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RL-80-010

QCD at Low Q^2 - A Correspondence Relation for Moments of Structure Functions

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January 1980

RL-80-010

80-5-201

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QCD AT LOW Q^2 - A CORRESPONDENCE RELATION FOR MOMENTS
OF STRUCTURE FUNCTIONS

B.Schrempf*† and F.Schrempf†

ABSTRACT

The precocious validity of QCD predictions in deep inelastic lepton nucleon scattering and e^+e^- annihilation is interpreted as a signal for an underlying "correspondence principle" relating perturbative and nonperturbative physics on the Q^2 average.

Correspondence relations for nonsinglet moments of deep inelastic structure functions are formulated, discussed and successfully tested against experiment. The relations provide an independent determination of the QCD Λ -parameter from low Q^2 data in perfect agreement with results from large Q^2 analyses.

Rutherford Laboratory,
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January, 1980

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1. INTRODUCTION

Predictions of renormalization group improved perturbative quantum chromodynamics (QCD), have turned out to be quite successful⁽¹⁾ for spacelike $q^2 = -Q^2$ in deep inelastic lepton nucleon scattering as well as for timelike q^2 in $e^+e^- \rightarrow$ hadrons. A striking result, which has emerged from phenomenological analyses is the precocious validity of these QCD predictions. This observation will form the starting point for our investigations.

In the timelike region it is known since a long time that already the lowest order prediction ($\equiv$ parton model)

$$R_{e^+e^-} = \frac{\sigma(e^+e^- \rightarrow \text{hadrons})}{\sigma(e^+e^- \rightarrow \mu^+\mu^-)} \underset{q^2 \rightarrow \infty}{\sim} \sum_{\text{quarks}} \frac{e_q^2}{q} \quad (1)$$

averages quite well the resonance region almost down to threshold, thus establishing a duality type relation. Various authors⁽²⁾ have shown that this duality relation remains valid and holds quantitatively well, if the $O(\alpha_s)$ corrections to eq. (1) are properly[†] taken into account.

†Straightforward extrapolation of the large q^2 prediction

$$R_{e^+e^-} \underset{q^2 \rightarrow \infty}{\sim} \sum_{\text{quarks}} \frac{e_q^2}{q} \left(1 + \frac{1}{\pi} \alpha_s(q^2)\right), \text{ with}$$

$$\alpha_s(q^2) \underset{q^2 \rightarrow \infty}{\sim} \frac{4\pi}{\beta_0} \frac{1}{\ln q^2/\Lambda^2}$$

to low q^2 , where $\alpha_s(q^2)$ exhibits a fake pole at $q^2 = \Lambda^2$, of course leads to a nonsense result. One rather has to remember the derivation of this equation⁽³⁾: analytic continuation of

the QCD expression for the vacuum polarization $\Pi(q^2)$ to positive q^2 gives (for massless quarks)

$$12\pi \operatorname{Im} \Pi(q^2) = R_{e^+e^-} \underset{q^2 > 0}{\sim} \sum_{\text{quarks}} \frac{e_q^2}{q} \left(1 + \frac{1}{\pi} \frac{4}{\beta_0} \arctan \frac{\pi}{\ln q^2/\Lambda^2}\right)$$

This leads to a correction of maximally 22% (at $q^2 = \Lambda^2$) to the lowest order prediction (1).

In the spacelike region, both, the leading and next to leading perturbative QCD predictions for nonsinglet moments of deep inelastic structure functions turned out to describe well⁽⁴⁻⁷⁾ the $(\bar{\nu})N, eN$ and μN data right down to about $Q^2 \approx 2-3 \text{ GeV}^2$.

An interesting possible interpretation of this precocious validity in the timelike and spacelike regions could be that there is again - as already in many other areas of physics - some kind of general "correspondence principle" at work, relating perturbative and nonperturbative physics on the average. To follow up this possibility in the spacelike region ($Q^2 > 0$) will be the purpose of this paper. We shall refrain from using the word "duality" in this context, since it is reserved typically for applications in resonance regions. Rather we shall invoke the general kind of correspondence arguments put forward in Ref. 8. The guiding principle is the demand of a certain continuity of the dynamical descriptions in adjacent kinematical regions - in our case the perturbative, large Q^2 region and the nonperturbative, low Q^2 region.

It is obvious that an extrapolation of the perturbative expression for the moments themselves into the nonperturbative region $Q^2 \lesssim 1 \text{ GeV}^2$ cannot be taken seriously, at least not on the quantitative level. Eventually the running coupling constant $\bar{g}^2(Q^2)$ becomes infinite (Landau ghost⁽⁹⁾) and so do the perturbative moments. However, Q^2 integrals over the perturbative expressions may well continue to exist and make sense. So we are led to consider a correspondence relation of the type

$$\int_{\ln Q^2}^{\ln Q^2} d(\ln Q'^2) M_n^{\text{data}}(Q'^2) \approx \int_{\ln Q^2}^{\ln Q^2} d(\ln Q'^2) M_n^{\text{pert}}(m) \left(\ln \frac{Q'^2}{\mu^2}, g^2 \right) \quad (2)$$

for increasing order $m = 1, 2, \dots$ of the perturbative expansion, where the integration runs over some Q^2 interval in the non-perturbative Q^2 region. As they stand these correspondence relations are rather ill defined and a lot of immediately arising questions remain to be answered. In the following sections we shall try to clarify them step by step before confronting the relations (2) with experiment in Section 7:

- (i) the appropriate choice of the lower integration limit $\ln Q_0^2$ will be settled first in Section 2.
- (ii) there are a number of effects which become relevant in the low Q^2 region and which will be considered in Sections 3 and 5
- $O(Q^2/\mu^2) = O(x^2 m^2/Q^2)$ effects ("target mass corrections") at low Q^2 ;

- possible contributions from higher twist (>2) operators on the r.h.s. of relation (2) at low Q^2 ;
- renormalization scheme dependence (for $m \geq 2$).

Having "settled" all these questions and taking an optimistic outlook we may expect the following pattern to emerge: for increasing order m of perturbation theory the correspondence relation (2) should improve until for $m \rightarrow \infty$ it becomes an identity. Unfortunately only the first and second order predictions of (renormalization group improved) perturbative QCD have been worked out (10,11) and a mathematically selfconsistent procedure for computation of higher order contributions is lacking⁽¹²⁾. Our phenomenological analysis in Section 7 will lead, however, to the reassuring result that in fact already for the first two known orders a correspondence type relation (2) holds convincingly well.

Let us finally glance at the experimental data for the moments and point out a purely phenomenological support for the idea of "continuity in dynamics" underlying our relation (2). Figure 1 displays the nonsinglet Nachtmann moments obtained by Duke and Roberts⁽⁷⁾ in an analysis of combined e, μ -N data of $F_2(x, Q^2)$, which exhibit the following two important features:

- (i) a dramatic and fairly sudden increase for decreasing values of $Q^2 \lesssim 2 \text{ GeV}^2$ (which is of course mainly due to the rapidly increasing importance of the elastic contribution);
- (ii) this increase gets significantly steeper with increasing order n of the moments (as clearly signalled by the

movement of the maximum towards higher values of Q^2). This effect is not a property of the elastic contribution, but due to an interplay of elastic and inelastic contributions.

Let us furthermore point out that qualitatively both these characteristic features appear in the nonsinglet and singlet moments for both, the Nachtmann as well as the Cornwall-Norton versions of the moments.

Both observed features (i) and (ii) qualitatively reflect the behaviour of a typical perturbative expression near the Landau ghost position Q_0^2

$$M_n^{\text{pert}(m)} \propto \frac{Q^2 + Q_0^2}{(\ln Q^2 - \ln Q_0^2)^{-\delta(m)} n} \quad (3)$$

with the anomalous dimension powers increasing for increasing n .

So altogether the singularity of the moment due to the Landau ghost of the coupling constant - even though clearly unphysical - may well mimic on the Q^2 average a more complex reality in the confinement sensitive region of low Q^2 .

2. CHOICE OF THE LOWER INTEGRATION LIMIT

At first sight a natural choice for the lower integration limit in the correspondence relation (2) may seem the point $Q_0^2 = 0$, where the exact moments $M_n(Q^2)$ are known to develop (for the massless case) a branchcut extending along $Q^2 = -q^2 < 0$. However, the perturbative moments instead exhibit a fake threshold at the position $Q_0^2 (> 0)$ of the Landau ghost, where the running coupling constant $\bar{g}^2/16\pi^2$ becomes infinite;

below this fake threshold $M_n^{\text{pert}(m)}$ develops an imaginary part. Hence, for consistency reasons, the position of this fake threshold

$$Q_0^2 = Q_{0,n}^2(m) \text{ defined by } \bar{g}^2(Q_{0,n}^2(m)) = \infty \quad (4)$$

should be inserted into the lower integration limit in relation (2); in general Q_0^2 depends on the order m of perturbation theory, on the order n of the moment considered and - as will be discussed in Section 5 - on the renormalization scheme (for $m \geq 2$).

3. TARGET MASS CORRECTIONS; HIGHER TWIST CONTRIBUTIONS

We consider (kinematical) target mass corrections to be taken into account by using Nachtmann moments⁽¹³⁾ on the l.h.s. of the correspondence relation (2).

As a working hypothesis we shall ignore contributions from higher twist operators on the r.h.s. of relation (2). The justification for this rather drastic assumption will (at best) come a posteriori from the phenomenological success of the correspondence relations. In the QCD framework successively higher twist contributions involve an increasing number of gluons tying the struck quark to the spectator quarks. Intuitively this type of interaction should mainly account for the resonance fluctuations in the γ^*N channel. The QCD interpretation⁽¹⁴⁾ of local Bloom Gilman duality⁽¹⁵⁾ was accordingly that down to $Q^2 \approx 1 \text{ GeV}^2$ such fluctuations average out, if integrated over the scaling variable x (or, to be more precise, the scaling

variable ξ).

Our working hypothesis implies that they continue to average out even for $Q^2 \ll 1\text{GeV}^2$ after the additional integration over Q^2 .

