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THE U(1) PROBLEM AND ANOMALOUS WARD IDENTITIES ⁺⁾

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ABSTRACT

An effective Lagrangian is developed using the $1/N$ expansion of QCD which includes the anomaly and θ dependent effects of the fundamental Lagrangian. The effective Lagrangian contains the full solution of the U(1) problem so that there is no light singlet pseudo-scalar meson in the chiral symmetry limit. Anomalous Ward identities to all n point order and arbitrary q^2 are shown to hold for this Lagrangian. The mass of the η' in the chiral symmetry limit is related to the properties of the vacuum energy in the absence of quarks as for the fundamental Lagrangian. The effective interaction of the strong CP violation is also obtained and is expressed directly in terms of the phenomenological mesic fields.

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1. INTRODUCTION

As we all know the $U(1)$ problem has been with us for many years. It was first pointed out by Glashow¹⁾ in 1967 that in a QCD theory with three light quarks one has nine rather than eight light pseudo-scalar particles. This argument was later shown by Weinberg²⁾ to lead to an upper bound on the mass of the singlet pseudo-scalar meson, the η' , so that $m_{\eta'} \leq \sqrt{3}m_{\pi}$. A second aspect of the $U(1)$ problem concerns the vanishing of the $\eta \rightarrow 3\pi$ decay in the soft pion limit³⁾. Both of the above results are of course in contradiction with experiment⁴⁾.

There have been several suggestions since the discovery of the $U(1)$ problem regarding its possible resolution⁵⁾⁻⁷⁾. In particular it was pointed out by Kogut and Susskind⁶⁾ that the necessary ingredient for the resolution was the existence of singular matrix elements of non-gauge invariant operators with singularities at $q^2 = 0$. In 1976 it was observed by 't Hooft⁷⁾ in his work on instantons that there exists an anomaly in the axial $U(1)$ channel and this led to the hope for a period of time that instantons would resolve the paradox. However, Crewther⁸⁾ then showed that the contributions arising from the instantons alone were not sufficient to satisfy the anomalous Ward identities in a normal way. Recently, a very interesting proposal has been made by Witten⁹⁾ to achieve a resolution by analysing the effect of the $U(1)$ anomaly in the framework of the $1/N$ expansion^{10),11)}. In the language of the $1/N$ expansion, the anomaly produces effects which are an order $1/N$ smaller than the leading terms. However, this is found sufficient to break the singlet-octet degeneracy and produce a significant non-vanishing value for the singlet mass in the chiral symmetry limit^{9),12),13)}.

A second phenomena which is related to the $U(1)$ problem concerns the appearance in QCD of the vacuum angle θ . The vacuum angle introduces in the fundamental Lagrangian a CP violating interaction proportional to θ so that the QCD Lagrangian involving gluonic variables only has the form

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \theta \left(\frac{g^2}{32\pi^2} \right) \tilde{F}_{\mu\nu} F^{\mu\nu} \quad (1)$$

where $\tilde{F}_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\alpha\beta} F^{\alpha\beta}$, $\epsilon_{0123} = +1$ is the dual of $F_{\mu\nu}$: we assume also an $SU(N)$ colour so that one may consider the large N limit¹⁰⁾. A priori, there appears to be no reason to believe that the θ phenomena and the $U(1)$ problem are interrelated. However, this observation is only superficial and there exists a relation between the two phenomena at a deeper level. In fact,

Witten⁹⁾ has shown that the mass of the η' in the chiral symmetry limit is directly proportional to the double derivative of the vacuum energy $E(\theta)$ in the absence of quarks at $\theta = 0$, i.e.,

$$m_{\eta'}^2 = 4 N_\ell F_\pi^{-2} \left(\frac{d^2 E}{d\theta^2} \right)_{\theta=0}^{\text{no-quark}} \quad (2)$$

where N_ℓ is the number of light quarks (i.e., $N_\ell = 3$).

Next we discuss the work we have done regarding the $U(1)$ problem recently¹⁴⁾. Using the $1/N$ expansion of QCD we have developed an effective Lagrangian formalism which possesses the following properties. First, it contains a full solution of the $U(1)$ problem so that there is no massless η' particle in the chiral symmetry limit. The solution to the η' problem is achieved through the introduction of a new axial four vector anomaly field $K^\mu(x)$ with mesonic interactions proportional to the topological charge density $\partial_\mu K^\mu(x)$ appearing in the Lagrangian. The effective Lagrangian contains the observed meson fields so that it can be used directly for the computation of processes that involve the anomaly effects. The second property that the effective Lagrangian possesses is that it reproduces all the θ properties of the fundamental Lagrangian. In particular the properties of the vacuum energy as a function of the vacuum angle θ is reproduced and one also obtains an effective interaction for the strong CP violation effects.

