + i) \sqrt{\frac{2R_0}{\beta (n - Q_y) \sigma \frac{1}{b^2}}} \right]
\]

- \(a\) is the coasting beam height [m],
- \(b\) is the half-height of the chamber [m].
In the case of cross-section variations\(^7\), the approximation of the low frequency impedance for short circular cavities and for perfectly conducting walls gives

\[
Z_\perp = -\frac{i}{b} \frac{Z_0}{\varepsilon} \left( \frac{B^2 - b^2}{B^2 + b^2} - 0.851 \frac{b^2}{\varepsilon} \right),
\]

where:

\[
\varepsilon = \frac{B}{L} > 1,
\]

while the approximation for long circular cavities and perfectly conducting walls gives

\[
Z_\perp = \frac{i}{b^2 \varepsilon} \frac{Z_0}{b \varepsilon} \left( \frac{B^2 - b^2}{B^2 + b^2} \right),
\]

where:

\[
\varepsilon = \frac{B}{L} \ll 1.
\]

\(b\) is the half-height of the chamber [m],
\(B\) is the half-height of the enlarged chamber [m],
\(L\) is the length of the chamber enlargement [m].

In the case of a cylindrical resonance cavity in its lowest mode, the transverse impedance is

\[
|Z_\perp| = 1.02 \frac{d}{r} \sqrt{\frac{Z_0^2 \rho \gamma (n-Q_y)}{2R}} \left( \frac{\sin \theta/2}{\theta/2} \right)^2
\]

where:

\[
\theta = \frac{d}{\beta r} j_1,
\]

\(j_1\) being the first zero of the Bessel function of the first order \(J_1\)\(^7,16\).
\(r\) is the radius of the cavity [m],
\(d\) is the length of the cavity [m].

For a beam surrounded by a circular metal envelope of radius \(r\), the transverse impedance is related to the longitudinal one \(Z_{\parallel}\) by

\[
Z_\perp = \frac{2R}{r^2} \left| n - Q_y \right| Z_{\parallel}.
\]

For the ISR: The conductivity of stainless steel and \(R\) are given in Table 2. The values of \(Q_y\) and of the relativistic parameters are given in Tables 4 and 5, respectively.
11.3 Stability criterion for longitudinal resistive wall instabilities of coasting beams

This instability is associated with self-bunching of the beam due to the fact that the revolution frequency above transition decreases when the energy increases. The longitudinal instability above transition is also called the "negative mass instability" when it is due to space charge effects in a perfectly conducting pipe.

The following stability criterion applies to a single coasting pulse\(^{a1}\):

\[
\left| \frac{Z_{\|}}{n} \right| \leq k F' \frac{\beta^2 \gamma \rho |n|}{I} \left( \frac{\Delta p}{p} \right)^2,
\]

while the stability criterion for a stacked coasting beam is\(^{a2}\)

\[
\left| \frac{Z_{\|}}{n} \right| \leq k F \frac{\beta^2 \gamma \rho |n|}{I} \left( \frac{\Delta p_s}{p} \right) \left( \frac{\Delta p_f}{p} \right),
\]

where:

- \(k = 4 \frac{E_0}{e} = 3.753024 \times 10^8 \text{ [V]}\),
- \(Z_{\|}\) is the longitudinal impedance [\(\Omega\)],
- \(\Delta p/p\) is the half relative momentum spread at half-height,
- \(\Delta p_s/p\) is the half relative momentum spread at half-height of the stack,
- \(\Delta p_f/p\) is the relative momentum width at half-height of the low energy flank,
- \(F\) is the form factor depending on the aspect ratio of the coasting stack beam, given in Section 11.1,
- \(F'\) is a form factor depending on the particle distribution in a single pulse.

The minimum form factors \(F'\) associated with the worst possible phase of \(Z_{\|}\) for different distributions are given in the table below\(^{a8}\):

<table>
<thead>
<tr>
<th>Distribution</th>
<th>Form factor (F')</th>
</tr>
</thead>
<tbody>
<tr>
<td>Triangular with rounded edges</td>
<td>0.555</td>
</tr>
<tr>
<td>Quartic</td>
<td>1.073</td>
</tr>
<tr>
<td>Circular limit</td>
<td>1.061</td>
</tr>
<tr>
<td>Very smooth</td>
<td>0.644</td>
</tr>
</tbody>
</table>

\(F'\) is very close to unity for a reasonable coasting pulse distribution where no sharp edges and infinite slopes appear in practice.

As in Section 11.1, the first criterion can be applied to a bunched beam if the rise time of the longitudinal instability is short compared with the period of the radio-frequency system. The current to be considered in this case is the peak current of one bunch (Section 3.10).

For the ISR: Table 5 gives the relativistic parameters and Table 6 gives the values of \(n\).
11.4 Expressions for the longitudinal impedance\textsuperscript{43,49}

In the case of a perfectly conducting circular pipe ("negative mass instability"), the following expression applies:

\[
\frac{Z_{//}}{n} = \frac{i \pi}{2} \frac{Z_0}{\beta \gamma^2} \left[ 1 + 2 \ln \left( \frac{b}{a} \right) \right],
\]

\(b\) is the radius of the circular pipe [m],
\(a\) is the radius of a circular beam [m].

For a resistive circular vacuum chamber, the impedance is

\[
\frac{Z_{//}}{n} = \frac{1 - i}{\sqrt{2} b} \sqrt{\frac{R Z_0}{\pi \sigma}},
\]

\(b\) is the radius of the vacuum chamber [m].

