Parametric Analysis of Forces and Stresses in Superconducting Quadrupole Sector Windings

P. Fessia, F. Regis, E. Todesco

Abstract

We first present a review of the existing analytical equations for electromagnetic forces and stress in a sector coil quadrupole, evaluating their extent of application in function of the coil geometrical layout. We analyze the distribution of stress provided on the coil retaining structure as well as on the coil mid plane, this one related to the degradation of critical current in the superconductor. We focus on the maximum compressive stress dependence on the magnetic gradient at short sample conditions, considering two possible low temperature superconductors: Nb-Ti and Nb3Sn. In the last part the effect of an iron yoke on the magnetic field and forces is presented, proposing a correction of the critical current density expression when an infinite permeable iron yoke is used.
1 Introduction

In superconducting quadrupoles, electromagnetic forces and associated stresses are generated by the interaction of the cable current with the magnetic field. The stress produced are generally of the order of 10-100 MPa. Two main issues have to be analyzed: first, a mechanical structure must be envisaged to contain these forces and to limit the cable movements during the magnet ramp; secondly, the stresses inside the coil must not exceed the limits beyond which insulation can start creeping and the superconductor properties are degraded. This second aspect is critical for the Nb\textsubscript{3}Sn which, according to measurements carried out on cables, cannot tolerate compressive stresses larger than 150 MPa \cite{1}. In this paper we aim at analyzing how forces and stresses depend on the quadrupole aperture, on the width of the coil, and on the superconducting material. We used a simplified coil layout made up of a sector of inner radius \( r_i \), radial width \( w \) and an angular extension \( \alpha_0 = 30^o \) (thus canceling the sixth order field harmonic). A uniform current density \( j \) is applied (see fig. 1-b). This simple geometry has the advantage of being closer to a real coil than the classical cos2\( \theta \), whilst still allowing an analytical approach. In \cite{2} it has been shown that a similar lay-out well represents, from e. m. point of view, several quadrupoles based on the shell geometry that have been built in the last 30 years.

The main steps carried out in this analysis have been the following ones:

- an analytical estimation by using the formalism developed in \cite{3}-\cite{5} of the electromagnetic forces and the induced stresses in the sector coil as a function of \( w \) and \( r_i \) for a given \( j \). In particular, we will focus on the evaluation of the position and of the value of the maximum compressive stress on the coil mid plane. In this section, no iron yoke is introduced. A similar approach has been developed for the dipoles using a cos\( \phi \) model in \cite{6}-\cite{7}
- the evaluation of the critical current density relative to a given coil lay-out and to the specific superconductor (either Nb-Ti or Nb\textsubscript{3}Sn). The relationship between the peak compressive stress and the obtained gradient will be studied
- the development of the formulae for the critical current density in presence of an iron yoke, revising the peak stress behavior in this new condition
- the analytical formulae are cross-checked with a two-dimensional finite element model coded in ANSYS of one quadrupole octant. In the first part of the work the magnetic model of the winding built in ANSYS is completely surrounded by air, and the "width" of the air section has been optimized in ANSYS to get a good convergence for the values of the magnetic field. In the last part, the iron yoke has been implemented in the model.

2 Analytical formulae of magnetic field and Lorentz force components

To define the magnetic field components and the related e.m. forces, we consider an approximation with constant current density sector coil. We also consider the cos\( \phi \) approach for the sense of completeness, since it is a reference case in literature.

2.1 Cos\( \phi \) winding

The magnetic field produced by a quadrupole magnet powered by a current density distributed as \( j = j_0 \cos\phi \), constant in radius can be derived from the definition of the scalar potential \cite{3}. The equations for the field within the useful aperture (\( 0 \leq r \leq r_i \)), developed only at the first term, are as follows:
Figure 1: LHC quadrupole cross section (a) and sector coil model adopted (b).

\[
\begin{align*}
\{ B_r, B_\varphi \} &= -\frac{\mu_0 j r}{2} \ln \left( \frac{r + w}{r_i} \right) \left\{ \begin{array}{c}
\sin 2\varphi \\
\cos 2\varphi
\end{array} \right\} \\
\end{align*}
\]

within the coil \((r_i \leq r \leq r_o = (r_i + w))\), one has

\[
\begin{align*}
\{ B_r, B_\varphi \} &= -\frac{\mu_0 j}{2} \left[ r \ln \left( \frac{r + w}{r} \right) + \frac{r^{-3}r_i^4}{4} \pm \frac{r}{4} \right] \left\{ \begin{array}{c}
\sin 2\varphi \\
\cos 2\varphi
\end{array} \right\} \\
\end{align*}
\]

whereas outside the winding \((r \geq r_o)\):

\[
\begin{align*}
\{ B_r, B_\varphi \} &= -\frac{\mu_0 j}{8} \left( (r_i + w)^4 - r_i^4 \right) \left\{ \begin{array}{c}
\sin 2\varphi \\
\cos 2\varphi
\end{array} \right\} \\
\end{align*}
\]

The equations of the e.m. forces produced by the magnetic field are as follows (see Appendix A):

\[
\begin{align*}
F_x &= -\frac{j^2 \mu_0 \sin \alpha_0}{72(r_i + w)} f_x(r_i^4, w^4) \\
F_y &= \frac{j^2 \mu_0}{63(r_i + w)} f_y(r_i^4, w^4)
\end{align*}
\]

2.2 Sector coil

The expressions for the magnetic field provided by a constant current density distribution can be derived by starting from the definition of the magnetic vector potential [5]. Consequently the field \(B_r\) within the aperture \((0 \leq r \leq r_i)\) is as follows:

\[
\begin{align*}
B_r &= -\frac{j \mu_0}{2\pi} \left\{ 4r \ln \left( \frac{r + w}{r_i} \right) \sin(2\alpha_0) \sin 2\varphi \right. + \\
&\left. \sum_{m=1}^{\infty} \frac{2r_i^{4m+1}}{-m(4m+2)} [(r_i + w)^{-4m} - r_i^{-4m}] \sin(4m+2) \alpha_0 \sin(4m+2) \varphi \right\}
\end{align*}
\]

where \(\alpha_0\) is the angle at the pole \((\pi/6)\).
The integration of the field $B_r$ outside the winding $(r \geq r_i + w)$ leads to:

$$B_r = -\frac{j\mu_0}{2\pi} \left\{ r^{-3} \{ (r_i + w)^4 - r_i^4 \} \sin(2\alpha_0)\sin2\varphi + \sum_{m=1}^{\infty} \frac{2^{r-4m-3}}{(m+1)(4m+2)} [(r_i + w)^{4m+4} - r_i^{4m+4}] \sin(4m+2)\alpha_0\sin(4m+2)\varphi \right\} \tag{7}$$

Summing up these two components by imposing that the first contribution is 0 at $r = r_i$ and the second is 0 at $r = (r_i + w)$ we get the radial field $B_r$ inside the coil:

$$B_r(r, \varphi) = -\frac{j\mu_0}{2\pi} \left\{ 4\ln \left( \frac{r_i + w}{r} \right) + r^{-3}(r^4 - r_i^4) \right\} \sin(2\alpha_0)\sin2\varphi + \sum_{m=1}^{\infty} \frac{2^{r-4m+1}}{-m(4m+2)} [(r_i + w)^{-4m} - r_i^{-4m}] \sin(4m+2)\alpha_0\cos(4m+2)\varphi \right\} \tag{8}$$