This is incidentally not the first time that non-leading terms - even though expected - do not show up in a correspondence type relation. In conventional finite energy sum rules the leading Regge behaviour - extrapolated down to low energies - averages well the resonance behaviour; contributions from daughter trajectories, which are only non-leading for high energies and which had been expected to be important at low energies, were never found to play any significant rôle.

4. SPECIFYING THE PERTURBATIVE MOMENTS

The (Nachtmann) moments of non-singlet combinations of structure functions are proportional to the coefficient functions $C_n^{NS}(t, g^2)$ of the spin n , non-singlet operators in the Wilson operator product expansion

$$M_n(t, g^2) = A_n^{NS} C_n^{NS}(t, g^2) \quad (5)$$

with

$$t = \ln \frac{Q^2}{\mu^2}, \quad (6)$$

g being the renormalized coupling at subtraction point μ^2 and A_n^{NS} unknown constants. As argued in Section 3 we shall only consider contributions from twist 2 operators. Since the

Wilson coefficient functions satisfy the renormalization group equation

$$\left[-\frac{\partial}{\partial t} + \beta(g) \frac{\partial}{\partial g} - \gamma^n(g) \right] C_n^{NS}(t, g^2) = 0 \quad (7)$$

the moments may be represented as

$$M_n(t, g^2) = A_n^{NS} C_n^{NS}(0, \bar{g}^2(t)) \exp \left[- \int_0^t dg' \frac{\gamma^n(g')}{\beta(g')} \bar{g}(t) \right] \quad (8)$$

in terms of the standard β -function, the anomalous dimension $\gamma^n(g)$ of the spin n Wilson operator and the running coupling constant $\bar{g}(t)$, satisfying

$$\frac{1}{2} t = \int_0^t \frac{dg'}{\beta(g')} \quad (9)$$

or equivalently

$$\frac{d\bar{g}^2}{dt} = \bar{g}\beta(\bar{g}) \text{ with } \bar{g}(t=0) = g. \quad (10)$$

By inserting the perturbative expansions up to order m

$$g\beta(g) = -g^2 \left[\beta_0 \frac{g^2}{16\pi^2} + \beta_1 \left(\frac{g^2}{16\pi^2} \right)^2 + \dots \right] \quad (11)$$

$$\gamma^n(g) = \gamma_0^n \frac{g^2}{16\pi^2} + \gamma_1^n \left(\frac{g^2}{16\pi^2} \right)^2 + \dots \quad (12)$$

$$C_n^{NS}(0, g^2) = 1 + B_n \frac{g^2}{16\pi^2} + \dots \quad (13)$$

into representation (8) the renormalization group improved perturbative expressions $M_n^{(m)}(t, g^2)$ for the moments are obtained. In particular the second order result takes the form

$$M_n^{pert(m=2)} \approx A_n^{NS} \left(\frac{\bar{g}^2}{16\pi^2} \right)^{d_n} \left[1 + \left(B_n + \frac{\gamma_1^n}{2\beta_0} - \frac{\gamma_{0,0}^n}{2\beta_1} \right) \frac{\bar{g}^2}{16\pi^2} + O\left(\left(\frac{\bar{g}^2}{16\pi^2}\right)^2\right) \right] \quad (14)$$

with $d_n = \gamma_0^n / 2\beta_0$ (15)

and second order $\bar{g}^2$ being given as implicit solution of

$$\frac{16\pi^2}{\beta_0 \bar{g}^2} - \frac{\beta_1}{\beta_0^2} \ln \left(\frac{16\pi^2}{\beta_0 \bar{g}^2} + \frac{\beta_1}{\beta_0^2} \right) = \ln \frac{Q^2}{\Lambda^2} \quad (16)$$

where Λ is defined by equation (16) evaluated for $Q^2 = \mu^2$ and $\bar{g}^2 = g^2$.

Altogether we have closely followed the notation and conventions of Ref. 11, where also the explicit expressions for the coefficients $\beta_{0,i}^n$, $\gamma_{0,i}^n$ and $B_n (= B_n^{NS})$ may be found.

5. RENORMALIZATION SCHEME DEPENDENCE

If all orders of perturbation theory could be taken into account, the resulting moments would of course be independent of the renormalization scheme used to define the running coupling constant $\bar{g}$. Truncation at a given (m 'th) order, however, introduces a renormalization scheme dependence

in the higher order $[O(\bar{g}^2)^{m+d_n}]$ corrections of the perturbative moments. In going to low Q^2 (large $\bar{g}^2$) these neglected corrections may become important. A comparison with data is then expected - in the ideal case - to single out the scheme, which minimizes the neglected higher order corrections. In this Section we shall discuss the repercussions of this scheme dependence on the correspondence relations for $m=2$.