For a simple cylindrical cavity, the expression for the lowest mode is

\[
Z_{//} = 72 \lambda Q \frac{d}{r^2} \left[ \frac{\sin \left( \pi d/\lambda \right)}{\pi d/\lambda} \right]^2,
\]

\(\lambda\) is the wavelength [m],
\(Q\) is the quality factor of the cavity,
\(d\) is the length of the cavity [m],
\(r\) is the radius of the cavity [m].

In the case of cross-section variations and bellows, the impedance can be approximated by the expression for corrugations with rectangular cross-section

\[
\frac{Z_{//}}{n} = -i C \beta Z_0 \ln \left( 1 + \frac{\tau}{\beta} \right),
\]

\(C\) is the fraction of the circumference occupied by these elements,
\(\tau\) is the depth of the corrugations [m],
\(b\) is the half-height of the chamber [m].

In the case of plates, such as those formed by clearing electrodes or pick-up stations, the impedance can be approximated by

\[
\frac{Z_{//}}{n} = -i C \beta Z_0 \frac{w_p}{2 \pi} \ln \frac{b}{h_p},
\]

\(C\) and \(b\) are defined as in the preceding expression,
\(w_p\) is the plate-width [m],
\(h_p\) is the distance of the plate from the chamber centre [m].

For the ISR: The conductivity of stainless steel and \(R\) are given in Table 2. The relativistic parameters are given in Table 5.

In the special case of a cylindrical cavity in stainless steel and with \(R = 150\) m, the following expression applies:
\[
\frac{z}{n} = 7.6 \times 10^3 \frac{r^{5/2}}{d + r} \sin^2 \left(1.2 \frac{d}{r}\right),
\]

d is the cavity length [m],

r is the cavity radius [m].

11.5 Stability criterion for electron-proton oscillations without frequency spreads

In coating high energy proton beams, transverse vertical oscillations can be due to electrons moving in the potential well of the protons. These oscillations become unstable beyond a threshold in the frequency change of protons. Neglecting the effect of particles of the same species and the presence of walls, this threshold and the growth rate of the consequent instability can be approximated, in the absence of frequency spread, by

\[
Q_p \leq \frac{(n - Q_e)^2 - Q_e^2}{2 Q_e (n - Q_e)}
\]

\[
\frac{1}{\tau} = \pi f_{\text{rev}} Q_p \sqrt{\frac{Q_e}{n - Q_e}},
\]

1/\tau is the growth rate of the unstable modes [s^{-1}]
n is the unstable oscillation mode, such that n - Q is close to Q,

Q_e and Q_p are the vertical tune changes of electrons and protons, respectively, in the potential well of the other species

\[
Q_e^2 = k_e \frac{N}{R b(a+b) f_{\text{rev}}^2}
\]

\[
Q_p^2 = k_p \frac{n_e N}{R b(a+b) f_{\text{rev}}^2}
\]

where:

\[
k_e = \frac{r_e c^2}{2 \pi^3} = 4.08407 \text{[m}^3\text{s}^{-2}],
\]

\[
k_p = \frac{r_p c^2}{2 \pi^3} = 2.22425 \times 10^{-3} \text{[m}^3\text{s}^{-2}],
\]

\(n_e\) is the neutralization factor defined in Section 4.8,

N is the number of protons (Section 3.11),

a is the half-width of the beam [m],

b is the half-height of the beam [m].

For the ISR: Table 2 gives the values of \(R\) and \(f_{\text{rev}}\), while Table 5 gives \(\gamma\).

11.6 Stability criterion for electron-proton oscillations with frequency spreads

Considering the oscillations described in Section 11.5, it is now assumed that the frequencies of oscillation of single particles will depend both on the energy and the amplitude of the particles. Corresponding to these frequency spreads there are tune spreads which are normally assumed to have a parabolic distribution with a half-width \(\Gamma\).
\[ \rho(Q) = \frac{3}{4\pi} \left( 1 - \left( \frac{Q - Q_0}{\Gamma} \right)^2 \right), \]

where \( Q \) stands for \( Q_e, Q_p \) and \( Q_z \) (Section 11.5) and where \( \Gamma \) stands for \( \Gamma_e, \Gamma_p \) and \( \Gamma_z \).

Neglecting the spread in the small quantity \( Q_p \) and assuming that the two spreads \( Q_e \) and \( Q_z \) overlap in frequency space, the stability threshold approximately corresponds to

\[ Q_p \leq \frac{8}{3\pi} \sqrt{\frac{Q_z}{Q_e}} \sqrt{\Gamma_z \Gamma_e}, \]

\( Q_e \) and \( Q_p \) are given in Section 11.5, \( \Gamma_z \) and \( \Gamma_e \) are the widths of the distributions \( \rho(Q_e) \) and \( \rho(Q_p) \), respectively.

12. **FORMULAE IN CONNECTION WITH BUNCHED BEAM INSTABILITIES**\(^{50,51,52,53,54,55}\)

**Symbols frequently used in this chapter**

- \( V \): cavity voltage [V]
- \( \phi_s \): phase of the synchronous particle
- \( h \): harmonic of the radio frequency system
- \( M \): number of bunches
- \( B_f \): bunching factor defined in Section 3.12
- \( f_p \): phase oscillation frequency (Section 3.3) [s\(^{-1}\)]
- \( f_{\text{rev}} \): revolution frequency (Section 3.3) [s\(^{-1}\)]
- \( Z_0 = \mu_0 c \): impedance of free space (Table 1) [\( \Omega \)]
- \( S \): bunch shape factor (Section 3.12).