In the same way as before, the integration of the field $B_\varphi$ within the useful aperture $(0 \leq r \leq r_i)$ leads to:

$$B_\varphi = -\frac{j\mu_0}{2\pi} \left\{ 4\ln \left( \frac{r_i + w}{r_i} \right) \sin(2\alpha_0)\cos2\varphi + \sum_{m=1}^{\infty} \frac{2^{r-4m+1}}{-m(4m+2)} [(r_i + w)^{-4m} - r_i^{-4m}] \sin(4m+2)\alpha_0\cos(4m+2)\varphi \right\} \tag{9}$$

The integration of the field $B_\varphi$ outside the winding $(r \geq r_i + w)$ leads to:

$$B_\varphi = \frac{j\mu_0}{2\pi} \left\{ r^{-3} \{ (r_i + w)^4 - r_i^4 \} \sin(2\alpha_0)\cos2\varphi + \sum_{m=1}^{\infty} \frac{2^{r-4m-3}}{(m+1)(4m+2)} [(r_i + w)^{4m+4} - r_i^{4m+4}] \sin(4m+2)\alpha_0\cos(4m+2)\varphi \right\} \tag{10}$$

Summing up these two components by imposing that the 1st contribution is 0 at $r = r_i$ and the 2nd is 0 at $r = (r_i + w)$ we get the azimuthal field $B_\varphi$ within the coil $(r_i \leq r \leq r_i + w)$:

$$B_\varphi(r, \varphi) = \frac{j\mu_0}{2\pi} \left\{ 4\ln \left( \frac{r_i + w}{r} \right) - r^{-3}(r^4 - r_i^4) \right\} \sin(2\alpha_0)\cos2\varphi + \sum_{m=1}^{\infty} \frac{2^{r-4m+1}}{m(4m+2)} [(r_i + w)^{-4m} - r_i^{-4m}] + \frac{2^{r-4m-3}}{(m+1)(4m+2)} [(r_i + w)^{4m+4} - r_i^{4m+4}] \sin(4m+2)\alpha_0\cos(4m+2)\varphi \right\} \tag{11}$$
The equations of the magnetic force resultants in the cartesian system are as follows (see Appendix A):

\[
F_x = -j^2 \mu_0 \cos \alpha_0 \sin^2 \alpha_0 \frac{f_x(r_i^4, w^4)}{9\pi(r_i + w)}
\]

\[
F_y = j^2 \mu_0 \sin(2\alpha_0) \frac{f_y(r_i^4, w^4)}{18\pi(r_i + w)}
\]

(12) (13)

2.3 E.m. forces in \( \cos \varphi \) vs. sector model

The two formulations provide an estimation of forces whose ratio is the following:

\[
F_{x,sec.}/F_{x,\cos \varphi} = \frac{9\cos \alpha_0 \sin \alpha_0}{\pi}
\]

(14)

For a sector coil at 30 degrees, the latter expression is equal to 1.24. In the same way, the ratio of \( F_y \) due to the different approaches is equal to (eq: 49-51):

\[
F_{y,sec.}/F_{y,\cos \varphi} = 4\sin(2\alpha_0)/\pi \sim 1.102
\]

(15)

3 Comparison with the Finite Element model

3.1 Magnetic Field

In order to verify the validity of the analytical formulae given in sections 2.1 and 2.2, a magnetic finite element model has been built with an aperture \( r_i=84 \text{ mm} \), a coil width \( w=20 \text{ mm} \) and a constant current density \( j=1000 \text{ A/mm}^2 \).

Figure 2: \( B_r \) distribution within the aperture and inside the coil for a sector winding of \( r_i=84\text{mm}, w=20\text{mm} \).

(a) \( B_r @ 30 \text{ mm} \) within the useful aperture. (b) \( B_r @ 92 \text{ mm} \) within the coil.

The results obtained can be summarized as follows:
- the formulae for a sector coil have been evaluated considering firstly the main term of the series expansion of the magnetic field, and then adding the second one (i.e. \( m=1 \)). The first term only provides a good description of the magnetic field
- the sector coil approach gives reliable results for the field produced within the aperture and outside the coil. The \( \cos \varphi \) model shows 10% maximum difference from the numerical model due the different current distribution
- the \( \cos \varphi \) approach gives a difference from the sector coil of about 10-20%.
Due to the discrepancy between the field values within the coil, an analysis of the magnetic energy $U_m$ has been done to verify that the field inside the coil leads to the same energy integral. The magnetic energy stored in a sector coil of volume $V$ is given by:

$$U_m = \int_V \frac{B^2}{2\mu_0} \, d\tau$$

(16)

Explicit equations are given in Appendix B. The discrepancy between FEM of a sector coil and the analytical approximation of both a sector and a $\cos \varphi$ coil have been estimated for different ratios $w/r_i$ (0.5, 1, 2) and aperture radii of 15, 30, 60 mm.

<table>
<thead>
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<th>$w$</th>
<th>Ansys</th>
<th>Sector Coil</th>
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<td>233</td>
</tr>
<tr>
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<td>1659</td>
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</tr>
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</tr>
<tr>
<td>60</td>
<td>120</td>
<td>3125300</td>
<td>3028581</td>
</tr>
</tbody>
</table>

Table 1: Magnetic Energy $U_m$ for a $[0 - 30^\circ]$ sector coil: comparison between numerical results (ANSYS) and sector coil approach.

As we can see in table 1 the average difference between numerical and analytical results given by the sector coil formula is about 5%, while the $\cos \varphi$ formula leads to percentage difference of about 20%.

3.2 Magnetic forces

The resultants of the e.m. forces have been analytically computed for different geometrical layouts, and then compared to the results given by the FE model in air. In order to reduce possible errors, the following model parameters have been optimized:
– width of the air to be placed externally to the coil
– radial and azimuthal mesh of the coil
– radial and azimuthal mesh of the air

The model has been studied for the following sets of radial dimensions and coil widths:

– \( r_i: [14, 28, 56, 84, 112, 140, 168, 196] \) mm
– \( w: [5, 10, 15, 20, 25, 30, 35, 40] \) mm

all the possible combinations have been explored for a total of 64 permutations.

Figure 4: e.m. forces varying the geometrical parameters. Figures (a)-(c): the coil width has been set to 30mm, varying the aperture from 14 to 196mm. Figures (b)-(d): \( r_i = 28\)mm, the coil width varies from 5 to 40mm.

The current density has been set as a constant (\( j=1000 \) A/mm\(^2\)) for the whole set of configurations, neglecting the real possibility of using such a current with the layout proposed. Notwithstanding the approximation shown for the magnetic field, the magnetic forces (sector coil approach) show a good match with numerical data, revealing a general trend that can be summarized as follows:

– \( F_{x,y} \) vs. \( r_i \): for coil widths smaller than 20 mm the relation

\[
\frac{F_{x,y}}{F_{x,y,\text{ref}}} = \frac{Kr}{r_{\text{ref}}} \tag{17}
\]

where \( F_{x,y,\text{ref}} \) is a reference value for a given aperture and coil width. The trend is approximately linear with the aperture radius and \( K=1 \); if the width increases (up to 40 mm), the relation remains linear, but with \( K \) lower then unity.
\( F_{x,y} \) vs. \( w \): for small radii (14 mm) the relation
\[
\frac{F_{x,y}}{F_{x,y,ref}} = K \left( \frac{w}{w_{ref}} \right)^2
\]

is a parabolic function of the width; \( K \geq 1 \) for apertures smaller than 60mm, decreasing for larger apertures.

