Contents

Preface XIII

Part One  Coherent States  1

1  Introduction  3
1.1  The Motivations  3

2  The Standard Coherent States: the Basics  13
2.1  Schrödinger Definition  13
2.2  Four Representations of Quantum States  13
2.2.1  Position Representation  14
2.2.2  Momentum Representation  14
2.2.3  Number or Fock Representation  15
2.2.4  A Little (Lie) Algebraic Observation  16
2.2.5  Analytical or Fock–Bargmann Representation  16
2.2.6  Operators in Fock–Bargmann Representation  17
2.3  Schrödinger Coherent States  18
2.3.1  Bergman Kernel as a Coherent State  18
2.3.2  A First Fundamental Property  19
2.3.3  Schrödinger Coherent States in the Two Other Representations  19
2.4  Glauber–Klauder–Sudarshan or Standard Coherent States  20
2.5  Why the Adjective Coherent?  20

3  The Standard Coherent States: the (Elementary) Mathematics  25
3.1  Introduction  25
3.2  Properties in the Hilbertian Framework  26
3.2.1  A “Continuity” from the Classical Complex Plane to Quantum States  26
3.2.2  “Coherent” Resolution of the Unity  26
3.2.3  The Interplay Between the Circle (as a Set of Parameters) and the Plane (as a Euclidean Space)  27
3.2.4  Analytical Bridge  28
3.2.5  Overcompleteness and Reproducing Properties  29
3.3  Coherent States in the Quantum Mechanical Context  30
3.3.1  Symbols  30
3.3.2  Lower Symbols  30
5.2 A Bayesian Probabilistic Duality in Standard Coherent States  70
5.2.1 Poisson and Gamma Distributions  70
5.2.2 Bayesian Duality  71
5.2.3 The Fock–Bargmann Option  71
5.2.4 A Scheme of Construction  72
5.3 General Setting: “Quantum” Processing of a Measure Space  72
5.4 Coherent States for the Motion of a Particle on the Circle  76
5.5 More Coherent States for the Motion of a Particle on the Circle  78

6 The Spin Coherent States  79
6.1 Introduction  79
6.2 Preliminary Material  79
6.3 The Construction of Spin Coherent States  80
6.4 The Binomial Probabilistic Content of Spin Coherent States  82
6.5 Spin Coherent States: Group-Theoretical Context  82
6.6 Spin Coherent States: Fock–Bargmann Aspects  86
6.7 Spin Coherent States: Spherical Harmonics Aspects  86
6.8 Other Spin Coherent States from Spin Spherical Harmonics  87
6.8.1 Matrix Elements of the $SU(2)$ Unitary Irreducible Representations  87
6.8.2 Orthogonality Relations  89
6.8.3 Spin Spherical Harmonics  89
6.8.4 Spin Spherical Harmonics as an Orthonormal Basis  91
6.8.5 The Important Case: $\sigma = j$  91
6.8.6 Transformation Laws  92
6.8.7 Infinitesimal Transformation Laws  92
6.8.8 “Sigma-Spin” Coherent States  93
6.8.9 Covariance Properties of Sigma-Spin Coherent States  95