12.1 **Growth-rate of the longitudinal motion**

The motion considered here consists of rigid-bunch oscillations, i.e. dipole mode \((m = 1)\), and of higher bunch-shape oscillations of the individual bunches, i.e. 2m-pole modes \((m \geq 2)\), plus perhaps coupled motions of the different bunches, i.e. coupled-bunch modes \((n)\). The index \( m \) specified the harmonic of the bunch-shape oscillation, while \( n \) gives the phase difference 2m/M between adjacent bunches.

In the longitudinal phase plane, this motion corresponds to oscillations of the particle distributions with the frequency [s\(^{-1}\)]

\[ f = mf_p + \Delta f_m, \]

where \( \Delta f_m \) is the coherent frequency shift of mode \( m \), explicitly given below for some practical cases (Sections 12.2 to 12.4).

The growth-rate of such a motion in the absence of frequency spreads is
\[ \frac{1}{\tau} = 2\pi \text{Im} (\Delta f_m) \]

and the longitudinal motion is unstable when \( \text{Im}(\Delta f_m) \) is positive. Another stability criterion and a decoupling criterion, both valid in the presence of frequency spreads, are given in Sections 12.5 and 12.6.

12.2 Frequency shift in bunch oscillations due to perfectly conducting walls

For a bunch with approximately parabolic line density, the frequency shift of the bunch oscillations due to perfectly conducting walls is given by

\[ \Delta f_m = \sqrt{m} \Delta f_{sc}, \]

where the space charge frequency shift is for equidistant bunches

\[ \Delta f_{sc} = \frac{1}{4\pi} \frac{I M^2}{\beta \gamma^2 (B_S)^2 h V \cos \phi_s} \left[ 1 + 2 \ln \left( \frac{b}{a} \right) \right], \]

\( m \) is the mode of the oscillation (Section 12.1),
\( b \) is the radius of the round vacuum chamber [m],
\( a \) is the beam radius [m].

By virtue of Section 3.12, \( S \) is roughly equal to 1.5.

For the ISR: The values of \( h, V \) and \( \beta, \gamma \) are given in Tables 2 and 5, respectively.

12.3 Frequency shift in bunch oscillations due to resistive walls

For a smooth round vacuum chamber, the frequency shift of the dipole mode (Section 12.1) is

\[ \Delta f_1 = \frac{1}{24\pi} \frac{I M^{1/2}}{b h V \cos \phi_s} \sqrt{\frac{R \beta Z_o}{\sigma g}} \left( \frac{f_p}{M f_{rev}} + \frac{n}{N} \right), \]

\( n \) is the coupled-bunch mode (Section 12.1),
\( \sigma \) is the conductivity of the chamber material [\( \text{m}^{-1} \text{m}^{-1} \)],
\( b \) is the radius of the vacuum chamber [m].

The bunch function \( G(q) \) is given in Fig. 23 and the motion is unstable for \( q < \frac{1}{2} \) below transition and for \( q > \frac{1}{2} \) above transition, by virtue of the stability criterion for the buckets (Section 3.4).

The frequency shifts for the higher modes \( m \geq 2 \) (Section 12.1) are estimated for equidistant bunches to be

\[ \Delta f_m \approx (B_S/m) \Delta f_1, \]

\( \Delta f_1 \) being given above.

For the ISR: The values of \( \sigma, h, V \) and \( f_{rev} \) are given in Table 2. Table 5 gives the values of \( \beta \).
12.4 Frequency shift in bunch oscillations due to resonant elements

A resonator is characterized by a shunt impedance and induces for equidistant bunches the following frequency shifts in the bunch oscillations described in Section 12.1:

$$\Delta f_m = \frac{1}{2\pi} \frac{R_{ph} I M f}{E_{ph} V \cos \phi} D F_m (\Delta \phi)$$

$m$ is the mode of the oscillation (Section 12.1),
$R_{ph}$ is the shunt impedance of the resonant element [Ω],
$F_m$ is a form factor that specifies the efficiency with which the resonator can drive a given mode. It is given in Fig. 24. The maximum values $\hat{F}_m$ are approximately $1/\sqrt{m}$.
$\Delta \phi$ is the phase change that occurs during the passage of a bunch. It is defined by

$$\Delta \phi = k f_{\text{res}} \frac{\delta}{B}$$
where:

\[ k = \frac{2\pi}{c} = 2.095845 \times 10^{-8} \text{[m}^{-1}\text{s}], \]
\[ f_{\text{res}} \] is the resonant frequency of the resonator [s\(^{-1}\)],
\[ \xi \] is the full bunch length [m].

The complex factor \( D \) is given by the following expression:

\[ D = i \alpha \left( \frac{1}{1 - e^{a_+}} - \frac{1}{1 - e^{a_-}} \right), \]

where:

\[ a_\pm = \frac{2\pi}{M} \left( n \pm \frac{f_{\text{res}}}{f_{\text{rev}}} \right) - \alpha \]

and

\[ \alpha = \pi \frac{f_{\text{res}} t_{\text{bb}}}{Q}, \]

\( \alpha \) is the attenuation factor of the induced signal between bunches,
\( n \) is the coupled bunch mode (Section 12.1). Modes \( n = 0 \) and \( n = M/2 \) for \( M \) even are not excited,
\( Q \) is the quality factor of the resonator,
\( t_{\text{bb}} \) is the time between bunch centres [s].