### 4 Mechanical stresses

The e.m. forces distribution produce a squeeze of the coil in the azimuthal direction as well as they push the coil outward leading to a compressive state both inside the coil and on the mechanical confinement structure (i.e. collars). By considering the stress balance in a sector winding element and neglecting the effect of shear components (see Appendix C), one can derive the expression of the azimuthal stress for a 30° sector coil as follows:

\[
\text{Sector: } \sigma_{\varphi,s} = -j^2\mu_0\sqrt{3} \frac{1}{16\pi r^2} \left[ r^4 - r_1^4 + 4r^4 \ln \left( \frac{r_i + w}{r} \right) \right]
\]

\[
\text{Cos} \varphi: \sigma_{\varphi,\cos \varphi} = -j^2\mu_0 \frac{32}{32r^2} \left[ r^4 - r_1^4 + 4r^4 \ln \left( \frac{r_i + w}{r} \right) \right]
\]

The stress profile, the position and the maximum value of the compressive stress are well represented (see fig. 5). The expression of the radial stress produced at the interface with the mechanical structure is as follows:

\[
\text{Sector: } \sigma_{r,s} = -j^2\mu_0 \sin(2\alpha_0) \frac{36\pi}{(r_i + w)^2} f_{pr}(r_i^4, w^4, \varphi)
\]

\[
\text{Cos} \varphi: \sigma_{r,\cos \varphi} = -j^2\mu_0 \frac{144}{(r_i + w)^2} f_{pr}(r_i^4, w^4, \varphi)
\]

where \( f_{pr}(r_i^4, w^4, \varphi) \) is defined in Appendix C. The numerical model has been modified in order to perform a coupled analysis (magnetic and mechanical). A constraint along the azimuthal direction has been applied on the coil mid plane, thus reproducing the structural symmetry with the lower coil block. The winding has been constrained along the radial direction, thus simulating the contact with the retaining structure, i.e. the collar, with an infinite stiffness.

By comparing the numerical results to the analytical ones (fig. 5-6), it can be observed that:

- the analytical formulae give a good approximation on the maximum compressive stress overestimating the absolute value of about 5%. The position of the maximum is evaluated with a maximum difference of about -10% (small apertures and big widths). A larger error is committed in estimating the value of the stress at the inner radius. This effect is mainly due to the fact that neglecting the shear stress in balance equations, we do not take into account the role that the material plays in the distribution of the compressive stress
- the maximum radial stress value is at the pole and is overestimated of about +10% in the worst case. This occurs for small apertures and large coil widths. For larger apertures, the stress profile diverges from the numerical one, overestimating the stress towards the coil mid-plane, underestimating it at the pole.
Figure 5: $\sigma_\varphi$ distribution on the coil mid plane. (a)$r_i=28\text{mm}, w=30\text{mm}$; (b) $r_i=14\text{mm}, w=40\text{mm}$.

Figure 6: $\sigma_r$ distribution on the outer coil edge. (a)$r_i=28\text{mm}, w=30\text{mm}$; (b) $r_i=196\text{mm}, w=5\text{mm}$.

5 Influence on the stress distribution induced by an anisotropic material

The formulae derived before are applicable in case of isotropic material and are not affected by change of the Young’s modulus $E$. In this section, the mechanical stresses in case of an orthotropic material ($E_\varphi = E_z \neq E_r$) are studied. This approach is more representative of a superconducting cable. The reference Young’s modulus is about 13 GPa, which typical of a Nb-Ti superconductor; then different ratios $E_r/E_\varphi$ – respectively: $0.5, 1, 2, 4, 6, 8$ – have been imposed to study the stress distribution on the coil mid plane, i.e. the azimuthal internal stress, and the radial stress on the contour line, i.e. on the contact area between the coil and the collar.

The study has been carried out analyzing the maximum value assumed by both the distribution, since the maximum should be kept low to avoid any degradation of the conductor. It has been found out that the shift between the isotropic case and the orthotropic one in terms of $|\sigma_\varphi|$ depends on the $w/r_i$ ratio; nevertheless it is always less then 2.5%. Concerning the position of the maximum compressive stress $r(\sigma_{\varphi,max})$, the difference with respect to the isotropic case depends on the aspect ratio $w/r_i$, being in any case less than 10%. In general, the larger is $E_r/E_\varphi$, the larger is the error committed.
Figure 7: $|\sigma_\varphi|$ distribution on the coil mid plane. The Young’s modulus has been increased in the radial direction form an initial value $E_y$ of about 13GPa.

Figure 8: $|\sigma_r|$ distribution on the collar contact surface. The Young’s modulus has been increased in the radial direction form an initial value $E_y$ of about 13GPa.

6 Forces and related stresses induced by the critical current

Here we introduce the critical current density expressions for both Nb-Ti and Nb$_3$Sn conductors into the equations for the e.m. forces and mechanical stress. The aim of this section is to address the mechanical stress produced at the short sample condition for a given layout (sector coil approach).

6.1 Nb-Ti

In a superconducting quadrupole $j$ is limited by the critical current density $j_c$, i.e. the current corresponding to a peak field on the critical surface. In order to estimate the critical current, following the approach of [2] we define the lay-out parameters:

- critical gradient: $G = \gamma(r_i, w)j$
- peak field: $B_p = \beta(r_i, w)j$

For a sector coil of $30^\circ$ one has:

$$\gamma(r_i, w) = \gamma_0 \ln \left(1 + \frac{w}{r}\right)$$
and $\beta$ is well fit by an empirical expression:

$$\beta(r_i, w) = r_i \gamma_0 \ln \left(1 + \frac{w}{r_i}\right) \lambda(r_i, w) = r_i \gamma_0 \ln \left(1 + \frac{w}{r_i}\right) \left(a_1 \frac{r_i}{w} + 1 + a_1 \frac{w}{r_i}\right)$$  \hspace{1cm} (26)

where $a_1, a_1, \gamma_0$ are constants related to the 30° sector layout. In this study they are set to:

$$a_1 = 0.06 \quad a_1 = 0.1 \quad \gamma_0 = 0.693$$  \hspace{1cm} (27)

The Nb-Ti critical surface is fit by a linear relation:

$$j_{sc,c} = \kappa c (B^*_{c2} - B) \quad B < B^*_{c2}$$ \hspace{1cm} (28)

The equation is written for the overall current density $j$, i.e. the current divided by the area of the insulated conductor. The filling ratio $\kappa$ is a dilution factor taking into account the presence of copper, voids and insulation in the coil; $\kappa$ is equal to 1 for a coil made only of superconductor and in the real case is in a range between 0.23 and 0.35 [2]. The factor $c$ is the slope of the fitting function. Introducing the critical surface fit in (23)-(27), we obtain the expression for the critical current of a sector winding of inner radius $r_i$ and width $w$ made of Nb-Ti conductor:

$$j_{c,Nb-Ti} = \kappa c B^*_{c2} \gamma_0 \ln \left(1 + \frac{w}{r_i}\right) \frac{1}{1 + \kappa c r_i \lambda(r_i, w) \gamma_0 \ln \left(1 + \frac{w}{r_i}\right)}$$ \hspace{1cm} (29)

And the critical gradient $G_c = \gamma(r_i, w)j_c$ can be obtained by multiplying (29) by (23):

$$G_c(r, w) = \frac{\kappa c B^*_{c2} \gamma_0 \ln \left(1 + \frac{w}{r_i}\right)}{1 + \kappa c r_i \lambda(r_i, w) \gamma_0 \ln \left(1 + \frac{w}{r_i}\right)}$$ \hspace{1cm} (30)

The parameters of a Nb-Ti superconducting cable are listed in table 2 [2].