Of course, one may ask what is gained by using an effective Lagrangian rather than the fundamental Lagrangian and chiral perturbation theory. While the advantages of the effective Lagrangian approach are well known, it is worth while to recount them here in the context of the solution to the $U(1)$ problem. Perhaps the most obvious advantage of the effective Lagrangian in the present context is the guaranteed satisfaction of the general n point anomalous Ward identities for all q^2 . The previous analyses thus far in this field have been successful in satisfying the anomalous Ward identities only up to two point order and $q^2 = 0$. Further, the framework of the effective Lagrangian allows one to include an arbitrary amount of $SU(3)$ and chiral symmetry breakdown. In contrast in chiral perturbation theory one must sum over many orders of perturbation to approach the experimental value of physical parameters. In an effective Lagrangian the physical parameters assume their experimental values from the very beginning. Finally, in the effective Lagrangian approach one does not need to use the soft pion approximations. There exists a well-defined set of correction terms given

by the effective Lagrangian which are consistently omitted in the soft pion limits and which may sometimes be non-negligible. The use of the effective Lagrangian includes automatically all such corrections over and above the results of the soft pion approximation.

Before we proceed to formulate the effective Lagrangian including the effects of the axial $U(1)$ anomaly we recall first the formulation of an effective Lagrangian in the absence of the anomaly. The construction of such a Lagrangian in terms of the physically observed mesons and obeying the principles of current algebra and the PCAC condition has been known for many years¹⁵⁾. On the other hand, it has been shown recently by Witten⁹⁾ that the $1/N$ expansion of QCD leads one to an effective Lagrangian involving physically observed mesonic interactions. One may also expect this $1/N$ expansion of QCD to obey current algebra since the fundamental QCD Lagrangian obeys current algebra. This expectation would be rendered invalid only if the current algebra constraints were to mix different orders in $1/N$.

To examine the question of whether current algebra obeys the $1/N$ expansion, one looks at the structure of the current algebra effective Lagrangian where the interaction part of the Lagrangian can be expanded in terms of a sum of three-point, four-point, etc., interactions so that

$$\mathcal{L}_I = \sum_n \mathcal{L}_n, \quad \mathcal{L}_n = g^{(n)} \phi_1 \cdots \phi_n \quad (3)$$

where ϕ 's represent the full set of observed mesic fields appearing in the effective Lagrangian and $g^{(n)}$ is the strength of the n th order interaction terms. The constraints of current algebra determine $g^{(n)}$ so that $g^{(n)} \sim F^{2-n}$ where F 's represent the strength of the interpolating constants that define the matrix elements of the bilinear quark currents between vacuum and one-meson states, e.g.,

$$\langle 0 | \bar{q} \gamma^\mu \gamma_5 \frac{1}{2} \lambda_a q | b(q) \rangle = i q^\mu F_{ab} \quad (4)$$

where the one-particle states refer to the 0^- nonets of mesons. In the $1/N$ expansion F 's obey the relation $F \sim \sqrt{N}$ so that

$$g^{(n)} \sim N^{1-\frac{n}{2}} \quad (5)$$

Equation (5) implies that current algebra predicts that the three-point mesic vertices behave as $1/\sqrt{N}$ and so on. In the $N = \infty$ limit all the amplitudes vanish and one has a set of stable and non-interacting meson states. These are precisely the constraints required by the $1/N$ expansion of QCD obtained by Witten¹¹⁾. However, we note that the current algebra analysis determines not only the $1/N$ order of the n point amplitudes correctly but also their relative strengths. We also note that the current algebra effective Lagrangian discussed above does not contain gluonic mesons or glueball states. This is consistent with the fact that the couplings of the glueball states to physical mesons are of higher order in $1/N$ ¹⁶⁾.

The discussion in the remaining part of this paper consists of the following topics. First we discuss the structure of the effective Lagrangian including the anomaly effects and satisfaction of the anomalous Ward identities. The satisfaction of the anomalous Ward identities requires the introduction in the Lagrangian of an anomaly field and gives rise to ghost poles in the matrix elements of non-gauge invariant operators though, of course, the ghosts decouple from all physical gauge invariant amplitudes. Next we shall investigate the θ dependence of the vacuum energy and its relation to the η' mass in the absence of quarks. Finally we shall discuss the form of the strong CP violating interaction that arises in the theory.