The extreme modulus \( |\hat{D}| \) as a function of the factor \( 1/\alpha \) is given in Fig. 25.
Fig. 25 Extreme modulus of the complex factor $D$
(formula in Section 12.4)

For large values of $Q$, i.e. $\alpha \ll 1$, the expression of $D$ simplifies to

$$D \approx \frac{f_{\text{res}}}{2Q (f_{\text{res}} - j f_{\text{rev}}) - i f_{\text{res}}} ,$$

$j$ being an integer. The coupled bunch mode $n$ is excited when $f_{\text{res}} = j M f_{\text{rev}} \pm n f_{\text{rev}}$

For small values of $Q$, i.e. $\alpha \gg 1$, the expression of $D$ becomes

$$D \approx -2 \alpha e^{-\alpha} e^{i2\pi n/M} \sin \left(2\pi \frac{f_{\text{res}}}{M} f_{\text{rev}} \right) ,$$

and coupled bunch modes near $n = \pm \frac{M}{4}$ are most strongly excited.

For the ISR: The values of $h, V$ and $f_{\text{rev}}$ are given in Table 2. Table 5 gives the values of $\beta$.

12.5 Decoupling criterion for longitudinal instabilities of bunched beams

A rule-of-thumb for decoupling the bunches is that the r.m.s. spread in individual bunch frequencies should exceed the frequency shift $\Delta f_m$ due to the coupling force. When

the spread arises naturally from a difference $\Delta N$ in bunch populations, this criterion can be written as follows:
\[ |\Delta f_m| < \left( \frac{\Delta f_{SC}}{N} \right)_{r.m.s.} |m\Delta f_{SC}|, \]

\(m\) is the oscillation mode (Section 12.1), \(m > 1\),
\(\Delta f_{SC}\) is the space charge frequency shift (Section 12.2) \(s^{-1}\),
\(N\) is the number of particles in a bunch,
\(\Delta f_m\) contains terms similar to those given in Sections 12.3 and 12.4, excluding the term in Section 12.2 which does not contribute to the coupling.

Since this rule-of-thumb is based on bunch-to-bunch differences in the incoherent motion, it does not apply to the dipole mode and is somewhat optimistic for the higher modes.

12.6 Stability criterion for longitudinal instabilities of bunched beams

The stability depends on the spread in synchrotron frequency of the bunch due to the nonlinearity of the force. For a given mode \(m\) (Section 12.1), the stability criterion becomes

\[ |\Delta f_m| < \frac{\sqrt{m}}{4} F_p, \]

\(\Delta f_m\) is the sum of all interactions, i.e. space charge, resistive wall, resonators, etc.
\(F_p\) is the full spread in \(f_p\) (Section 3.3) between centre and edge of the bunch. It is plotted in Fig. 26 as a function of the product \(B_p S\) (Section 3.12) and the parameter \(\Gamma\) (Section 3.1).

Fig. 26 Full spread in synchrotron frequency (formula in Section 12.6)
12.7 Radio-frequency voltage inside the bunches with inductive walls

For bunched beams, the space charge effects and the presence of inductive walls modify the radio-frequency voltage inside the bunches. Assuming a parabolic line density distribution \( \lambda [m^{-1}] \)

\[
\lambda(s) = k \frac{I}{M f_{\text{rev}}} s^2 \left( \frac{s^2}{4} - s^2 \right),
\]

where:

\[
k = \frac{6}{e} = 3.74508 \times 10^{19} A^{-1} s^{-1},
\]

the voltage \( U [V] \) seen by a particle at a distance \( s [m] \) from the bunch centre is:

\[
U = V \left( \sin \Phi_s - s \frac{h f_p^2}{R f_{p_0}^2} \cos \Phi_s \right),
\]

\( l \) is the full bunch length \([m]\),
\( s \) is the distance from the centre of the bunch \([m]\): \(- \frac{l}{2} \leq s \leq \frac{l}{2} \),
\( f_{p_0} \) is the phase oscillation frequency in the absence of self-forces and wall inductance (Section 3.3) \([s^{-1}]\),
\( f_p \) is the incoherent phase oscillation frequency in the presence of self-forces and wall inductance \([s^{-1}]\)

\[
f_p^2 = f_{p_0}^2 \left[ 1 - 24 \frac{\pi}{\pi} \frac{IR^3}{hM} \frac{1}{2} \frac{2g}{hM} \left( \frac{gZ_0}{2} - \frac{Z}{2} \right) \right].
\]

Considering the coherent oscillations of the bunches described in Section 12.1, the dipole mode and quadrupole mode frequency shifts are

\[
\Delta f_1 = 0
\]

\[
\Delta f_2 = \frac{1}{2} \left( f_p - f_{p_0} \right).
\]

The wall properties appear in the coupling coefficient \( g \) and the total inductance \( L \).

With a circular symmetry, the following formulae apply:

\[
g = 1 + 2 \ln \frac{b}{a}
\]

\[
\left| \frac{Z}{n} \right| = 2 \pi f_{\text{rev}} L,
\]

\( a \) is the radius of the circular beam \([m]\),
\( b \) is the radius of the circular pipe \([m]\),
\( \left| \frac{Z}{n} \right| \) is the inductive impedance of the pipe \([\Omega]\).
For the ISR: The values of $R$, $h$, $f_{\text{rev}}$ and $V$ are given in Table 2. The values of $\beta$ and $\gamma$ are given in Table 5.

A measurement made in the ISR\textsuperscript{2}) gives $|Z/n| \approx 26 \ \Omega$ or $L \approx 13 \ \mu H$.