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<tr>
<th>Temp (K)</th>
<th>1.9</th>
<th>4.4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c$ (A/Tmm$^2$)</td>
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<td>600</td>
</tr>
<tr>
<td>$B_{c2}$ (T)</td>
<td>13</td>
<td>10</td>
</tr>
</tbody>
</table>

Table 2: Nb-Ti: $j_c$ vs. $B$ characteristics.

The distribution of the e.m. forces has been analyzed by setting different aspect ratios $w/r_i$, varying the aperture radius and coil width independently as well as the dilution factor $\kappa$. As a general remark, if we vary $\kappa$ starting from a given configuration, we have that the e.m. forces vary with:

$$\frac{F_{x,y}(k_0)}{F_{x,y}(k_1)} \sim \left(\frac{\kappa_0}{1 + C(r_i, w)\kappa_0}\right)^2 \left(\frac{\kappa_1}{1 + C(r_i, w)\kappa_1}\right)^{-2}$$  \hspace{1cm} (31)

The equations of e.m. forces have been applied to different real Nb-Ti quadrupoles layouts at short sample current, showing a general good agreement within 10%, exception made for the ISR quadrupole, where the current density law is less accurate. The magnetic forces have been computed in ROXIE.
Figure 9: $F_{x,y}$ as a function of the coil width assuming $\kappa$ equal to 0.25.

Table 3: Characteristics for eight Nb-Ti quadrupoles cross sections; the last three quadrupoles have current grading (no iron; computations made at short sample current).

<table>
<thead>
<tr>
<th>r_i (mm)</th>
<th>w_{eq} (mm)</th>
<th>$\kappa$</th>
<th>T (K)</th>
<th>j (A/mm²)</th>
<th>$G_c$ (T/m)</th>
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7 Maximum azimuthal stress as a function of the critical current density

As for the magnetic forces, we can compute the value of the azimuthal stress on the coil mid-plane as a function of the critical current density.

The maximum stress inside the coil can be computed from the analytic expression
<table>
<thead>
<tr>
<th></th>
<th>Fx (MN/m)</th>
<th>Fy (MN/m)</th>
<th>Fx,an (MN/m)</th>
<th>Fy,an (MN/m)</th>
<th>%Dfx</th>
<th>%Dfy</th>
</tr>
</thead>
<tbody>
<tr>
<td>LHC-MQ</td>
<td>0.69</td>
<td>-1.22</td>
<td>0.63</td>
<td>-1.17</td>
<td>-8.9</td>
<td>-4.1</td>
</tr>
<tr>
<td>LHC-MQM</td>
<td>0.38</td>
<td>-0.73</td>
<td>0.34</td>
<td>-0.70</td>
<td>-10.2</td>
<td>-4.4</td>
</tr>
<tr>
<td>RHIC MQ-ARC</td>
<td>0.09</td>
<td>-0.21</td>
<td>0.08</td>
<td>-0.20</td>
<td>-8.5</td>
<td>-5.9</td>
</tr>
<tr>
<td>HERA MQ</td>
<td>0.30</td>
<td>-0.61</td>
<td>0.27</td>
<td>-0.58</td>
<td>-9.7</td>
<td>-4.6</td>
</tr>
<tr>
<td>ISR MQ</td>
<td>1.22</td>
<td>-2.53</td>
<td>0.93</td>
<td>-2.17</td>
<td>-23.4</td>
<td>-14.1</td>
</tr>
<tr>
<td>Tevatron MQ</td>
<td>0.17</td>
<td>-0.35</td>
<td>0.15</td>
<td>-0.33</td>
<td>-9.7</td>
<td>-5.4</td>
</tr>
<tr>
<td>LHC-MQXA</td>
<td>1.10</td>
<td>-2.04</td>
<td>1.04</td>
<td>-1.93</td>
<td>-5.1</td>
<td>-5.2</td>
</tr>
<tr>
<td>LHC-MQXB</td>
<td>0.76</td>
<td>-1.49</td>
<td>0.72</td>
<td>-1.41</td>
<td>-5.4</td>
<td>-5.4</td>
</tr>
</tbody>
</table>

Table 4: Analytical force estimation and comparison with eight Nb-Ti quadrupoles; the last three quadrupoles have current grading (no iron; computations made at short sample current).

Figure 10: $\sigma_{\varphi,\text{max}}$ as a function of the aperture radius $r_i$ (a) and of the coil width (b), assuming $\kappa$ equal to 0.25 (LHC-MQ cable).

derived in Section 4, eq. 19.

$$
\sigma_{\varphi}(j_c) = -\frac{\sqrt{3}j_c^2\mu_0}{16\pi r^2} \left[ r^4 - r_i^4 + 4r^4\ln\left(\frac{r_i + w}{r}\right) \right]
$$  (32)
One has to solve a transcendental function, recurring to the Lambert W-Function (see Appendix D):

\[
r = \exp \left[ -\frac{1}{4} + \frac{1}{4} \text{ProductLog} \left( \frac{e r_i^4}{(r_i + w)^4} \right) \right] (r_i + w)
\]

(33)

So the maximum azimuthal stress on the coil mid plane:

\[
\sigma_{\phi,\text{max}} = -\frac{\sqrt{3} j_c^2 \mu_0}{16 \pi (r_i + w)^2} \left[ -r_i^4 + e^{-1+\text{ProductLog} \left( \frac{e r_i^4}{(r_i + w)^4} \right)} (r_i + w)^4 + 4e^{-1+\text{ProductLog} \left( \frac{e r_i^4}{(r_i + w)^4} \right)} \cdot (r_i + w)^4 \ln \left( e^{\frac{1}{4} \text{ProductLog} \left( \frac{e r_i^4}{(r_i + w)^4} \right)} \right) \right]
\]

(34)

In fig. 10 we plot the maximum stress (\(\kappa=0.25\), LHC-MQ cable), varying the aperture radius for different values of coil width: 20, 30, 40, 60 mm respectively and viceversa, setting the coil width equal to 30, 40, 50, 60 and 70 mm.

Since the most important design parameter for a quadrupole is the gradient, an analysis of \(|\sigma_{\phi,\text{max}}|\) vs. the critical gradient \(G_c\) has been done. Quadrupole apertures between \(r_i=20\) mm and \(r_i=70\) mm have been analyzed; coil widths \(w\) have chosen between 5 mm up to the value corresponding to a saturation of the critical gradient, which in our case is one-two times the inner radius.

The curves in fig. 11 correspond to quadrupoles with the same aperture and different coil widths. For larger coil widths one obtains larger \(G_c\) (in the analyzed range). For small apertures, larger coil widths and larger \(G_c\) correspond to a saturation of the stress values. On the other hand, for very large apertures the stress reaches a peak for a given coil width and then decreases; this means that adding more material and more cable one can reduce stress and still gain in gradient.

The results obtained from the analytical approach are then compared to the numerical ones given by a mechanical model of the winding built in ANSYS (see fig. 11), revealing a good agreement with the latter ones. In general in the largest analyzed case \(r_i=70\) mm it is just below 110 MPa.

Since the function of the maximum stress is rather complicated, we can make considerations on the stress at the outer radius because it follows an analogous trend as \(\sigma_{\phi,\text{max}}\) (see fig. 12). This can help in understanding the behavior of the maximum stress without giving indications on the value assumed by the stress at the local maximum point.