In second order the freedom of choosing the renormalization scheme to define $\bar{g}$ amounts to the freedom of choosing an arbitrary integration constant $1/2\tau$ in eq. (9)

$$\frac{1}{2} (t-\tau) = \int_{\bar{g}}^g \frac{dg'}{\beta(g')} \quad (17)$$

This is equivalent to the freedom of

- (i) choosing the point $Q^2 = \mu^2$, at which $\bar{g}$ is equal to the renormalized coupling constant g : $\mu^2 \rightarrow \mu^2 \exp \tau$
 - (ii) redefining the unknown scale Λ^2 : $\Lambda^2 \rightarrow \Lambda^2 \exp \tau$.
- The second order moment $M_n^{(m=2)}$ - written as a function of $\bar{g}^2$ - then exhibits a renormalization scheme dependence (i.e. τ dependence) in the coefficient f_n of $\bar{g}^2$

$$M_n^{(m=2)} = A_n \left(\frac{\bar{g}^2}{16\pi^2} \right)^{d_n} \left[1 + f_n(\tau) \frac{\bar{g}^2}{16\pi^2} + O(\bar{g}^4) \right] \quad (18)$$

with

$$f_n(\tau) = B_n + \frac{\gamma_1^n}{2\beta_0} - \frac{\gamma_{0,0}^n}{2\beta_1} - \frac{1}{2} \tau \gamma_0^n \quad (19)$$

(and scheme independent combination $B_n + \gamma_1^n / 2\beta_0$). Of course the full second order contribution to $M_n^{(m=2)}$ is invariant under

the simultaneous redefinition of Λ and f_n ; different choices for τ correspond to different estimates of the $O(\bar{g}^4)$ corrections only.

In principle the freedom of choosing τ can be exercised separately for each n when solving the renormalization group equation, leading to n -dependent μ_n 's, Λ_n 's and $\bar{g}_n$'s. This n -dependence is, however, calculable in terms of τ_n :

$$\mu^2(\tau_n) = \mu^2(0) \exp \tau_n, \quad (20)$$

$$\Lambda^2(\tau_n) = \Lambda^2(0) \exp \tau_n,$$

$\bar{g}(t-\tau_n)$ given by eq. (16) with Λ replaced by $\Lambda(\tau_n)$.

In the following we shall arbitrarily associate $\tau=0$ with the minimal subtraction scheme (MS-scheme) and express other schemes with respect to $\tau=0$.

The quality of large Q^2 data ($Q^2 \gtrsim 2-3 \text{ GeV}^2$) was not sufficient to clearly single out the scheme, in which the $O(\bar{g}^4)$ corrections in eq. (18) happen to be minimal (11,1).

At first sight one might expect that our correspondence relations (2) should become very sensitive to the renormalization scheme chosen, since their evaluation includes high precision low Q^2 data on the l.h.s. and contributions from large $\bar{g}$ -values on the r.h.s. However, this gain in sensitivity is to a certain extent lost again by the "smearing effect" of the integration over Q^2 . In fact we shall find in Section 7 that the correspondence relations hold well for a similar range of schemes (with similar values for the Λ 's) as typically found

in large Q^2 moment analyses. The only distinguishing feature between different schemes will turn out to be the locality, with which the correspondence relations hold.

In later applications we shall consider the following four particular schemes frequently used in the literature

(i) minimal subtraction (MS) scheme, characterized by

$$\tau = 0 \quad (21)$$

(ii) $\overline{\text{MS}}$ scheme characterized by $\tau = 4n4\pi - \gamma_E$ (22)

(iii) Λ_n -scheme characterized[†] by $\tau_n = 2B_n/\gamma_0^n$ (23)

(iv) $\hat{\Lambda}_n$ -scheme characterized by $\tau_n = \frac{2}{\gamma_0^n} \left(B_n + \frac{\gamma_1^n}{2\beta_0} - \frac{\gamma_{0\beta_1}^n}{2\beta_0^2} \right)$ (24)

6. SECOND ORDER CORRESPONDENCE RELATIONS

The extrapolated second order perturbative moments (18) behave as

$$M_n^{(m=2)}(Q^2) \stackrel{Q^2 \rightarrow Q_0^2, \text{ i.e., }}{\sim} \frac{1+d_n}{2} (\ln Q^2 - \ln Q_0^2) - \frac{1+d_n}{2} \quad (25)$$

at the lower integration limit $Q^2 = Q_0^2$. Hence the integral on the r.h.s. of the correspondence relation (2) ceases to exist for $d_n \geq 1$, i.e., $n \geq 6$. For practical applications it is therefore convenient to replace the relations (2) by "weighted" correspondence relations: the moments in both integrands are

[†]with B_n calculated in the MS-scheme

multiplied by suitable weights, which

- (i) guarantee the existence of the integral for all values of n ,
- (ii) lead to similar results as the "unweighted" relations (2) for small values of n ,
- (iii) do not suppress unduly the contribution from the low

Q^2 data ($Q^2 \lesssim 2-3 \text{ GeV}^2$) in the l.h.s. integral.

All three conditions are well satisfied for weights

$$\bar{g}^2(Q^2)^{-d_n} \propto (\ln Q^2 - \ln Q_0^2) d_n/2 \text{ with } \bar{g} = \bar{g}^{(m=2)} \quad (26)$$

to be multiplied to both integrands $M_n^{\text{data}}(Q^2)$ and $M_n^{(m=2)}(Q^2)$ in relation (2). All second order integrands on the r.h.s. remain thus with the same singularity $\propto (\ln Q^2 - \ln Q_0^2)^{-\frac{1}{2}}$ for $Q^2 \rightarrow Q_0^2$. (To further remove this singularity by a stronger weight-function would come close to violating condition (iii)).