2. EFFECTIVE LAGRANGIAN WITH ANOMALY EFFECTS

As pointed out by 't Hooft⁷⁾ the divergence of the ninth axial current for the fundamental QCD Lagrangian is modified from the usual soft (chiral symmetry) breaking part by an additional term so that

$$\partial_\mu A^\mu_a = D_a + 2N_\ell (g^2/32\pi^2) \tilde{F}_{\mu\nu} F^{\mu\nu} \delta_{a9} \quad (6)$$

where D_a is the so-called soft part which breaks chiral symmetry and N_ℓ is the number of light quarks. The effect of the anomaly actually turns out to be an order $1/N$ smaller than the leading meson terms⁹⁾. Even so the anomaly contribution is seen to be the dominant term in chiral symmetry limit and of sufficient strength to break the singlet-octet degeneracy.

To account for the anomaly effect in the effective Lagrangian one introduces an additional anomaly field $K^\mu(x)$ in the theory. The anomaly field $K^\mu(x)$ is in effect an axial four-vector ghost field defined so that $\partial_\mu K^\mu(x)$ represents the topological charge density, i.e.,

$$\partial_\mu K^\mu = \text{topological charge density} \quad (7)$$

The effective Lagrangian which contains the anomaly effects is then given by

$$\begin{aligned} \mathcal{L} = & \frac{1}{2C} (\partial_\mu K_\nu)^2 + G(\chi_a) \partial_\mu K^\mu - \theta \partial_\mu K^\mu \\ & - \phi_a^\mu \partial_\mu \phi_a + \frac{1}{2} \phi_a^\mu \phi_{a\mu} - \frac{1}{2} \phi_a \mu_{ab} \phi_b + \mathcal{L}_{\text{C.A.}}(\chi_A, \partial_\mu K^\mu) \end{aligned} \quad (8)$$

where $\chi_a \equiv (\phi_a, \sigma_a)$ represent the 18-plet of $0^\pm$ scalar fields and χ_A represent the full set of all mesic fields. The quadratic part of the 0^- scalar fields ϕ_a has been exhibited explicitly since these enter in an important way in our discussion while $\mathcal{L}_{\text{C.A.}}$ represents the remaining part of the current algebra Lagrangian. The mass matrix μ_{ab} in Eq. (8) arises due to the breakdown of chiral symmetry, is diagonal in $a, b = 1 \dots 7$ and is allowed to be non-diagonal in the sectors $a, b = 8, 9$. This latter aspect of non-diagonality of μ_{ab} in the 8,9 sectors is found necessary if one is to accommodate features of a broken $SU(3)$ symmetry in the effective Lagrangian with anomaly. The parameter C appearing in Eq. (8) also plays a crucial role in our discussion and it represents essentially the strength of the topological charge.

In analogy with Eq. (6), the PCAC condition that the Lagrangian of Eq. (8) is required to obey has the form

$$\partial_\mu A^\mu_a = F_{ab} \mu_{bc} \phi_c + \delta a_q 2N_\ell \partial_\mu K^\mu \quad (9)$$

From Eq. (8) we see that $G(\chi_a) \partial_\mu K^\mu$ represents that part of $\mathcal{L}$ which depends on the anomaly where $G(\chi_a)$ is some polynomial in $\chi_a \equiv (\phi_a, \sigma_a)$ with non-derivative couplings. Equation (9) then implies that G obey the condition

$$F_{ab} \frac{\partial G}{\partial \phi_b} = -Z_{1abc} \frac{\partial G}{\partial \chi_b} \chi_c - 2N_\ell \delta a_q \quad (10)$$

Here Z_{1abc} is the constant that enters in the axial current A_a^0 (17):

$$A_a^0(x) = -F_{ab} \chi_{ob} - Z_{1abc} \chi_{ob} \chi_c \quad (11)$$

and is calculated in Ref. 22). Equation (10) may be integrated to arbitrary n point order. Thus to quadratic order one has¹⁸⁾

$$G = -2N_c \bar{F}'_{a9} \phi_a + 2N_c \phi_a \bar{F}'_{a9} Z_{1rbc} \bar{F}'_{b9} \sigma_c + f(\sigma^2) \quad (12)$$

where $a, b, c, r = 1 \dots 9$ and $f(\sigma^2)$ is a quadratic function of σ_c .

The essential new element in the effective Lagrangian of Eq. (8) is that in addition to the usual current algebra parts one has now a new set of terms involving the interaction of the topological charge density directly with the meson fields. One may determine the effect of the anomaly on the meson masses by examining the quadratic part of the Lagrangian and determining the free field propagators. For the propagators corresponding to the pseudo-scalar fields one finds $\Delta_{ab} = \delta_{ab} (q^2 + m_q^2)^{-1}$ where

$$m_a^2 \delta_{ab} = \mu_{ab} + 4N_c^2 C \bar{F}'_{a9} \bar{F}'_{b9} \quad (13)$$

From Eq. (13) we find that the (bare) chiral symmetry breaking masses represented by μ_{ab} are clothed by an anomaly part to generate the physical masses represented by m_a .