The stress at the outer radius reads:

\[
|\sigma_{\phi,r_{\text{ext}}}| = \frac{\sqrt{3} j_c^2 \mu_0}{16 \pi (r_i + w)^2} \left[ (r_i + w)^4 - r_i^4 \right] \cong \phi_1 \phi_2
\]

(35)

where: \(\phi_1 \propto j_c^2\) and \(\phi_2 \propto \left[ (r_i + w)^4 - r_i^4 \right] / (r_i + w)^2\).

The limits for small and large widths are as follows:

\[
w \to 0 \quad \phi_1 \phi_2 = r_i w \to 0; \quad w \to \infty \quad \phi_1 \phi_2 = 1/(w \ln(w)) \to 0
\]

(36)
Figure 11: $|\sigma_{\varphi,\text{max}}|$ vs. the critical gradient $G_c$, varying the dilution factor $\kappa$, equal to 0.25 (a) and 0.35 (b).

Between these limits, the stress behaves differently according on the aperture, the coil width and the dilution factor $\kappa$ and depends on how they combine in the product $\varphi_1 \varphi_2$; for an increasing stress the geometrical parameter $\varphi_2$ rules over the decreasing $j_c$. When the stress decreases, it is the decrease of $j_c$ due to the higher $B_p$ ruling over the augmented dimension of the coil. The stress shows a local maximum depending also on the dilution factor $\kappa$ (see fig. 13): the higher is $\kappa$, the higher $j_c$ and so $\varphi_1$. Therefore, the higher is the dilution factor, the smaller is the ratio $w/r_i$ where the stress changes in behavior.

8 Nb3Sn

The critical surface for the Nb3Sn can be approximated by an hyperbolic law as follows:

$$j_c(r, w) = \kappa c \left( \frac{B_{c2}^*}{B} - 1 \right) \quad B < B_{c2}^*$$

which is accurate within 5% with respect to the usual Summer law [8] between 11 and 17
Figure 12: Comparison between $\sigma_{\phi,r_o}$ and $\sigma_{\phi,max}$ for two different apertures: 30 and 60mm respectively.

Figure 13: The aspect ratio $w/r_i$ where the maximum value of $\sigma_{\phi,max}$ occurs has been determined as a function of the aperture radius. The higher the dilution factor the lower is the aspect ratio where the compressive stress stands at his maximum value.

T at 1.9 K, and has the advantage of allowing an explicit solution for the critical current:

$$ j_c(r, w) = \frac{kc}{2} \left[ \sqrt{\frac{4B^*_c}{kc\beta}} + 1 - 1 \right] $$

(38)

| Temp (K) | 1.9 | 4.4 |
| c (A/Tmm$^2$) | 4000 | 3900 |
| B$_{c2}$ (T) | 23.1 | 21 |

Table 5: Nb$_3$Sn: $j_c$ vs. $B$ characteristics.

The same analysis performed for a Nb-Ti cable has been done for Nb$_3$Sn varying only the cable features, using the data in table 5, at 4.2K. In general, set a geometrical layout,
the increase of the e.m. forces for a Nb$_3$Sn cable are proportional to \((j_{c,Nb-Ti}/j_{c,Nb_3Sn})^2\).

![Graph](image)

**Figure 14:** $F_{x,y}$ varying the coil width, assuming $\kappa=0.3$ (CERN-Elin cable).

We can observe that with a dilution factor $\kappa$ equal to 0.3 (CERN-Elin cable) [10], the maximum stress for an aperture of 60 mm reaches the maximum value admissible for a Nb$_3$Sn cable. The same considerations made on the behavior of $\sigma_{\varphi,max}$ for a Nb-Ti cable apply for the Nb$_3$Sn, the only difference is that a maximum of $\sigma_{\varphi,max}$ appears for $r_i > 50$ mm.

### 9 Comparison between Nb – Ti and Nb$_3$Sn

The aim of this section is to compare the performances of a sector winding constituted by either Nb-Ti or Nb$_3$Sn superconducting cable. By setting a particular geometrical layout, the two superconductors have been first compared at their nominal operating temperatures: 1.9K (Nb-Ti) and 4.2K (Nb$_3$Sn) separately, then at the same temperatures.

For instance, cooling a Nb$_3$Sn from 4.2K to 1.9K leads to a low increase in $j_c$ of about 7%. At 1.9K, being constant the coil layout and the dilution factor as well, a Nb$_3$Sn cable produces a $j_c$ higher than a Nb-Ti cable one of about 40%, on the other hand the peak stress doubles. Nevertheless, the same gradient can be achieved with a thinner cable:
Figure 15: Comparison of $|\sigma_{\phi,max}|$ vs. $G_c$ ($\kappa=0.3$) between analytical and numerical results.

Figure 16: $|\sigma_{\phi,max}|$ vs. the critical gradient $G_c$: comparison between Nb-Ti and Nb$_3$Sn winding in a sector coil at 1.9K.

e.g. to obtain 280 T/m with an aperture of 30mm, a coil width of 14 mm is needed (Nb$_3$Sn cable) instead of 40 mm (Nb-Ti cable, $\kappa=0.25$).

10 Iron effect

In this section the formulae for the magnetic field due to a magnetic coil with an iron yoke placed at a distance $R_s$ are evaluated. The approaches followed are exactly the same as before, first starting with a theoretical $\cos\phi$ distribution and secondly considering the case of a constant current distribution in a sector coil. Only the results of the sector coil are proposed here since they better represent the magnetic field distribution (see Appendix E for details about $\cos\phi$ approach).

10.1 Sector coil

The effect of the iron yoke can be evaluated by using the "Image current" approach: the additional field is produced by an imaginary coil enclosed between radii $r'_i = R^2_s/r_i$ and $r'_o = R^2_s/r_o$. The angular limits are the same as the quadrupole coil:
Figure 17: $|\sigma_{\phi,\text{max}}|$ vs. the critical gradient $G_c$: comparison between Nb-Ti and Nb$_3$Sn winding in a sector coil at 4.2K.

\[
\begin{align*}
\begin{bmatrix} B_{r,\text{iron}} \\ B_{\phi,\text{iron}} \end{bmatrix} &= \begin{bmatrix} \frac{\mu_r - 1}{\mu_r + 1} j \mu_0 & \frac{\infty}{\sum_{m=0}^{\infty}} \frac{r^{4m+1}(r_o^{4(m+1)} - r_i^{4(m+1)})}{(m + 1)(4m + 2)R_s^{4(2m+1)}} \sin(4m + 2)\alpha_0 \\
\sin(4m + 2)\phi & \cos(4m + 2)\phi \end{bmatrix} \\
\end{align*}
\]  

(39)

The results obtained by the analytical approximations have been compared to the ones given by a FE model, where the iron yoke has been implemented. The condition of perpendicularity of the field lines has been imposed on the model outer boundary. In the analytical approximations, $\mu_r$ has been set equal to 3000, thus describing a not saturated iron yoke. As in the case of coil in air, the sector coil approximation better represents the field inside the coil, even if the agreement with numerical results is always not so good.

10.2 Critical current density approximation

An iron yoke has the main function of closing the magnetic circuit, thus increasing the magnetic field produced for the same current density. This means that also the magnet critical current $j_c$ is reduced because the load line hits the critical surface at higher field. Since the analytical approximation of the field inside the coil is not reliable, we cannot analytically compute the peak field, neither in the case of coil in air, nor when an iron yoke is introduced. In the first case the peak field can be computed by using the formulae proposed [2], but when an iron screen is present a new formulation of the peak field and current density must be found.