12.8 Bunch lengthening due to inductive wall effects

The voltage perturbation mentioned in Section 12.7 provokes a bunch lengthening given by

$$l_b^o = l_b - 24 \pi \frac{IR^2}{hMV \cos \phi} \left( \frac{g}{2R^2} - \frac{|Z|}{n} \right) l,$$

$l$ is the full bunch length in the presence of self-forces and wall inductance [$m$],

$l_b^o$ is the full bunch length in the absence of self-forces and wall inductance [$m$],

$g$ and $|Z/n|$ are defined in Section 12.7.

For the ISR: The values of $R$, $h$ and $V$ are given in Table 2, while $\beta$ and $\gamma$ are given in Table 5. A value for $|Z/n|$ is given in Section 12.7.

12.9 Bucket area variations due to inductive wall effects

The voltage perturbation given in Section 12.7 provokes a change in the bucket area defined in Section 3.5. For $\Gamma = \sin \phi = 0$, this change is given by

$$A = A_0 \sqrt{1 \pm \frac{hI}{V} \left( \frac{g}{2R^2} - \frac{|Z|}{n} \right)}.$$

$A$ is the bucket area in the presence of self-forces and wall inductance,

$A_0$ is the bucket area in the absence of self-forces and wall inductance (Section 3.5),

$g$ and $|Z/n|$ are defined in Section 12.7,

the negative sign is valid below transition energy and the positive one above transition energy.

For the ISR: The values of $h$, $V$, $\beta$ and $\gamma$ are given in Tables 2 and 5. A value for $|Z/n|$ is given in Section 12.7.

Acknowledgements

The author is grateful to P.J. Bryant, O. Gröbner, A. Hofmann, E. Keil and B. Zotter who have checked and commented on parts of the manuscript.
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Chapter 12


Tables


APPENDIX

Analytic form of the integrals giving the maximum amplitude growth

The integrals in which we are interested are given in Section 10.7. Reference 5 gives the explicit expressions of these integrals for any values of $n_x$ and $n_z$. Since these expressions are quite complicated, it seems judicious to give only the analytic forms for $N \leq 5$.

Let us put

$$J(n_x, n_z, \text{sign}) = \int \frac{d g_x}{(k + g_x^2)^{n_x/2} (n_x - 1)} = \sqrt{|k|} (2 - N) \int \frac{d u}{(u^2 + \text{sign})^{n_z/2} (n_z - 1)} ,$$

where:

$$k = \frac{n_x}{n_z} - 1 ; \quad u = \frac{g_x}{\sqrt{|k|}} ,$$

and $\text{sign} = \pm 1$, depending on the sign of $k$.

Using the new variable $u$, the last integral on the right hand side (i.e. $J/\sqrt{|k|} (2 - N)$) is explicitly given below for different $n_x$, $n_z$ and $N (N \leq 5)$. The cases where $n_x$ or $n_z$ is zero do not appear, since these solutions are given in Section 10.7.

<table>
<thead>
<tr>
<th>$n_x, n_z$</th>
<th>$N - 1, 1$</th>
<th>$N - 2, 2$</th>
<th>$N - 3, 3$</th>
<th>$N - 4, 4$</th>
<th>Sign</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>$\ln\left[u + \sqrt{u^2 + \text{sign}}\right]$</td>
<td></td>
<td></td>
<td></td>
<td>$\pm 1$</td>
</tr>
<tr>
<td>3</td>
<td>$\ln\frac{\sqrt{u^2 + 1} - 1}{u}$</td>
<td>$\arctg u$</td>
<td></td>
<td></td>
<td>$+1$</td>
</tr>
<tr>
<td></td>
<td>$\arccos\left(\frac{1}{u}\right)$</td>
<td>$- \argth u$</td>
<td></td>
<td></td>
<td>$-1$</td>
</tr>
<tr>
<td>4</td>
<td>$-\text{sign} \frac{\sqrt{u^2 + \text{sign}}}{u}$</td>
<td>$\ln\frac{u^2}{u^2 + \text{sign}}$</td>
<td>$\frac{u}{\sqrt{u^2 + \text{sign}}}$</td>
<td></td>
<td>$\pm 1$</td>
</tr>
<tr>
<td>5</td>
<td>$-\frac{\sqrt{u^2 + 1}}{2u} - \ln\frac{\sqrt{u^2 + 1} - 1}{\sqrt{u^2 + 1} + 1}$</td>
<td>$-\frac{1}{u} - \arctg u$</td>
<td>$\frac{1}{\sqrt{u^2 + 1}} - \ln\frac{u + \sqrt{u^2 + 1}}{u}$</td>
<td>$\frac{u}{2(1 + u^2)} + \frac{1}{2} \arctg u$</td>
<td>$+1$</td>
</tr>
<tr>
<td></td>
<td>$\sqrt{u^2 - 1} + \frac{1}{2} \arccos\frac{1}{u}$</td>
<td>$\frac{1}{u} - \argth u$</td>
<td>$-\frac{1}{\sqrt{u^2 + 1}} - \frac{1}{\cos 1/u}$</td>
<td>$\frac{u}{2(u^2 - 1)} - \frac{1}{2} \argth u$</td>
<td>$-1$</td>
</tr>
</tbody>
</table>
Table 1