To see in a more transparent way how the anomaly clothing in the mass relation of Eq. (13) resolves the $U(1)$ problem related to light η' we consider the chiral and $SU(3)$ limit of Eq. (13). In the chiral and $SU(3)$ limit one has¹⁹⁾

$$\mu_{ab} = 0, \quad F_{89} = 0 = F_{98}, \quad F_{99} = \sqrt{N_c} F_\pi \quad (14)$$

Under the constraints of Eq. (14) one has then that $m_\pi = m_K = m_\eta = 0$ and $m_{\eta'}$ obeys

$$m_{\eta'}^2 = 4N_c C / F_\pi^2 \quad (15)$$

Thus in the chiral symmetry limit $m_{\eta'}$ is non-vanishing resolving the problem of the light η' . Further since $F_\pi \sim \sqrt{N_c}$ one has from Eq. (15) that $m_{\eta'}^2 \sim \sim O(1/N)$ for large N as in Witten's analysis⁹⁾.

To gain further insight into the structure of the field K^μ responsible for anomaly clothing, one may decompose the gauge invariant axial current into a symmetry current $\tilde{A}^\mu$ and an anomaly current. Thus one has

$$A_a^\mu = \tilde{A}_a^\mu + \delta_{aq} 2N_c K^\mu \quad (16)$$

One may compute next the vacuum to particle state matrix element of the anomaly current using the effective Lagrangian. Thus

$$\langle 0 | \delta_{aq} 2N_c K^\mu | b(q) \rangle = -i \frac{q^\mu}{q^2} 4N_c^2 C \delta_{aq} \bar{F}^{-1} b q \quad (17)$$

where $b(q)$ represents a 0^- state. We note that the matrix element in Eq. (17) contains a singularity at $q^2 = 0$. It is precisely this singularity that guarantees a non-vanishing topological charge as was first pointed out by Kogut and Susskind in the so called "seizing" of the vacuum effect in QCD. Further, to see how the symmetry currents which enter into the physical decay processes are effected by the presence of the anomaly, one can compute the matrix element analogous to Eq. (17) for the symmetry current. Again one finds a singularity at $q^2 = 0$ with exactly the same coefficient as in Eq. (17) but this time with an opposite sign. Thus the gauge invariant current correctly has singularity free matrix elements.

The propagator of the K_μ field is somewhat more complex. From the quadratic part of Eq. (8) one may compute $\Delta_{\mu\nu} = i \langle T(K_\mu K_\nu) \rangle$ and obtain

$$\Delta_{\mu\nu} = -C \frac{\eta_{\mu\nu}}{q^2} \left(1 - \frac{q_\mu q_\nu}{q^2} 4N_c^2 C (\bar{F}^{-1}_{aq})^2 (q^2 + m_a^2)^{-1} \right) \quad (18)$$

From Eq. (15) we see that the parameter C is positive so that Eq. (18) contains a monopole ghost²⁰⁾. It must be noted, however, that the ghost poles couple only to the anomaly currents and cancel out in the physical amplitudes since these involve only the divergence of K^μ . Thus only the residual effect of the ghosts remains.

We shall see in the next section that the physical meaning of the parameter C (which has figured prominently in our discussions thus far) is that it represents quantum fluctuations of the topological charge in the chiral symmetry limit. In addition to producing an anomaly clothing in the mass relations, the parameter C also enters in the solutions of the anomalous Ward identities. At the three-point level one obtains from the 8-9 sectors a set of three equations relating the four interpolating constants $F_{88}, F_{89}, F_{98}, F_{99}$ and the parameter C . One has²¹⁾

$$(m_\eta F_{88})^2 + (m_\eta' F_{89})^2 = \frac{4}{3} (C_K + C_X) - \frac{1}{3} C_\pi \quad (19)$$

$$F_{88}(F_{88} + \sqrt{2} F_{98}) m_\eta^2 + F_{89}(F_{89} + \sqrt{2} F_{99}) m_\eta'^2 = C_\pi \quad (20)$$

$$(F_{88} + \sqrt{2} F_{98})^2 m_\eta^2 + (F_{89} + \sqrt{2} F_{99})^2 m_\eta'^2 = 3C_\pi + 8N_c^2 C \quad (21)$$

where $C_\pi = m_\pi^2 F_\pi^2$, $C_K = m_K^2 F_K^2$ and²³⁾ $C_X = m_X^2 F_X^2$. Equations (19)-(21) allow one to solve for the four interpolating constants and C in terms of two angles ϕ and ϕ' . Thus