The gradient and the peak field are assumed to be linear functions of the current $j$, i.e. we neglect saturation. At first order one can expect that the gradient and the peak field have a similar relative increase when an iron yoke is added.

In fig. 19-(a) we plot the difference between the relative increase in the peak field $\Delta B_p/B_p$ and the relative increase of the gradient in the aperture $\Delta G/G$, normalized to $\Delta G/G$, for different ratios $w/r_i$ and collar thickness $w_{\text{coll}}$ (fig. 18). The increment of the gradient in the aperture $\Delta G$ has been analytically derived using the formulae of a sector coil with constant current density and iron yoke, whereas the increment of the field $\Delta B_p$ has been computed through the numerical code ANSYS.
It can be stated that:

– the relative difference mainly depends on the ratio \( w/r_i \), and is practically independent on the distance between the coil and the iron (i.e. the collar width), see fig. 19-(a);

– the 1\(^{st}\) order approximation \( \Delta B_p/B_p = \Delta G/G \) is correct within 10% in a large part of the domain of interest \( 0.5 < w/r_i < 1 \). Since this difference has to be applied on an increment, it represents a small error.

The critical current density formula has to be reviewed in order to take into account the effect of the iron yoke. Since the magnetic field reached is higher than the one obtained from a simple coil in air, the current density in the winding has to be lower in order to respect the limit imposed by the superconductor critical curve. This means that by keeping the peak field, we will have an increased factor \( \beta \) (magnetic field per unit current density).

The curve in fig. 19-(b), obtained by averaging the relative increment at a given aspect ratio, has been empirically fit with:

\[
\frac{\Delta B_p}{B_p} - \frac{\Delta G}{G} = \frac{\Delta G}{G} \left[ p_0 + p_1 \left( \frac{w}{r_i} \right)^q \right]
\]

(40)

where:

\[
p_0 = 30 \quad p_1 = 37.4 \quad q = 0.71
\]

(41)

Though the peak field is not well described by the analytical approach, the field in the center of the aperture is in good agreement with the numerical data, due to the reliability of the field expression inside the aperture. The field gradient can be defined as:

\[
G_{\text{iron}} = \frac{j \mu 10^6}{2 \pi} \left[ \left( \frac{\mu_r - 1}{\mu_r + 1} \right) \left( \frac{(r_i + w)^4 - r_i^4}{R_s^4} \right) + 4 \ln \left( \frac{r_i + w}{r_i} \right) \right] \sin(2\alpha_0)
\]

(42)

It is now possible to derive the expression of \( \beta_{\text{iron}} \) introducing (24) and (42) into (40) as follows (lengths are expressed in [mm]):
Figure 19: Difference between the relative increase in the peak field $\Delta B_p/B_p$ and the relative increase of the gradient in the aperture $\Delta G/G$, normalized to $\Delta G/G$. The relation with the aspect ratio $w/r_i$ is independent of the yoke radius $R_s$.

$$
\beta_{\text{iron}} = 10^{-5} r_i \lambda (r_i, w) \left\{ \gamma_0 \left( p_0 + p_1 \left( \frac{w}{r_i} \right)^q \right) \ln \left( 1 + \frac{w}{r_i} \right) + \frac{10^7 \mu_0 \sin(2\alpha_0)}{\left( \mu_r + 1 \right) R_s^4} \left( 4.456 + 2.382 \left( \frac{w}{r_i} \right)^q \right) \right. \\
\left. \left[ (\mu_r - 1) w (r_i^3 + 1.5 r_i^2 w + r_i w^2 + 0.25 w^3) + (\mu_r + 1) R_s^4 \ln \left( 1 + \frac{w}{r_i} \right) \right] \right\} 
$$

(43)

Now the new expression of $j_c$ can be derived introducing $\beta_{\text{iron}}$ in (29). This new current density can be used to define the e.m. forces as well as the peak stress acting on the coil mid-plane, both for a Nb-Ti and Nb$_3$Sn sector coil. By increasing the collar width the contribution of the iron yoke decreases, whilst the behavior of the magnetic forces is antithetic, due to the different field distribution in the coil.

The maximum stress was computed for two different apertures, scanning finely the coil width, for an ironless case and for a case with iron having a collar thickness of 20 mm.
Table 6: Magnetic forces computations for some superconducting magnets and comparison with analytical values. The data are derived at nominal operating conditions, assuming an infinitely permeable iron. Except for LHC-MQ and LHC-MQM, the magnetic forces at nominal operating conditions have been computed using the program ROXIE.

<table>
<thead>
<tr>
<th>Magnet Type</th>
<th>Rs (mm)</th>
<th>w_{coil} (mm)</th>
<th>F_x (MN/m)</th>
<th>F_y (MN/m)</th>
<th>F_{x,an} (MN/m)</th>
<th>F_{y,an} (MN/m)</th>
<th>%Df_x</th>
<th>%Df_y</th>
</tr>
</thead>
<tbody>
<tr>
<td>LHC-MQ</td>
<td>90</td>
<td>31</td>
<td>0.537</td>
<td>-0.732</td>
<td>0.515</td>
<td>-0.731</td>
<td>-4.2</td>
<td>-0.1</td>
</tr>
<tr>
<td>LHC-MQM</td>
<td>102</td>
<td>27</td>
<td>0.309</td>
<td>-0.446</td>
<td>0.300</td>
<td>-0.436</td>
<td>-2.9</td>
<td>-2.3</td>
</tr>
<tr>
<td>RHIC MQ-ARC</td>
<td>55</td>
<td>5</td>
<td>0.099</td>
<td>-0.0842</td>
<td>0.092</td>
<td>-0.077</td>
<td>-6.7</td>
<td>-8.3</td>
</tr>
<tr>
<td>HERA MQ</td>
<td>80</td>
<td>24</td>
<td>0.148</td>
<td>-0.187</td>
<td>0.134</td>
<td>-0.180</td>
<td>-9.5</td>
<td>-3.8</td>
</tr>
<tr>
<td>ISR MQ</td>
<td>176</td>
<td>22</td>
<td>0.911</td>
<td>-0.838</td>
<td>0.754</td>
<td>-0.685</td>
<td>-17.2</td>
<td>-18.2</td>
</tr>
<tr>
<td><em>Tevatron MQ</em></td>
<td>101</td>
<td>41</td>
<td>0.137</td>
<td>-0.209</td>
<td>0.121</td>
<td>-0.201</td>
<td>-11.4</td>
<td>-4.0</td>
</tr>
<tr>
<td>LHC-MQXA</td>
<td>92</td>
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<td>1.635</td>
<td>-1.573</td>
<td>1.356</td>
<td>-1.343</td>
<td>-17.1</td>
<td>-14.6</td>
</tr>
<tr>
<td>LHC-MQXB</td>
<td>92</td>
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<td>0.868</td>
<td>-1.13</td>
<td>0.704</td>
<td>-0.925</td>
<td>-18.9</td>
<td>-18.1</td>
</tr>
</tbody>
</table>

Figure 20: $F_{mag}$ for a Nb-Ti sector winding ($\kappa=0.254$ - LHC MQ) with infinitely permeable iron yoke. The coil width is equal to 15.4mm.
Results are shown in fig. 21 for Nb-Ti and Nb$_2$Sn: the iron acts as a larger coil width, but the relation stress/gradient remains the same.

![Graph](image)

Figure 21: $|\sigma_{\phi,\text{max}}|$ vs. $G_c$ for a Nb-Ti (a) and Nb$_2$Sn (b) sector winding ($\kappa=0.3$).