Table of physical constants

<table>
<thead>
<tr>
<th>Symbols</th>
<th>Meaning</th>
<th>Value</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>c</td>
<td>velocity of light</td>
<td>$2.9979246 \times 10^8$</td>
<td>ms$^{-1}$</td>
</tr>
<tr>
<td>e</td>
<td>electronic charge</td>
<td>$1.602189 \times 10^{-19}$</td>
<td>As</td>
</tr>
<tr>
<td>$N_m$</td>
<td>number of molecules per mole</td>
<td>$6.022169 \times 10^2$</td>
<td>-</td>
</tr>
<tr>
<td>$V_m$</td>
<td>molar volume of ideal gas at standard conditions</td>
<td>$2.241411 \times 10^{-2}$</td>
<td>m$^3$</td>
</tr>
<tr>
<td>$T_{st}$</td>
<td>standard temperature for gas</td>
<td>273</td>
<td>K</td>
</tr>
<tr>
<td>$P_{st}$</td>
<td>standard pressure for gas</td>
<td>760</td>
<td>Torr</td>
</tr>
<tr>
<td>$E_o$</td>
<td>rest mass of proton</td>
<td>938.256</td>
<td>MeV</td>
</tr>
<tr>
<td>$E_e$</td>
<td>rest mass of electron</td>
<td>0.511004</td>
<td>MeV</td>
</tr>
<tr>
<td>$r_p$</td>
<td>classical proton radius</td>
<td>$1.534697 \times 10^{-18}$</td>
<td>m</td>
</tr>
<tr>
<td>$r_e$</td>
<td>classical electron radius</td>
<td>$2.817938 \times 10^{-15}$</td>
<td>m</td>
</tr>
<tr>
<td>$C_H$</td>
<td>absolute atomic gas factor</td>
<td>10.552</td>
<td>-</td>
</tr>
<tr>
<td>$C_{He}$</td>
<td>for helium</td>
<td>39.448</td>
<td>-</td>
</tr>
<tr>
<td>$C_C$</td>
<td>for carbon</td>
<td>328.64</td>
<td>-</td>
</tr>
<tr>
<td>$C_N$</td>
<td>for nitrogen</td>
<td>442.27</td>
<td>-</td>
</tr>
<tr>
<td>$C_O$</td>
<td>for oxygen</td>
<td>554.09</td>
<td>-</td>
</tr>
<tr>
<td>$C_{Ne}$</td>
<td>for neon</td>
<td>878.57</td>
<td>-</td>
</tr>
<tr>
<td>$C_{Ar}$</td>
<td>for argon</td>
<td>2709.3</td>
<td>-</td>
</tr>
<tr>
<td>$\varepsilon_0$</td>
<td>permittivity of free-space</td>
<td>$8.854188 \times 10^{-12}$</td>
<td>AsVm$^{-1}$m$^{-1}$</td>
</tr>
<tr>
<td>$Z_0$</td>
<td>impedance of free-space</td>
<td>119.917$\pi$</td>
<td>$\Omega$</td>
</tr>
<tr>
<td>$\mu_0$</td>
<td>permeability of free-space</td>
<td>$4\pi \times 10^{-7}$</td>
<td>A$^{-1}$sVm$^{-1}$</td>
</tr>
<tr>
<td>$k_B$</td>
<td>Boltzmann constant</td>
<td>$8.617084 \times 10^{-11}$</td>
<td>MeV K$^{-1}$</td>
</tr>
</tbody>
</table>
Table 2
Table of parameters valid for the present ISR\textsuperscript{56)}

<table>
<thead>
<tr>
<th>Symbols</th>
<th>Meaning</th>
<th>Value</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\rho$</td>
<td>bending radius</td>
<td>78.5895</td>
<td>m</td>
</tr>
<tr>
<td>R</td>
<td>average machine radius</td>
<td>150.0253</td>
<td>m</td>
</tr>
<tr>
<td>$\ell$</td>
<td>machine circumference</td>
<td>942.6568</td>
<td>m</td>
</tr>
<tr>
<td>$r/\rho$</td>
<td>field index upon $\rho$ for F unit</td>
<td>$-3.136$</td>
<td>m\textsuperscript{-1}</td>
</tr>
<tr>
<td></td>
<td>for D unit</td>
<td>$3.020$</td>
<td>m\textsuperscript{-1}</td>
</tr>
<tr>
<td>$n'/\rho$</td>
<td>radial derivative of field-index upon $\rho$ for F unit</td>
<td>$-1.946$</td>
<td>m\textsuperscript{-2}</td>
</tr>
<tr>
<td></td>
<td>for D unit</td>
<td>$1.493$</td>
<td>m\textsuperscript{-2}</td>
</tr>
<tr>
<td>N</td>
<td>number of intersections</td>
<td>8</td>
<td>-</td>
</tr>
<tr>
<td>h</td>
<td>RF harmonic</td>
<td>30</td>
<td>-</td>
</tr>
<tr>
<td>$t_{\text{rev}}$</td>
<td>revolution time at $\beta = 1$</td>
<td>3.144298</td>
<td>$\mu$s</td>
</tr>
<tr>
<td>$f_{\text{rev}}$</td>
<td>revolution frequency at $\beta = 1$</td>
<td>3.18036 $\times 10^{5}$</td>
<td>Hz</td>
</tr>
<tr>
<td>V</td>
<td>cavity voltage at the beginning of the cycle</td>
<td>16</td>
<td>kV</td>
</tr>
<tr>
<td></td>
<td>at the end of the cycle</td>
<td>700</td>
<td>V</td>
</tr>
<tr>
<td>$\varepsilon_x$</td>
<td>normalized horizontal emittance</td>
<td>13$\pi$ to 18$\pi$</td>
<td>mm mrad</td>
</tr>
<tr>
<td>$\varepsilon_z$</td>
<td>normalized vertical emittance</td>
<td>8$\pi$</td>
<td>mm mrad</td>
</tr>
<tr>
<td>$\sigma$</td>
<td>stainless steel conductivity</td>
<td>7.6923 $\times 10^{5}$</td>
<td>$\Omega^{-1}$ m$^{-1}$</td>
</tr>
<tr>
<td>$\sigma_{pp}$</td>
<td>total proton-proton cross-section at the ISR energies</td>
<td>4.09 $\times 10^{-30}$</td>
<td>m$^2$</td>
</tr>
</tbody>
</table>
Table 3