$$\frac{\sqrt{2}}{3} \frac{C}{\sqrt{E_A E_B}} = \sin(\phi - \phi') \quad (22a)$$

$$F_{88} = (\sqrt{3} m_\eta)^{-1} [\sqrt{2} E_A \cos \phi + \sqrt{E_B} \sin \phi'] \quad (22b)$$

$$F_{98} = (\sqrt{3} m_\eta)^{-1} [-\sqrt{E_A} \cos \phi + \sqrt{2} E_B \sin \phi'] \quad (22c)$$

$$F_{89} = (\sqrt{3} m_{\eta'})^{-1} [-\sqrt{2} E_A \sin \phi + \sqrt{E_B} \cos \phi'] \quad (22d)$$

$$F_{99} = (\sqrt{3} m_{\eta'})^{-1} [\sqrt{E_A} \sin \phi + \sqrt{2 E_B} \cos \phi'] \quad (22e)$$

where E_A and E_B are defined by

$$E_A = 2 (C_K + C_X) - C_\pi + \frac{1}{3} C \quad (23)$$

$$E_B = C_\pi + \frac{2}{3} C \quad (24)$$

The angles ϕ and ϕ' are of course totally unconstrained, and one needs additional input such as the branching ratios of the $\eta \rightarrow 2\gamma$ and $\eta' \rightarrow 2\gamma$ decays. With this additional experimental input one has then a totally determined system. The resulting equations are somewhat involved and must await a full analysis to determine the numerical values of C and all the F_{ab} . However, one may make rough estimates which yield $C \approx C_\pi$ and hence $N_\ell^2 C \sim O(C_K)$. Thus the contribution of C to the Ward identities is non-negligible. Similarly one finds the value of the off-diagonal elements of the interpolating matrix, i.e., F_{89} and F_{98} are also non-negligible. Thus one has, for example, $F_{89} m_{\eta'} / (F_{99} m_\eta) \sim \frac{1}{2}$. In summary one finds that the effects of the anomaly on the solutions of the physical quantities is not necessarily small. One reason for this is that the anomaly effect in the Ward identity Eq. (21) is being magnified by a factor of N_ℓ^2 which is a rather large factor for $N_\ell = 3$.

One may also calculate, using the effective Lagrangian, the matrix element of the topological charge density between vacuum and η and η' states. One has then

$$\langle 0 | \partial_\mu K^\mu | \eta a \rangle = 2 N_\ell C F^{-1} a_9 \quad a=8,9 \quad (25)$$

The matrix element of Eq. (25) provides in some sense a measure of the "gluonic content" of η and η' . Thus, for example, in the $SU(3)$ limit since $F^{-1}89 = 0$, the matrix element

$$\langle 0 | \partial_\mu K^\mu | \eta \rangle_{SU(3) \text{ Limit}} = 0 \quad (26)$$

The result of Eq. (26) is expected since in the $SU(3)$ symmetry limit π, K and η form an $SU(3)$ multiplet and η should not have any "gluonic content" since π and K clearly do not. Similarly in the $SU(3)$ limit we have for the η' matrix element

$$\langle 0 | \partial_\mu K^\mu | \eta' \rangle_{SU(3) \text{ Limit}} = 2N_c C / F_{99} \quad (27)$$

The η' matrix element clearly is non-vanishing and large showing the important role the anomaly plays in the structure of η' .²⁴⁾

We proceed now to a discussion in greater detail of the anomalous Ward identities at higher point level. Of course, if the PCAC condition with anomaly, i.e., Eq. (9) is satisfied at higher point level, the satisfaction of the anomalous Ward identities is then automatically guaranteed. The satisfaction of Eq. (9) at higher point level brings into the effective Lagrangian a set of additional terms proportional to $\partial_\mu K^\mu$. The general form of the functional differential equation which gives rise to the new couplings in the presence of anomaly can be obtained from the old functional differential equation in the absence of anomaly by the following replacement²⁵⁾

$$\mu_a^2 \phi_a \rightarrow \mu_{ab} \phi_b + 2N_c F_{a9}^{-1} \partial_\mu K^\mu \quad (28)$$

The functional differential equations that result from the replacement of Eq. (28) can be integrated to an arbitrary order to generate the new couplings. Equation (28) guarantees that the new couplings are of the form $G(\chi_A) \partial_\mu K^\mu$ so that only the gauge invariant combination $\partial_\mu K^\mu$ couples to the mesonic fields. This is precisely the structure that is displayed in Eq. (8). We also note that the strength of the new couplings is determined through the PCAC equation in terms of the non-anomaly couplings. If one assumes a linear representation for the scalar fields and limits oneself to no derivatives for the new couplings one has that the new couplings then obey Eq. (10). It is these additional couplings that guarantee the satisfaction of the anomalous Ward identities for the higher point functions.