11 Conclusions

A simplified model of superconducting quadrupole has been analyzed, namely a 30° sector coil with uniform current density $j$. We outlined an analytical approach that allows to predict the stress distribution along the mid-plane with good precision, and that holds also for more realistic non-isotropic materials.

We computed the stress at the short sample condition, showing that it increases for larger quadrupole apertures, whereas the dependence on the coil thickness is more involved and for large apertures larger coils can give larger gradients and smaller stress. For the Nb-Ti the stress is below 100 MPa for apertures radii smaller than 60 mm ($\kappa=0.3$). For the Nb$_2$Sn the stress in aperture radii is smaller than 60 mm is below 150 MPa with the same cable features, which is taken as guideline for $\sigma_{\phi,\text{max}}$ before degradation of the superconducting properties of the material.

Introducing an iron screen, both the critical gradient and the peak stress increases to reach the same level as it would have in an ironless larger coil creating the same gradient.
So further studies can be accomplished on coil in air (easier to treat analytically) since the peak stress depends only on $G_c$.

References
APPENDIX A: E.M. FORCES

The equations of the e.m. forces produced by the magnetic field over a conductor of infinite area \( r \, dr \, d\phi \) are as follows:

\[
dF_r(r, \phi) = -j B_\phi r \, dr \, d\phi \\
dF_\phi(r, \phi) = j B_r r \, dr \, d\phi
\]

(44) (45)

We can now compute the components \( F_x \) and \( F_y \) referring to the Cartesian Reference System placed at the center of the aperture as follows:

\[
dF_x(r, \phi) = dF_r \cos \phi - dF_\phi \sin \phi \\
dF_y(r, \phi) = dF_r \sin \phi + dF_\phi \cos \phi
\]

(46) (47)

In order to define the resultant of the magnetic forces, one have to integrate magnetic field equations (46)-(47) for costheta winding and (8)-(9) for the sector coil between the geometrical limits of the winding. For the costheta we obtain:

\[
F_x = -\frac{j^2 \mu_0 \sin \alpha_0}{72 (r_i + w)} \left[ 2((r_i + w)^4 - 4(r_i + w)r_i^3 + 3r_i^4) \cos(2\alpha_0) + \\
+ 3\left( -(r_i + w)^4 + r_i^4 + 4(r_i + w)r_i^3 \ln \left( \frac{r_i + w}{r_i} \right) \right) \right]
\]

(48)

\[
F_y = \frac{j^2 \mu_0}{63 (r_i + w)} \left[ -5(r_i + w)^4 + 8(r_i + w)r_i^3 - 3r_i^4 + \\
+ ((r_i + w)^4 - 4(r_i + w)r_i^3 + 3r_i^4) \cos 3\alpha_0 + 12(r_i + w)r_i^3 \ln \left( \frac{r_i + w}{r_i} \right) + \\
+ 4(r_i + w) \cos \alpha_0 \left( (r_i + w)^3 - r_i^3 - 3r_i^3 \ln \left( \frac{r_i + w}{r_i} \right) \right) \right]
\]

(49)

whereas, for the sector coil:

\[
F_x = -\frac{j^2 \mu_0 \cos \alpha_0 \sin^2 \alpha_0}{9\pi (r_i + w)} \left[ 2((r_i + w)^4 - 4(r_i + w)r_i^3 + 3r_i^4) \cos(2\alpha_0) + \\
+ 3\left( -(r_i + w)^4 + r_i^4 + 4(r_i + w)r_i^3 \ln \left( \frac{r_i + w}{r_i} \right) \right) \right]
\]

(50)

\[
F_y = \frac{j^2 \mu_0 \sin(2\alpha_0)}{18\pi (r_i + w)} \left[ -5(r_i + w)^4 + 8(r_i + w)r_i^3 - 3r_i^4 + \\
+ ((r_i + w)^4 - 4(r_i + w)r_i^3 + 3r_i^4) \cos 3\alpha_0 + 12(r_i + w)r_i^3 \ln \left( \frac{r_i + w}{r_i} \right) + \\
+ 4(r_i + w) \cos \alpha_0 \left( (r_i + w)^3 - r_i^3 - 3r_i^3 \ln \left( \frac{r_i + w}{r_i} \right) \right) \right]
\]

(51)
APPENDIX B: MAGNETIC ENERGY

The magnetic energy stored in a coil of volume $V$ is as follows:

$$U_m = \int_V \frac{B^2}{2\mu_0} d\tau$$  \hspace{1cm} (52)

The resultant of magnetic field is given by:

$$\vec{B} = \vec{B}_r + \vec{B}_\varphi$$  \hspace{1cm} (53)

and the modulus is as follows:

Sector : $B = \sqrt{B_r^2 + B_\varphi^2} =
\begin{align*}
&= \frac{\sin(2\alpha_0)}{2\pi} \sqrt{\frac{j^2 \mu_0}{r^6}} \sqrt{\left(\left(r^4 - r_i^4\right) - 8r^4(r^4 - r_i^4)\cos4\varphi\ln\left(\frac{r_i + w}{r}\right) + 16r^8\ln\left(\frac{r_i + w}{r}\right)^2\right)} \\
&= 1 \sqrt{\frac{j^2 \mu_0}{r^6}} \sqrt{\left(\left(r^4 - r_i^4\right) - 8r^4(r^4 - r_i^4)\cos4\varphi\ln\left(\frac{r_i + w}{r}\right) + 16r^8\ln\left(\frac{r_i + w}{r}\right)^2\right)}
\end{align*}  \hspace{1cm} (54)

Cos$\varphi$ : $B = \sqrt{B_r^2 + B_\varphi^2} =
\begin{align*}
&= \frac{1}{8} \sqrt{\frac{j^2 \mu_0}{r^6}} \sqrt{\left(\left(r^4 - r_i^4\right) - 8r^4(r^4 - r_i^4)\cos4\varphi\ln\left(\frac{r_i + w}{r}\right) + 16r^8\ln\left(\frac{r_i + w}{r}\right)^2\right)}
\end{align*}  \hspace{1cm} (55)
APPENDIX C: MECHANICAL STRESS

By considering the stress balance in a sector winding element and neglecting the effect of shear components, one can define the relations between the stresses and volume e.m. forces ($F_r, F_\phi$) as follows:

\[ \sigma_\phi \, dr - d(r\sigma_r) + F_r r \, dr = 0 \]  
\[ -d\sigma_\phi + F_\phi r \, d\phi = 0 \]  \(56\)  \(57\)

By integrating eq. 57 between the coil angular limits (0, $\alpha_0$), we can get the stress resultant along the mid plane as a function of the radius r:

\[ \text{Sector : } \sigma_{\phi,s} = -j\frac{\mu_0}{\pi r^2} \cos(\alpha_0) \sin^3(\alpha_0) \left[ r^4 - r_i^4 + 4r^4 \ln\left(\frac{r_i + w}{r}\right) \right] \]  
\[ \text{Cos} \phi : \sigma_{\phi,\cos} = -j\frac{\mu_0}{8 r^2} \sin(\alpha_0) \left[ r^4 - r_i^4 + 4r^4 \ln\left(\frac{r_i + w}{r}\right) \right] \]  \(58\)  \(59\)

By replacing $\alpha_0$ with $\pi/6$, we can get the azimuthal stress for a 30° sector coil as follows:

\[ \text{Sector : } \sigma_{\phi,s} = -j\frac{\mu_0}{16\pi r^2} \sqrt{3} \left[ r^4 - r_i^4 + 4r^4 \ln\left(\frac{r_i + w}{r}\right) \right] \]  
\[ \text{Cos} \phi : \sigma_{\phi,\cos} = -j\frac{\mu_0}{32 r^2} \left[ r^4 - r_i^4 + 4r^4 \ln\left(\frac{r_i + w}{r}\right) \right] \]  \(60\)  \(61\)

The stress on the collar contact profile can be computed by introducing the expression of $\sigma_{\phi}$ in 56, considering that in this case $\sigma_{\phi}$ does not have to be computed as a resultant, but as a point value:

\[ \sigma_{\phi} = \int F_\phi r \, d\phi + \text{const}. \]  \(62\)

where const is set to have $\sigma_{\phi} = 0$ at a pole ($\pi/6$).