7 Selected Pieces of Applications of Standard and Spin Coherent States  97
7.1 Introduction  97
7.2 Coherent States and the Driven Oscillator  98
7.3 An Application of Standard or Spin Coherent States in Statistical Physics: Superradiance  103
7.3.1 The Dicke Model  103
7.3.2 The Partition Function  105
7.3.3 The Critical Temperature  106
7.3.4 Average Number of Photons per Atom  108
7.3.5 Comments  109
7.4 Application of Spin Coherent States to Quantum Magnetism  109
7.5 Application of Spin Coherent States to Classical and Thermodynamical Limits  111
7.5.1 Symbols and Traces  112
7.5.2 Berezin–Lieb Inequalities for the Partition Function  114
7.5.3 Application to the Heisenberg Model  116
<table>
<thead>
<tr>
<th>Chapter</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>SU(1,1) or SL(2,R) Coherent States</td>
<td>117</td>
</tr>
<tr>
<td>8.1</td>
<td>Introduction</td>
<td>117</td>
</tr>
<tr>
<td>8.2</td>
<td>The Unit Disk as an Observation Set</td>
<td>117</td>
</tr>
<tr>
<td>8.3</td>
<td>Coherent States</td>
<td>119</td>
</tr>
<tr>
<td>8.4</td>
<td>Probabilistic Interpretation</td>
<td>120</td>
</tr>
<tr>
<td>8.5</td>
<td>Poincaré Half-Plane for Time-Scale Analysis</td>
<td>121</td>
</tr>
<tr>
<td>8.6</td>
<td>Symmetries of the Disk and the Half-Plane</td>
<td>122</td>
</tr>
<tr>
<td>8.7</td>
<td>Group-Theoretical Content of the Coherent States</td>
<td>123</td>
</tr>
<tr>
<td>8.7.1</td>
<td>Cartan Factorization</td>
<td>123</td>
</tr>
<tr>
<td>8.7.2</td>
<td>Discrete Series of SU(1,1)</td>
<td>124</td>
</tr>
<tr>
<td>8.7.3</td>
<td>Lie Algebra Aspects</td>
<td>126</td>
</tr>
<tr>
<td>8.7.4</td>
<td>Coherent States as a Transported Vacuum</td>
<td>127</td>
</tr>
<tr>
<td>8.8</td>
<td>A Few Words on Continuous Wavelet Analysis</td>
<td>129</td>
</tr>
<tr>
<td>9</td>
<td>Another Family of SU(1,1) Coherent States for Quantum Systems</td>
<td>135</td>
</tr>
<tr>
<td>9.1</td>
<td>Introduction</td>
<td>135</td>
</tr>
<tr>
<td>9.2</td>
<td>Classical Motion in the Infinite-Well and Pöschl–Teller Potentials</td>
<td>135</td>
</tr>
<tr>
<td>9.2.1</td>
<td>Motion in the Infinite Well</td>
<td>136</td>
</tr>
<tr>
<td>9.2.2</td>
<td>Pöschl–Teller Potentials</td>
<td>138</td>
</tr>
<tr>
<td>9.3</td>
<td>Quantum Motion in the Infinite-Well and Pöschl–Teller Potentials</td>
<td>141</td>
</tr>
<tr>
<td>9.3.1</td>
<td>In the Infinite Well</td>
<td>141</td>
</tr>
<tr>
<td>9.3.2</td>
<td>In Pöschl–Teller Potentials</td>
<td>142</td>
</tr>
<tr>
<td>9.4</td>
<td>The Dynamical Algebra su(1,1)</td>
<td>143</td>
</tr>
<tr>
<td>9.5</td>
<td>Sequences of Numbers and Coherent States on the Complex Plane</td>
<td>146</td>
</tr>
<tr>
<td>9.6</td>
<td>Coherent States for Infinite-Well and Pöschl–Teller Potentials</td>
<td>150</td>
</tr>
<tr>
<td>9.6.1</td>
<td>For the Infinite Well</td>
<td>150</td>
</tr>
<tr>
<td>9.6.2</td>
<td>For the Pöschl–Teller Potentials</td>
<td>152</td>
</tr>
<tr>
<td>9.7</td>
<td>Physical Aspects of the Coherent States</td>
<td>153</td>
</tr>
<tr>
<td>9.7.1</td>
<td>Quantum Revivals</td>
<td>153</td>
</tr>
<tr>
<td>9.7.2</td>
<td>Mandel Statistical Characterization</td>
<td>155</td>
</tr>
<tr>
<td>9.7.3</td>
<td>Temporal Evolution of Symbols</td>
<td>158</td>
</tr>
<tr>
<td>9.7.4</td>
<td>Discussion</td>
<td>162</td>
</tr>
<tr>
<td>10</td>
<td>Squeezed States and Their SU(1,1) Content</td>
<td>165</td>
</tr>
<tr>
<td>10.1</td>
<td>Introduction</td>
<td>165</td>
</tr>
<tr>
<td>10.2</td>
<td>Squeezed States in Quantum Optics</td>
<td>166</td>
</tr>
<tr>
<td>10.2.1</td>
<td>The Construction within a Physical Context</td>
<td>166</td>
</tr>
<tr>
<td>10.2.2</td>
<td>Algebraic (su(1,1)) Content of Squeezed States</td>
<td>171</td>
</tr>
<tr>
<td>10.2.3</td>
<td>Using Squeezed States in Molecular Dynamics</td>
<td>175</td>
</tr>
<tr>
<td>11</td>
<td>Fermionic Coherent States</td>
<td>179</td>
</tr>
<tr>
<td>11.1</td>
<td>Introduction</td>
<td>179</td>
</tr>
<tr>
<td>11.2</td>
<td>Coherent States for One Fermionic Mode</td>
<td>179</td>
</tr>
<tr>
<td>11.3</td>
<td>Coherent States for Systems of Identical Fermions</td>
<td>180</td>
</tr>
<tr>
<td>11.3.1</td>
<td>Fermionic Symmetry SU(r)</td>
<td>180</td>
</tr>
<tr>
<td>11.3.2</td>
<td>Fermionic Symmetry SO(2r)</td>
<td>185</td>
</tr>
</tbody>
</table>
11.3.3 Fermionic Symmetry $SO(2r + 1)$ 187
11.3.4 Graphic Summary 188
11.4 Application to the Hartree–Fock–Bogoliubov Theory 189