3. $(d^2E/d\theta^2)\theta = 0$ IN QCD AND IN THE EFFECTIVE LAGRANGIAN

We proceed now to discuss the θ dependence of the vacuum energy. For the case of the fundamental QCD Lagrangian one can show in a straightforward fashion that $(d^2E/d\theta^2)_{\theta=0}$ is given by²⁶⁾

$$\left(\frac{d^2E}{d\theta^2}\right)_{\theta=0} = (g^2/8\pi^2)^2 \langle B_a^i(0) B_a^i(0) \rangle - \tau \quad (29)$$

where B_a^i are the colour magnetic fields $B_a^i = \frac{1}{2} \epsilon^{ijk} F_{jk}^a$ and τ represents the quantum fluctuations of the topological charge density as is given by

$$\tau = i \int d^4x \langle 0 | T(\partial_\mu K^\mu(x) \partial_\nu K^\nu(0)) | 0 \rangle \quad (30)$$

Here $K^\mu(x)$ is

$$K^\mu(x) = (g^2/32\pi^2) \epsilon^{\mu\alpha\beta\gamma} A_\alpha^a (F_{\beta\gamma}^a - \frac{1}{3} g C^{abc} A_\beta^b A_\gamma^c) \quad (31)$$

and is thus normalized so that

$$\partial_\mu K^\mu = (g^2/32\pi^2) \tilde{F}_{\mu\nu} F^{\mu\nu} \quad (32)$$

The first term on the right-hand side of Eq. (29) arises due to the dependence of the gluon canonical momentum on θ . This piece, however, cancels as one moves the gradients out of the time ordered product in Eq. (30) and one obtains the relation

$$\left(\frac{d^2E}{d\theta^2}\right)_{\theta=0} = -q^\mu q^\nu \Delta_{\mu\nu}(q) \Big|_{q^2=0} \quad (33)$$

where $\Delta_{\mu\nu} = i \langle T(K_\mu K_\nu) \rangle$. By explicit calculation, one finds that Eq. (33) also holds identically for the case of the effective Lagrangian. Further for the case of the effective Lagrangian we can compute explicitly the various quantities that we have dealt with in arriving at Eq. (33). First we compute τ . Using the definition of Eq. (30) and the effective Lagrangian of Eq. (8) we have that

$$\tau = 4N_\ell^2 C^2 \left(\frac{(F_{89}^{-1})^2}{m_\eta^2} + \frac{(F_{99}^{-1})^2}{m_{\eta'}^2} \right) \quad (34)$$

In the SU(3) and chiral symmetry limits one can show by using the Ward identities of Eqs (15)-(17) that τ has the following limit²⁷⁾

$$(\tau)^{\text{Chiral Limit}} = C \quad (35)$$

Thus the physical meaning of C is that it represents quantum fluctuations of the topological charge density in the chiral symmetry limit. Further, the effective Lagrangian also gives by using Eq. (33)

$$\left(\frac{d^2 E}{d\theta^2} \right)_{\theta=0} = C - \tau \quad (36)$$

By comparison of Eqs (29) and (35) one finds that the QCD analogue of C is the quantity $(g^2/8\pi^2)^2 \langle B_a^i B_a^i \rangle$. Since in the chiral symmetry limit τ limits to C one has from Eq. (36) that $(d^2 E/d\theta^2)_{\theta=0}$ vanishes in the chiral limit exactly as happens in QCD.

The explicit form of the effective Lagrangian of Eqs (34) and (36) also allows us to go to the no-quark limit quite easily. This can be done by simply setting $N_\ell = 0$ in Eqs. (34) and (36). One has then

$$(\tau)^{\text{no-quark}} = 0 \quad (37)$$

$$\left(\frac{d^2 E}{d\theta^2} \right)_{\theta=0}^{\text{no-quark}} = C \quad (38)$$

Equation (37) represents the vanishing of the quantum fluctuation of the topological charge density in the no-quark limit. Substituting the evaluation of C from Eq. (38) in Eq. (15) one obtains that the mass of η' in the symmetry limit is given by

$$m_{\eta'}^2 = (4N_\ell/F_\pi^2) \left(\frac{d^2 E}{d\theta^2} \right)_{\theta=0}^{\text{no-quark}} \quad (39)$$

As stated in the introduction Eq. (39) was first derived by Witten in the QCD framework. Here it follows from the effective Lagrangian in a rather direct manner. As also stated in the Section I, Eq. (39) provides the relationship between two different phenomena, i.e., that the non-vanishing of m_η , in the chiral symmetry limit needed to solve the $U(1)$ problem requires a non-vanishing $(d^2E/d\theta^2)_\theta = 0$.