Finally, let us get rid of the unknown constants A_n^{NS} in the perturbative moments. We divide the whole correspondence relation by the integrand taken at the largest value $Q^2 = Q_{\text{max}}^2$ where data are available. Since Q_{max}^2 is supposed to lie well within the perturbative region, we may assume

$$M_n^{\text{data}}(Q_{\text{max}}^2) \approx M_n^{\text{pert}(m=2)}(Q_{\text{max}}^2). \quad (27)$$

Using

$$\ln Q_0^2 \int_{\ln Q_0^2}^{\ln Q^2} d(\ln Q'^2) = \int_{-\infty}^{\infty} \frac{dg'^2}{g'^2 B(g')} \quad (28)$$

in order to perform the integration on the (perturbative) r.h.s. explicitly, we obtain the following final form of the second order correspondence relations.

Unweighted correspondence relations

(In practice applicable for small values of n)

$$\begin{aligned} & \ln Q_0^2 \int_{\ln Q_0^2}^{\ln Q^2} d(\ln Q'^2) \frac{M_n^{\text{data}}(Q'^2)}{M_n^{\text{data}}(Q_{\text{max}}^2)} \\ &= \frac{1}{\beta_0 (1-d_n)} \left(\frac{\bar{g}^2(Q^2)}{\bar{g}^2(Q_{\text{max}}^2)} \right)^{d_n} \left(\frac{\bar{g}^2(Q^2)}{16\pi^2} \right)^{-1} \\ & \quad \times \left[1 + \left(f_n - \frac{\beta_1}{\beta_0} \right) \frac{\bar{g}^2(Q^2)}{16\pi^2} \right] {}_2F_1 \left(1, 1, 2-d_n; \frac{1}{1 + \frac{\beta_1}{\beta_0} \frac{\bar{g}^2(Q^2)}{16\pi^2}} \right) \end{aligned} \quad (29)$$

involving the hypergeometric function ${}_2F_1$.

Weighted correspondence relations

(Applicable for all values of n)

$$\begin{aligned} & \ln Q_0^2 \int_{\ln Q_0^2}^{\ln Q^2} d(\ln Q'^2) \left(\frac{\bar{g}^2(Q'^2)}{\bar{g}^2(Q_{\text{max}}^2)} \right)^{-d_n} \frac{M_n^{\text{data}}(Q'^2)}{M_n^{\text{data}}(Q_{\text{max}}^2)} \\ &= \frac{1}{\beta_0} \frac{\left(\frac{\bar{g}^2(Q^2)}{16\pi^2} \right)^{-1}}{1 + f_n \frac{\bar{g}^2(Q_{\text{max}}^2)}{16\pi^2}} \\ & \quad \times \left[1 + \left(f_n - \frac{\beta_1}{\beta_0} \right) \frac{\bar{g}^2(Q^2)}{16\pi^2} \ln \left(1 + \frac{\beta_0}{\beta_1} \frac{16\pi^2}{\bar{g}^2(Q^2)} \right) \right] \end{aligned} \quad (30)$$

where everywhere the second order expression (16) for $\bar{g}^2$ has to be inserted.

Let us finally point out the various places, where scheme dependences enter (see eqs. (19), (20) and (16)). For any scheme characterized by a set of values τ_n , we have

$O(\bar{g}^2)$ -coefficients:

$$f_n(\tau_n) = f_n(\tau=0) - \frac{1}{2} \tau_n \gamma_0 \quad (31)$$

Λ -parameter in the definition of $\bar{g}^2$:

$$\Lambda_n^2(\tau_n) = \Lambda^2(\tau=0) \exp(\tau_n) \quad (32)$$

lower integration limit (defined by $\bar{g}^2(Q_0^2) = \infty$)

$$\begin{aligned} \ln Q_0^2(\tau_n) &= \frac{\beta_1}{\beta_0} \ln \frac{\beta_0^2}{\beta_1} + \ln \Lambda_n^2(\tau_n) \\ &= \frac{\beta_1}{\beta_0} \ln \frac{\beta_0^2}{\beta_1} + \ln \Lambda^2(\tau=0) + \tau_n \end{aligned} \quad (33)$$

For each scheme the whole set of n correspondence relations has only one free parameter $\Lambda(\tau=0)$, say. This parameter is however, not really free, since for consistency it should agree with the value for Λ obtained from large Q^2 moment analyses. Thus, altogether, the correspondence relations (29) and (30) are essentially parameter free.

7. ANALYSIS AND RESULTS

We base the test of our correspondence relations (29) or (30) on the nonsinglet moments extracted from the combined e, μ data (16,17) of the structure functions F_2^{p-n} by Duke and

Roberts⁽¹⁸⁾. They suit our purpose best, since they are statistically much more precise at low Q^2 than nonsinglet moments from $\nu, \bar{\nu}$ data (4,19). The results of this moment analysis for $Q^2 \gtrsim 1.5 \text{ GeV}^2$ have been presented in Refs. 7, the moments below $Q^2 \lesssim 1.5 \text{ GeV}^2$ (see Fig. 1) have been kindly made available to us by private communication.