Values of some transverse parameters
for standard working conditions in the ISR

<table>
<thead>
<tr>
<th>Working Line</th>
<th>( \gamma_t )</th>
<th>( \alpha_{p,x} )</th>
<th>( \beta_x )</th>
<th>( \beta_z )</th>
<th>( \beta_{z,1} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>FP</td>
<td>8.8172</td>
<td>1.9294</td>
<td>20.8185</td>
<td>22.9348</td>
<td>14.56</td>
</tr>
<tr>
<td>8C</td>
<td>8.7977</td>
<td>1.9380</td>
<td>20.8789</td>
<td>22.9427</td>
<td>14.56</td>
</tr>
<tr>
<td>5C</td>
<td>8.7800</td>
<td>1.9458</td>
<td>21.0724</td>
<td>23.0305</td>
<td>14.93</td>
</tr>
<tr>
<td>ELSA</td>
<td>9.0705</td>
<td>1.8232</td>
<td>20.2391</td>
<td>22.5412</td>
<td>12.17</td>
</tr>
<tr>
<td>Steel Low-( \beta )</td>
<td>9.0903</td>
<td>1.8152</td>
<td>20.5244</td>
<td>23.4407</td>
<td>2.63</td>
</tr>
</tbody>
</table>

Note: These values are given on the centre line.

Table 4

Values of the tunes and tune derivatives
for standard working conditions in the ISR

| Working Line | | H-plane | | | V-plane | |
|--------------|------------------|------------------|------------------|------------------|------------------|
|               | \( Q_x \) central | \( Q_x \) top(+40) | \( Q'_x \) | \( Q_z \) central | \( Q_z \) top(+40) | \( Q'_z \) |
| 8C           | 8.6145           | 8.6475           | 1.599           | 8.6245           | 8.6575           | 1.599           |
| 5C           | 8.550            | 8.615            | 1.970           | 8.600            | 8.657            | 3.259           |
| TW           | 8.637            | 8.6785           | 2.147           | 8.627            | 8.6685           | 2.147           |
| ELSA         | 8.9015           | 8.955            | 2.439           | 8.8815           | 8.935            | 2.439           |
| Steel Low-\( \beta \) | 8.9015        | 8.955            | 2.428           | 8.8815           | 8.935            | 2.428           |
Table 5

Values of magnetic rigidity and relativistic parameters at ISR energies

<table>
<thead>
<tr>
<th>p (GeV/c)</th>
<th>β</th>
<th>γ</th>
<th>B₀ (Tm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>11.776</td>
<td>0.99684</td>
<td>12.591</td>
<td>39.280</td>
</tr>
<tr>
<td>15.376</td>
<td>0.99814</td>
<td>16.418</td>
<td>51.288</td>
</tr>
<tr>
<td>22.505</td>
<td>0.99913</td>
<td>24.007</td>
<td>75.068</td>
</tr>
<tr>
<td>26.588</td>
<td>0.99929</td>
<td>28.355</td>
<td>88.687</td>
</tr>
<tr>
<td>31.400</td>
<td>0.99949</td>
<td>33.481</td>
<td>104.738</td>
</tr>
</tbody>
</table>

Table 6

Values of revolution frequency spread per unit of momentum spread for different working conditions in the ISR

<table>
<thead>
<tr>
<th>p (GeV/c)</th>
<th>η in 10⁻³ on FP</th>
<th>η in 10⁻³ on 8C</th>
<th>η in 10⁻³ on ELSA</th>
<th>η in 10⁻³ on Steel Low-β</th>
</tr>
</thead>
<tbody>
<tr>
<td>11.776</td>
<td>- 6.55507</td>
<td>- 6.61215</td>
<td>- 5.84669</td>
<td>- 5.79380</td>
</tr>
<tr>
<td>22.505</td>
<td>-11.12779</td>
<td>-11.18488</td>
<td>-10.41941</td>
<td>-10.36652</td>
</tr>
</tbody>
</table>