4. THE STRONG CP VIOLATION

The strong CP violation effects arise only with a broken chiral symmetry. This is so because in the chiral symmetry limit one can eliminate θ from the fundamental QCD Lagrangian completely by a $U(1)$ rotation. In the presence of a broken chiral symmetry one may still eliminate the θ dependence completely from the gluon sector but now the θ dependence reappears in the quark sector and one obtains²⁸⁾

$$\delta \mathcal{L}_{CP} = \theta m_u m_d m_s (m_u m_d + m_d m_s + m_s m_u)^{-1} \bar{q} i \gamma_5 q \quad (40)$$

The form of Eq. (40) rather than of Eq. (1) with quarks present is the one found more useful for carrying out chiral perturbation theory for the computation of strong CP violating effects.

The effective Lagrangian exhibits properties similar to those of the fundamental Lagrangian in this respect also. Thus in analogy to the fundamental Lagrangian one can make an identical $U(1)$ transformation to eliminate the θ term in the topological charge density sector, i.e., the term $-\theta \partial_\mu K^\mu$ in Eq. (8) and introduce it in the meson sector. We proceed now to compute the total $\partial \mathcal{L}_{CP}$ that arises when we make a $U(1)$ rotation on the symmetry breaking part of the Lagrangian $\mathcal{L}_1(\chi_a, \partial_\mu K^\mu)$ which we write in the form

$$\mathcal{L}_1(\chi_a, \partial_\mu K^\mu) = L(\chi_a) + G(\chi_a) \partial_\mu K^\mu - \theta \partial_\mu K^\mu \quad (41)$$

where $G(\chi_a)$ obeys Eq. (10) and $L(\chi_a)$ is the chiral symmetry breaking part of the current algebra Lagrangian and obeys the PCAC condition²⁹⁾

$$F_{ab} \frac{\partial L}{\partial \chi_b} = -Z_{1abc} \frac{\partial L}{\partial \chi_b} \chi_c - F_{ab} \mu_{bc} \chi_c \quad (42)$$

The $U(1)$ rotation which we wish to perform is associated with the generator of the symmetry transformation $\tilde{Q}_9^5$ where

$$\tilde{Q}_9^5 = \int d^3x \tilde{A}_9^0(x) ; \tilde{A}_a^\mu = A_a^\mu - 2N_\ell K^\mu \delta_{a9} \quad (43)$$

and A_a^μ are the nonet of gauge invariant axial currents. The desired $U(1)$ transformation is induced by the unitary operator

$$U(\theta) = \exp\left(i \frac{\theta}{2N_\ell} \tilde{Q}_9^5\right) \quad (44)$$

It is easily seen that³⁰⁾

$$U(\theta) \mathcal{L}_1 U(\theta)^{-1} = \mathcal{L}_1(\chi_a' - \theta(2N_\ell)^{-1} F_{9a}, \partial_\mu K^\mu) \quad (45)$$

The $U(1)$ transformation alone, however, is not enough. One must in addition make chiral $SU(3) \times SU(3)$ rotation to conform to the theorems of Dashen³¹⁾. This requires that one minimize the quantity

$$F(\beta_i) = \langle 0 | V^{-1}(\beta) \mathcal{L}_1 V(\beta) | 0 \rangle \quad (46)$$

with respect to β_i , where

$$V(\beta) = \exp(i \beta_i Q_i^5) \quad (47)$$

and Q_i^5 are the axial charges of $SU(3) \times SU(3)$. It is then easily seen that this procedure generates a $\delta \mathcal{L}_{CP}$ of the form³²⁾

$$\delta \mathcal{L}_{CP} = \theta(2N_\ell \bar{\mu}^{-1} q_9)^{-1} F_{9a} \sqrt{Z} q_a v_0 \quad (48)$$

where $v_0 = (\sqrt{Z}\phi)_9$ is the ninth pseudo-scalar density and $(\sqrt{Z})_{ab}$ is the wave-function renormalization matrix of Glashow and Weinberg³³⁾. The quantity μ_{ab} appearing in Eq. (48) is defined by

$$\bar{\mu}_{ab} = (Z^{-1/2})_{ac} \mu_{cd} (Z^{-1/2})_{db} \quad (49)$$

From Eq. (48) we see that the effective CP violating interaction that results after elimination of the θ dependence from the topological charge density sector consists of a linear term proportional to the 9th pseudo-scalar density. In analogy with the fundamental Lagrangian the CP violating terms vanish in the limit of chiral symmetry. Further, since one now has the form of the CP violating interaction both in the fundamental as well as the effective Lagrangian formalisms one can obtain a relation among parameters appearing in the two formalisms by equating the vacuum matrix element $\langle 0 | [\tilde{Q}_9^5, \delta \mathcal{L}_{CP}] | 0 \rangle$ in the two formalisms. For $N_\ell = 3$ one obtains

$$(\bar{u}^{-1} q q)^{-1} (\sqrt{Z_{qa}} F_{qa})^2 = 12 m_u m_d m_s (m_u m_d + m_d m_s + m_u m_s)^{-1} \langle 0 | \bar{q} q | 0 \rangle \quad (50)$$