As already argued in Sect. 3, we shall use the Nachtmann moments⁽¹³⁾

$$M_n^{\text{data}}(Q^2) = \int_0^1 dx \frac{\xi^{n+1}}{x^3} \frac{n(n+2)r^2 + 3(n+1)r + 3}{(n+2)(n+3)} F_2^{p-n}(x, Q^2)$$

with

$$r = \left(1 + \frac{4m^2 x^2}{Q^2} \right)^{\frac{1}{2}}, \quad \xi = \frac{2x}{1+r}, \quad m = m_{\text{nucleon}} \quad (34)$$

on the l.h.s. of the correspondence relations in order to take into account (kinematical) target mass effects.

Since the emphasis of the correspondence relations is on the low Q^2 regime, we shall assume 3 flavours in all perturbative QCD expressions.

The moments have been interpolated by hand (see Fig. 1); for integration over moments we use this interpolation without error analysis. Q_{max}^2 in eqs. (29) and (30) was chosen to be 22.5 GeV^2 throughout.

As we have seen in Section 6 the correspondence relations (29) and (30) depend in a well-defined way on the order n of the moment and on the renormalization scheme chosen. For each scheme there is one free parameter (Λ) which is, however fixed (within

errors) from large Q^2 moment analyses. In our subsequent analysis we shall test the various predictions step by step in order to make the results as transparent as possible.

- (i) For each value of n and for each of the four schemes (21-24) we separately treat

$$\Lambda = \Lambda_n(\tau_n)$$

as an independent free parameter, where the set of values τ_n characterizes the scheme under consideration.

We determine the parameters $\Lambda_n(\tau_n)$ from the requirement that each correspondence relation holds individually in its weakest form, i.e. with upper integration limit

$$Q^2 = Q_{\max}^2 \quad \text{l.h.s. of corr. rel. (29,30)} \quad \stackrel{Q^2=Q_{\max}^2}{=} \quad \text{r.h.s. of corr. rel. (29,30)} \quad (35)$$

Let us emphasize that the existence of a solution ($\Lambda_n(\tau_n)$) is highly nontrivial. Remember that the integrands are required to be equal at the upper integration limit $Q^2 = Q_{\max}^2$ and that the integrand on the r.h.s. has necessarily a singularity at the lower integration limit $Q_{0,n}^2(\tau_n)$. A solution is in fact only possible because of the enormous increase of the experimental moments for decreasing Q^2 (see Fig. 1).

For the actual determination of the values $\Lambda_n(\tau_n)$ we use the weighted correspondence relations (30); for low values of n (in particular for $n=2$) they come out to be very close to the ones obtained from the unweighted relations (29). We have convinced ourselves that in all cases the low Q^2 portion of the integral, from $Q_{0,n}^2(\tau_n)$ to 3GeV^2 , say, makes up about half of the total integral from Q_0^2 to $Q_{\max}^2$; i.e., low Q^2 data play as important a role in this analysis as high Q^2 data (see e.g. Fig. 3):

- (ii) Having thus fixed the values $\Lambda_n(\tau_n)$ the r.h.s. integrands are completely determined for each n and each scheme considered. We can then check for the locality of the validity of the correspondence relations as a function of the scheme for fixed n and as a function of n for a fixed scheme.

Fig. 2 shows the unweighted integrands - data (18) and QCD prediction - for $n=2,3$ and various schemes[†]. Here we show the results of the unweighted relations (29), since they are most instructive. (Notice the importance of the low Q^2 contributions!)

Fig. 3 shows the (weighted) integrals as a function of the upper integration limit Q^2 for a fixed scheme

[†]The $\overline{\text{MS}}$ scheme is not shown, because it is very close to the Λ_n -scheme for $n=2,3$.

(Λ_n -scheme) and various values of n ($n=2,3,4,7$).

Figs. 2 and 3 reflect a general pattern, which applies to all schemes and all values of n and is in fact not at all surprising: the correspondence relations hold the more locally, the less influential the singularity of the r.h.s. QCD-integrand at $Q^2 = Q_0^2$ (i.e. $\bar{g}^2 = \infty$). In practice this means that the smaller the second order coefficient $f_n(\tau_n)$ in eq. (18) the more local is the validity.

From Figs. 2 and 3 we also observe that different schemes probe different regions in the low Q^2 regime (due to the scheme dependence of the lower integration limit Q_0^2).

(iii) Next, we check, whether - within each given scheme - the values $\Lambda_n(\tau_n)$ follow the predicted n -dependence

$$\Lambda_n(\tau_n) \propto \exp(\tau_n/2). \quad (36)$$

Fig. 4a shows the values[†] of $\Lambda_n(\tau_n)$ in our four schemes (21-24), as obtained according to prescription (i). We expect n -independence for the $\overline{MS}$ and $\overline{MS}$ schemes and

$$\Lambda_n \propto \exp(B_n/\gamma_0^n) \quad (37)$$

$$\Lambda_n \propto \exp\left(\frac{1}{\gamma_0^n} \left(B_n + \frac{\gamma_1^n}{2\beta_0} - \frac{\gamma_{O1}^n}{2\beta_0^2}\right)\right). \quad (38)$$

[†]Our analysis does not provide errors.