Part Two Coherent State Quantization 191

12 Standard Coherent State Quantization: the Klauder–Berezin Approach 193
12.1 Introduction 193
12.2 The Berezin–Klauder Quantization of the Motion of a Particle on the Line 193
12.3 Canonical Quantization Rules 196
12.3.1 Van Hove Canonical Quantization Rules [161] 196
12.4 More Upper and Lower Symbols: the Angle Operator 197
12.5 Quantization of Distributions: Dirac and Others 199
12.6 Finite-Dimensional Canonical Case 202

13 Coherent State or Frame Quantization 207
13.1 Introduction 207
13.2 Some Ideas on Quantization 207
13.3 One more Coherent State Construction 209
13.4 Coherent State Quantization 211
13.5 A Quantization of the Circle by $2 \times 2$ Real Matrices 214
13.5.1 Quantization and Symbol Calculus 214
13.5.2 Probabilistic Aspects 216
13.6 Quantization with $k$-Fermionic Coherent States 218
13.7 Final Comments 220

14 Coherent State Quantization of Finite Set, Unit Interval, and Circle 223
14.1 Introduction 223
14.2 Coherent State Quantization of a Finite Set with Complex $2 \times 2$ Matrices 223
14.3 Coherent State Quantization of the Unit Interval 227
14.3.1 Quantization with Finite Subfamilies of Haar Wavelets 227
14.3.2 A Two-Dimensional Noncommutative Quantization of the Unit Interval 228
14.4 Coherent State Quantization of the Unit Circle and the Quantum Phase Operator 229
14.4.1 A Retrospective of Various Approaches 229
14.4.2 Pegg–Barnett Phase Operator and Coherent State Quantization 234
14.4.3 A Phase Operator from Two Finite-Dimensional Vector Spaces 235
14.4.4 A Phase Operator from the Interplay Between Finite and Infinite Dimensions 237

15 Coherent State Quantization of Motions on the Circle, in an Interval, and Others 241
15.1 Introduction 241
15.2 Motion on the Circle 241
Contents

18 Conclusion and Outlook  287

Appendix A The Basic Formalism of Probability Theory  289
A.1 Sigma-Algebra  289
A.1.1 Examples  289
A.2 Measure  290
A.3 Measurable Function  290
A.4 Probability Space  291
A.5 Probability Axioms  291
A.6 Lemmas in Probability  292
A.7 Bayes's Theorem  292
A.8 Random Variable  293
A.9 Probability Distribution  293
A.10 Expected Value  294
A.11 Conditional Probability Densities  294
A.12 Bayesian Statistical Inference  295
A.13 Some Important Distributions  296
A.13.1 Degenerate Distribution  296
A.13.2 Uniform Distribution  296

Appendix B The Basics of Lie Algebra, Lie Groups, and Their Representations  303
B.1 Group Transformations and Representations  303
B.2 Lie Algebras  304
B.3 Lie Groups  306
B.3.1 Extensions of Lie algebras and Lie groups  310

Appendix C SU(2) Material  313
C.1 SU(2) Parameterization  313
C.2 Matrix Elements of SU(2) Unitary Irreducible Representation  313
C.3 Orthogonality Relations and 3 j Symbols  314
C.4 Spin Spherical Harmonics  315
C.5 Transformation Laws  317
C.6 Infinitesimal Transformation Laws  318
C.7 Integrals and 3 j Symbols  319
C.8 Important Particular Case: j = 1  320
C.9 Another Important Case: σ = j  321

Appendix D Wigner–Eckart Theorem for Coherent State Quantized Spin Harmonics  323

Appendix E Symmetrization of the Commutator  325

References  329

Index  339