Note: These values are given on the centre line.
Table 7

Factor between moving and stationary bucket area

<table>
<thead>
<tr>
<th>( \Gamma )</th>
<th>( \alpha(\Gamma) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>0.1</td>
<td>0.8039</td>
</tr>
<tr>
<td>0.15</td>
<td>0.7294</td>
</tr>
<tr>
<td>0.2</td>
<td>0.6612</td>
</tr>
<tr>
<td>0.25</td>
<td>0.5981</td>
</tr>
<tr>
<td>0.3</td>
<td>0.5389</td>
</tr>
<tr>
<td>0.35</td>
<td>0.4833</td>
</tr>
<tr>
<td>0.4</td>
<td>0.4306</td>
</tr>
<tr>
<td>0.45</td>
<td>0.3807</td>
</tr>
<tr>
<td>0.5</td>
<td>0.3333</td>
</tr>
<tr>
<td>0.52</td>
<td>0.3151</td>
</tr>
<tr>
<td>0.54</td>
<td>0.2972</td>
</tr>
<tr>
<td>0.56</td>
<td>0.2799</td>
</tr>
<tr>
<td>0.58</td>
<td>0.2627</td>
</tr>
<tr>
<td>0.60</td>
<td>0.2461</td>
</tr>
<tr>
<td>0.62</td>
<td>0.2297</td>
</tr>
<tr>
<td>0.64</td>
<td>0.2168</td>
</tr>
<tr>
<td>0.66</td>
<td>0.1981</td>
</tr>
<tr>
<td>0.68</td>
<td>0.1828</td>
</tr>
<tr>
<td>0.70</td>
<td>0.1678</td>
</tr>
<tr>
<td>0.72</td>
<td>0.1534</td>
</tr>
<tr>
<td>0.74</td>
<td>0.1393</td>
</tr>
<tr>
<td>0.76</td>
<td>0.1255</td>
</tr>
<tr>
<td>0.78</td>
<td>0.1121</td>
</tr>
<tr>
<td>0.80</td>
<td>0.0991</td>
</tr>
<tr>
<td>0.82</td>
<td>0.0865</td>
</tr>
<tr>
<td>0.84</td>
<td>0.0744</td>
</tr>
<tr>
<td>0.85</td>
<td>0.0686</td>
</tr>
<tr>
<td>0.86</td>
<td>0.0627</td>
</tr>
<tr>
<td>0.87</td>
<td>0.0571</td>
</tr>
<tr>
<td>0.88</td>
<td>0.0515</td>
</tr>
<tr>
<td>0.89</td>
<td>0.0462</td>
</tr>
</tbody>
</table>
Table 8
Ionization cross-sections at 26 GeV/c\textsuperscript{57}) and molecular masses

<table>
<thead>
<tr>
<th>Gas</th>
<th>Molecular mass M</th>
<th>Ionisation cross-section $\sigma_+$ in $10^{-23} \text{ m}^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>H$_2$</td>
<td>2</td>
<td>1.8</td>
</tr>
<tr>
<td>He</td>
<td>4</td>
<td>1.8</td>
</tr>
<tr>
<td>CH$_4$</td>
<td>16</td>
<td>9.7</td>
</tr>
<tr>
<td>H$_2$O</td>
<td>18</td>
<td>7.5</td>
</tr>
<tr>
<td>N$_2$</td>
<td>28</td>
<td>8.2</td>
</tr>
<tr>
<td>CO</td>
<td>28</td>
<td>8.2</td>
</tr>
<tr>
<td>O$_2$</td>
<td>32</td>
<td>9.1</td>
</tr>
<tr>
<td>Ar</td>
<td>40</td>
<td>8.9</td>
</tr>
<tr>
<td>CO$_2$</td>
<td>44</td>
<td>13.0</td>
</tr>
</tbody>
</table>

Table 9
Nuclear cross-sections for protons at ISR energies\textsuperscript{58})

<table>
<thead>
<tr>
<th>Atom</th>
<th>Symbol</th>
<th>Value in mb = $10^{-31} \text{ m}^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hydrogen</td>
<td>$\sigma_{n,H}$</td>
<td>40</td>
</tr>
<tr>
<td>Carbon</td>
<td>$\sigma_{n,C}$</td>
<td>340</td>
</tr>
<tr>
<td>Nitrogen</td>
<td>$\sigma_{n,N}$</td>
<td>390</td>
</tr>
<tr>
<td>Oxygen</td>
<td>$\sigma_{n,O}$</td>
<td>440</td>
</tr>
<tr>
<td>Argon</td>
<td>$\sigma_{n,Ar}$</td>
<td>890</td>
</tr>
</tbody>
</table>
Table 10

Principal beam parameters in the SFM of the ISR

<table>
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<tr>
<th>SFM field T</th>
<th>Parameter</th>
<th>31.460</th>
<th>26.588</th>
<th>22.505</th>
<th>15.376</th>
<th>11.780</th>
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<td>1.00</td>
<td>$d_{\text{max}}$ [mm]</td>
<td>10.7</td>
<td>12.6</td>
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<td></td>
<td>$s_{\text{max}}$ [m]</td>
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<td>3.1</td>
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<tr>
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<td>$d_{\text{min}}$ [mm]</td>
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<td>80.4</td>
<td>95</td>
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<tr>
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<td>2.9</td>
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<tr>
<td></td>
<td>$x_I$ [mm]</td>
<td>29.3</td>
<td>34.7</td>
<td>40.9</td>
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<tr>
<td></td>
<td>$\psi$ [degree]</td>
<td>17.477</td>
<td>17.971</td>
<td>18.552</td>
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<tr>
<td>0.85</td>
<td>$d_{\text{max}}$</td>
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<td>10.2</td>
<td>12.1</td>
<td>17.8</td>
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<tr>
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<td>17.998</td>
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<td>16.647</td>
<td>17.516</td>
<td>18.353</td>
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</table>

Notes:

i) Without field in the SFM, the colliding angle $\psi$ is 14.773$^\circ$.

ii) The distances $s$, $d$ and $l$ given in the table are defined in fig. 27.

iii) Heavy boxes indicate standard operating conditions.
Trajectories in SFM at 100 T and 22.5 GeV/c

Unperturbed ISR trajectories

Fig. 27 Orbits in intersection 4 of the ISR with the SFM switched on

The parameters appearing on the figure are given in Table 10 for different SFM fields and beam momenta.