Fig. 4a shows that all four predictions are amazingly well fulfilled.

(iv) As a last check we compare the absolute values of the Λ parameters for each scheme with the corresponding values obtained from large Q^2 moment analyses. Our results are summarized in table 1. Table 2 shows a comparison with the Λ parameters from large Q^2 . Fig. 4b compares the individual Λ_n values of the Λ_n -scheme with the corresponding large Q^2 results (the comparison with the $\nu, \bar{\nu}$ data should not be taken literally, since the predicted n -dependences for moments from F_2 and xF_3 are not quite the same). In both comparisons our Λ values have been reduced by 6%; according to a private communication by R. Roberts this reduction makes up phenomenologically for the difference in the numbers of flavours entering the analyses in the two different Q^2 regimes (3 flavours at low Q^2 , 4 flavours at large Q^2).

In all cases we find good agreement of our low Q^2 Λ values with the ones from large Q^2 moment analyses.

As a final result we may now conclude that the proposed correspondence relations (2) - relating low Q^2 non-perturbative physics (represented by data) and large Q^2 perturbative QCD physics on the average - hold very well indeed already for second order ($n=2$).

Let us finally mention two interesting points. One might ask how sensitive our correspondence relations are to the specific QCD predicted numbers $B_n + \gamma_1^n/2\beta_0$, $\gamma_{0,1}^n\beta_1/\beta^2_0$ entering the second order coefficients f_n in eq. (19). One might suspect that integration washes out the sensitivity to a certain extent. This is, however, not the case. We remember that different schemes shuffle different (n-dependent) portions of the f_n 's into the $\Lambda_{(n)}$ -dependences. As stressed in (iii) we indeed found for each scheme the expected relation between the input set of f_n 's and the output set of $\Lambda_{(n)}$'s. Since, moreover, for each different scheme we are sensitive to different low Q^2 regions (see (ii)), each of these regions of data have been shown to be sensitive to the details of the second order coefficients f_n .

As a second point we notice from table 1 that the $\Lambda(\tau=0)$ values in the three schemes $\overline{MS}$, Λ_n and Λ_n^0 are equal (within errors). This is equivalent to scheme independence of the correspondence relations within the band of the three above-mentioned schemes.

8. CONCLUSIONS

Since at the present stage our correspondence relations are still based on intuitive rather than rigorous arguments, the consistent results which have emerged cannot (yet) be interpreted as a successful test of QCD. On the other hand they demonstrate that the gap between the well understood perturbative regime and the theoretically yet inaccessible non-perturbative regime is much smaller than may have been expected. In our view, the relations represent a step towards

the exciting possibility to use perturbation theory for estimating Q^2 averaged non-perturbative quantities.

ACKNOWLEDGEMENTS

We thank D.W. Duke, M.R. Pennington, R.G. Roberts and H. Rubinstein for valuable discussions. Moreover, we are grateful to D.W. Duke and R.G. Roberts for making available to us the low Q^2 results of their moment analysis. Finally we should like to acknowledge the warm hospitality of the theory group of Rutherford Laboratory, where most of this work has been done, and of the Physics Department of Durham University, where it has been completed.

TABLE 2 : comparison of Λ -parameters obtained from low Q^2 and large Q^2 analyses.

scheme	values* of Λ	
	correspondence relations involving low Q^2 data (F_2)	large Q^2 moment analyses
MS-scheme	0.34	BEBC $\nu, \bar{\nu}$; $x F_3$; Ref. 4 analysis Ref. 11 0.4
$\overline{MS}$ -scheme	0.52	BEBC $\nu, \bar{\nu}$; $x F_3$; Ref. 4 analysis Ref. 11 0.52
Λ_n -scheme with parametrization $\Lambda_n = \Lambda \exp \frac{\bar{B}_n}{\gamma_n \gamma_0}$	0.46	BEBC $\nu, \bar{\nu}$; $x F_3$; Ref. 4 analysis Ref. 11 0.4-0.5
$\bar{B}_n = B_n$ $-\frac{1}{2} \gamma_0^n (\ln 4\pi - \gamma_E)$		e, μ ; F_2 ; Ref. 16; 17 analysis Ref. 7 0.43
		CDHS; $x F_3$; Ref. 19 analysis Ref. 7 0.46

*For this comparison the Λ values from low Q^2 correspondence relations have been reduced by 6% to simulate the transition from 3 to 4 flavours. (This phenomenological rule of thumb has been communicated to us by R. Roberts).
Substantial errors [$\sim 20\%$] should of course be allowed in all cases.