Equation (50) represents a sum rule for the chiral mass matrix of the η and η' .

5. CONCLUDING REMARKS

In summary we have obtained from the $1/N$ expansion of QCD a realistic form of an effective Lagrangian involving the QCD $U(1)$ anomaly and the θ dependent effects. This Lagrangian automatically contains the full solution of the $U(1)$ problem. Further, since the Lagrangian contains physically observed meson states such as π, ρ, A_1 etc., one can use it directly for the computation of physical processes. The problems that remain now concern detailed computations. We mention here two such applications.

The first, and perhaps the most interesting, concerns the application of the method developed above to compute the neutron electric dipole moment without recourse to the bag²⁸⁾ or the soft pion approximations³⁴⁾. Such a calculation is of interest since the weak contribution to the dipole moment in the standard Kobayashi-Maskawa theory³⁵⁾ is very small, i.e., of the order of 10^{-30} cm. This value is, of course, several orders of magnitude smaller than the present experimental limit of $\leq 10^{-24}$ cm. for the magnitude of the dipole moment. Thus an observation in excess of the weak dipole moment prediction might indicate a real strong interaction θ effect, provided of course the standard Kobayashi-Maskawa mechanism is right. The second application concerns an improved calculation of the theoretical estimate of $\eta \rightarrow 3\pi$ decay. One expects current algebra calculations to be accurate to within 10-15% and so this decay represents a strong test of the validity of the above theoretical ideas.

Finally, after completion of this work we have learnt of some work, in the same spirit as discussed here by other authors^{36),37)}. The analyses of Refs 36) and 37) also satisfy anomalous Ward identities to arbitrary order. However, these analyses are carried out in the framework of the sigma models and do not discuss the solutions in the general current algebra framework. A second essential difference with the work of Ref. 36) lies in their treatment of the anomaly field which only enters in the form $\partial_\mu K^\mu$ in the Lagrangian. Consequently, the formalism of Ref. 36) involves variables K^i which are dynamically not determined.

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- 16) While the meson-meson-meson coupling is of order $1/\sqrt{N}$, the meson-meson-glueball coupling is of order $1/N$.
- 17) Terms depending on other field variables in Eq. (11) are suppressed.
- 18) F^{-1}_{ab} is the matrix inverse of F_{ab} .
- 19) The last relation in Eq. (14) is obtained in the large N limit, See Ref. 9).
- 20) A similar monopole ghost was also observed by Veneziano in his analysis of Ref. 12).
- 21) Equations (19)-(21) were first obtained without anomaly effects in Ref. 22). Modifications due to anomaly were considered in Ref. 8). Equations similar to Eqs (19)-(21) but involving different interpolating constants were obtained by H. Goldberg, Northeastern University preprint NUB Nr. 2411 (1979).

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- 23) This relation is obtained under the assumption that the κ field is linearly realized in the strangeness changing vector sector. More generally C_κ measures the breakdown of SU(3) symmetry and can be obtained from the 0^+ strangeness changing spectral function.
- 24) Even in the SU(3) limit it is inappropriate to construe η' as a glueball since the right-hand side of Eq. (27) vanishes when $N_\ell = 0$, i.e., when the no-quark limit is taken.
- 25) Thus the general functional differential equation in the presence of anomaly can be obtained by making the replacement of Eq. (28) in Eq. (6.4) of Ref. 22).
- 26) This result is established most easily in the temporal gauge. See Ref. 9).
- 27) The SU(3) and chiral symmetry limit of Eq. (34) is somewhat subtle since one must evaluate F_{89}^{-1}/m_η . Both F_{89}^{-1} and m_η approach zero in these limits and one can verify that the ratio also vanishes.
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- 29) For the sake of simplicity we restrict our discussion here to the case where $L(\chi_a)$ is independent of the gradients of χ_a so that $L = -\frac{1}{2} \chi_a^{\mu} \chi_{ab}^{\mu} \chi_b + \dots$. Under this restriction L obeys Eq. (42) rather than the more general form which is Eq. (6.4) of Ref. 22).
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