In this case, we obtain:

\[ \sigma_{\phi,s} = -j\frac{\mu_0}{4\pi r^2} (\cos(2\alpha_0) + \cos 2\phi) \left[ r^4 - r_i^4 + 4r^4 \ln\left(\frac{r_i + w}{r}\right) \right] \sin(2\alpha_0) \]  \(63\)
\[ \sigma_{\phi,\cos} = -j\frac{\mu_0}{16r^2} (\cos(2\alpha_0) + \cos 2\phi) \left[ r^4 - r_i^4 + 4r^4 \ln\left(\frac{r_i + w}{r}\right) \right] \]  \(64\)

By introducing the expression for the azimuthal stress on a sector winding element into 56, we can derive the expression for the radial reaction stress. By integrating between the inner and outer radius, we finally obtain:

\[ \text{Sector : } \sigma_{r,s} = -j\frac{\mu_0}{36\pi (r_i + w)^2} \int_{\phi} \int_{r_i}^{r_i + w} \frac{1}{r^4} f_{pr}(r_i, w, \phi) = \]  
\[ -j\frac{\mu_0}{36\pi (r_i + w)^2} \left\{ \left[ 12r_i^3 (r_i + w) \ln\left(\frac{r_i + w}{r_i}\right) - w(12r_i^3 + 42r_i^2 w + 28r_i w^2 + 7w^3) \right] \cos(2\alpha_0) + \right. \]  
\[ + \left[ 9w(2r_i + w)(2r_i^2 + 2r_i w + w^2) - 36r_i^3 (r_i + w) \ln\left(\frac{r_i + w}{r_i}\right) \right] \cos 2\phi \right\} \]  \(65\)
\[
\cos \varphi : \sigma_{r,\cos \varphi} = - \frac{j^2 \mu_0}{144(r_i + w)^2} f_{pr}(r^4_i, w^4, \varphi) = \\
= - \frac{j^2 \mu_0}{144(r_i + w)^2} \left\{ \left[ 12r_i^3(r_i + w) \ln \left( \frac{r_i + w}{r_i} \right) - w(12r_i^3 + 42r_i^2w + 28r_iw^2 + 7w^3) \right] \cos(2\alpha_0) - \\
+ \left[ 9w(2r_i + w)(2r_i^2 + 2r_iw + w^2) - 36r_i^2(r_i + w) \ln \left( \frac{r_i + w}{r_i} \right) \right] \cos 2\varphi \right\} 
\] (66)
APPENDIX D: MAXIMUM AZIMUTHAL STRESS

The equation of the compressive stress on the coil mid plane is as follows:

\[
\sigma_\varphi(j_c) = -\frac{\sqrt{3} j_c^2 \mu_0}{16\pi r^2} \left[ r^4 - r_i^4 + 4r^4 \cdot \ln\left(\frac{r_i + w}{r}\right) \right]
\]  

(67)

By deriving the eq. (67), we obtain:

\[
\frac{d\sigma_\varphi(j_c)}{dr} = -\frac{\sqrt{3} j_c^2 \mu_0}{8\pi r^3} \left[ -r^4 + r_i^4 + 4r^4 \cdot \ln\left(\frac{r_i + w}{r}\right) \right]
\]  

(68)

By imposing eq. (68) equal to 0 and \(r \neq 0\), we get the transcendental equation:

\[
r^4 - r_i^4 - 4r^4 \cdot \ln\left(\frac{r_i + w}{r}\right) = 0
\]  

(69)

that can be solved by mean of the Lambert W-Function (also called \textit{Product log} function), i.e. the inverse function of:

\[
f(W) = We^W
\]  

(70)

Rearranging eq. 69, we get:

\[
r^4 \cdot \left[ 1 + 4\ln\left(\frac{r}{r_i + w}\right) \right] = r_i^4
\]  

(71)

Multiplying both members times \(e/(r_i + w)^4\), we get:

\[
\frac{e}{(r_i + w)^4} \cdot r^4 \cdot \left[ 1 + 4\ln\left(\frac{r}{r_i + w}\right) \right] = r_i^4 \cdot \frac{e}{(r_i + w)^4}
\]  

(72)

that is equal to:

\[
e^{1+4\ln\left(\frac{r}{r_i + w}\right)} \cdot \left[ 1 + 4\ln\left(\frac{r}{r_i + w}\right) \right] = \frac{e \cdot r_i^4}{(r_i + w)^4}
\]  

(73)

The radius where the azimuthal stress is maximized is then given by:

\[
r = \exp\left[ -\frac{1}{4} + \frac{1}{4} \cdot \text{ProductLog}\left(\frac{e \cdot r_i^4}{(r_i + w)^4}\right) \right] \cdot (r_i + w)
\]  

(74)

by introducing this expression into eq. (67), we can get the maximum azimuthal stress on the coil mid plane:

\[
\sigma_{\varphi,\text{max}} = -\frac{\sqrt{3} j_c^2 \mu_0}{16\pi (r_i + w)^2} \left[ -r_i^4 + \right. \\
\left. + e^{-1+\text{ProductLog}\left(\frac{e \cdot r_i^4}{(r_i + w)^4}\right)} (r_i + w)^4 + 4e^{-1+\text{ProductLog}\left(\frac{e \cdot r_i^4}{(r_i + w)^4}\right)} \right] \\
\cdot (r_i + w)^4 \ln\left(\frac{1}{4} - \frac{1}{4} \cdot \text{ProductLog}\left(\frac{e \cdot r_i^4}{(r_i + w)^4}\right) \right)
\]  

(75)
APPENDIX E: IRON YOKE EFFECT

.1 Cosϕ winding

As it has been shown for the sector coil approach, in the same way can be derived the expressions for the cosϕ approach. The additional terms to be summed up to expressions 1, 2, 3 are as follows:

\[
\begin{align*}
\{ B_{r,iron} & \} = -\frac{\mu_r - 1}{\mu_r + 1} \frac{\mu_0 j r}{4 R_s} \ln \left( \frac{r_o^2 - r_i^2}{4 R_s^2} \right) \frac{\sin 2\varphi}{\cos 2\varphi} \\
\{ B_{\varphi,iron} & \} = \frac{\mu_r - 1}{\mu_r + 1} \frac{\mu_0 j r}{4 R_s} \ln \left( \frac{r_o^2 - r_i^2}{4 R_s^2} \right) \frac{\sin 2\varphi}{\cos 2\varphi}
\end{align*}
\]

The results in term of magnetic field distribution are shown in fig. 22. The same considerations made for the coil in air can hold in this case.

Figure 22: \( B_{iron} \) inside the coil \( r_i = 30 \text{ mm}, w = 30 \text{ mm}, R_s = 90 \text{ mm} \). The magnetic field has been evaluated at the pole \( \alpha_0 = \pi/6 \).