TABLE 1 : Results for Λ -parameters from low Q^2 correspondence relations (3 flavours)

Λ parameter in	expressed with respect to	expressed with respect to
	MS-scheme: $\Lambda = \Lambda(\tau_n) \exp(-\tau_n/2)$	$\overline{MS}$ -scheme: $\Lambda = \Lambda(\tau_n) \exp(\frac{1}{2}(-\tau_n + \ln 4\pi - \gamma_E))$
MS scheme	0.36	0.96
$\overline{MS}$ scheme	0.21	0.55
Λ_n scheme	0.19	0.49
γ_n scheme	0.19	0.49

REFERENCES

1. recent reviews:
A.J. Buras, Fermilab - PUB-79/17-THY (1979);
L.F. Abbott and R.M. Barnett, SLAC-PUB-2325 (1979);
J. Ellis, talk at Int. Symp. on Lepton and Photon Interactions at High Energies, Fermilab, August 23-39 (1979);
A. Peterman, Physics Reports 53C, 157 (1979);
R. Shankar, Phys. Rev. D15, 755 (1977);
R.G. Moorhouse, M.R. Pennington and G.G. Ross, Nucl. Phys. B124, 285 (1977);
A.I. Sanda, Phys. Letters 85B, 273 (1979); Phys. Rev. Letters 42, 1658 (1979).
2. T. Appelquist and H. Georgi, Phys. Rev. D8, 4000 (1973);
A. Zee, Phys. Rev. D8, 4038 (1973).
3. P. Bosetti et al., Nucl. Phys. B142, 1 (1978).
4. J.G.H. de Groot et al., Phys. Letters 82B, 292 (1979).
5. H.L. Anderson et al., Phys. Rev. Letters 40, 1061 (1978).
6. D.W. Duke and R.G. Roberts, Phys. Letters 85B, 289 (1979); RL-79-073.
7. J.D. Bjorken and J. Kogut, Phys. Rev. D8, 1341 (1973).
8. L.D. Landau, A.A. Abrikosov and I.M. Khalartnikov, Doklady Acad. Nauk. URSS 95, 773 (1954).
9. E.G. Floratos, D.A. Ross and C.T. Sachrajda, Nucl. Phys. B129, 66 (1977). [erratum - B139, 545 (1978)] Phys. Letters 80B, 269 (1979).
10. W.A. Bardeen, A.J. Buras, D.W. Duke and T. Muta, Phys. Rev. D18, 3998 (1978).
11. See e.g. G. 't Hooft in "Deeper Pathways in High Energy Physics", Proc. of Orbis Scientiae, Coral Cables 1977, Chairman B. Kursunoglu, p. 699 (1977).
12. O. Nachtmann, Nucl. Phys. B63, 237 (1973); B78, 455 (1974).
13. A. De Rujula, H. Georgi and H.D. Politzer, Phys. Letters 64B, 428 (1976); Annals of Physics 103, 315 (1977).
H.D. Politzer in "Deeper Pathways in High Energy Physics" Proc. of Orbis Scientiae, Coral Cables, 1977, Chairman B. Kursunoglu, p.621 (1977).
14. E.D. Bloom and F.J. Gilman, Phys. Rev. Letters 25, 1140 (1970); Phys. Rev. D4, 2901 (1971).
15. B.A. Gordon et al., Phys. Rev. Letters 41, 615 (1978).
16. J.S. Poucher et al., Phys. Rev. Letters 32, 118 (1974); SLAC-PUB-1309 (1973);
M. Mestayer, SLAC-Rep. No. 214 (1978).
A. Bodek et al., SLAC-PUB-2248 (1979).
17. Ref. 7, and D.W. Duke and R.G. Roberts, private communication.
18. J.G.H. de Groot, Z. Phys. C1, 143 (1979).

FIGURE CAPTIONS

FIG. 1 : F_2^{p-n} nonsinglet moments (data and interpolation), normalized by the interpolated value at $Q^2 = Q_{\max}^2 = 22.5\text{GeV}^2$. The moments are from an analysis (18) of the combined e, μ data (16,17) for deep inelastic structure functions F_2^{p-n} .

FIG. 2 : Same moments and interpolations as in Fig. 1 for $n=2,3$. Integrands of (unweighted) correspondence relations for different schemes are shown. The shaded areas illustrate the averaging property and the varying degree of locality in different schemes.

FIG. 3 : Test for locality of the (weighted) correspondence relations in the Λ_n -scheme. The l.h.s. integrals (data, $\bullet$) and r.h.s. integrals (second order QCD, ---) are displayed as functions of the upper integration limit $\ln Q^2$. Decreasing locality for increasing n becomes apparent.

FIG. 4a : The Λ -values $[\Lambda_n(\tau_n)]$ as determined individually for each n and four schemes from (weighted) correspondence relations. They are in good agreement with the n -dependences predicted from second order QCD (---).

FIG. 4b : The Λ values ($\bullet$) in the Λ_n -scheme, as obtained from (weighted) correspondence relations (reduced by 6% to account for the transition from $n_f=3$ to $n_f=4$). Comparison with the values obtained from large Q^2 nonsinglet moment analyses of e, μ and $\nu, \bar{\nu}$ data (Refs. 16,17,4,19,18).

Fig.1

Fig.2

Fig.3

Fig.4a

Fig.4b