07 AVR. 1980

Arkiv for Det Fysiske Seminar i Trondheim

No 1 - 1980

Correlated Percolation on a Quadratic Lattice

by

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In the standard site percolation problem the sites on the lattice are randomly occupied [1]. A natural generalization, in which the lattice sites are occupied according to a probability distribution corresponding to a thermal equilibrium system with a nearest-neighbour interaction $(-J)$, has recently received some interest [2-4]. The standard site percolation problem is then recovered in the infinite-temperature limit.

The critical percolation probability p_c will depend upon the dimensionless nearest-neighbour coupling strength $K=J/kT$, and the present note comments upon the function $p_c(K)$ for the quadratic lattice. We show first an exact equivalence between the strongly negatively correlated system $(J \rightarrow -\infty)$ and an uncorrelated percolation problem on a lattice with a different connectivity. From this equivalence the numerical value of $p_c(-\infty)$ is determined with good precision. Secondly, we employ a simple real-space renormalization method to give a rough qualitative determination of the function $p_c(K)$.

For very strong repulsive interactions and a number density of one-half, it is clear that the particles will fill one of the two sublattices (Fig.1a). This configuration is of course not percolating so a density $p > \frac{1}{2}$ is required to reach the percolation threshold, and the additional particles are filled in on the other sublattice. All configurations of these excess particles have the same energy, so these particles occupy the sites of the second sublattice randomly.

Consider now the uncorrelated percolation problem on the second sublattice, with the first sublattice fully occupied. It is clear from Fig.1 that filling an empty site on the second

sublattice provides a connection between both the nearest-neighbour and the next-nearest-neighbour sites on this sublattice (Fig.1b).

Fig.1. Configuration for $K \rightarrow -\infty$ and $p = \frac{1}{2}$. (Occupied sites are black). The percolation problem connected with occupying the second sublattice in (a) is equivalent to the uncorrelated site percolation problem on the lattice shown in (b).

Thus, the strongly negatively correlated percolation problem on the quadratic lattice is equivalent to the uncorrelated percolation problem on the square lattice with diagonal bonds present. Denoting the critical probability for the latter problem by $\tilde{p}_c$ we have

$$p_c(-\infty) = \frac{1}{2}(1 + \tilde{p}_c) \quad (1)$$

A series-expansion estimate [6] and a real-space renormalization estimate [7] both yield $p = 0.41$. Using the error bounds (± 0.01) for the series-expansion results [1] we obtain

$$p_c(-\infty) = 0.705 \pm 0.005 \quad (2)$$

As far as we know no estimate exists in the literature for this limiting value.

A rough estimate of the influence of correlations upon the critical percolation density p_c may be obtained by a simple real-space renormalization group [8] calculation in which four sites of the original lattice constitute a new renormalized site.

Several prescriptions have been proposed [9-12] for a suitable definition of when a cell percolates (in other words, when the renormalized cell is occupied). For the present purpose it is convenient to define a cell as percolating when a vertical path of occupied sites exists within the cell [13] (Fig.2a). The weight of a configuration is determined by a hamiltonian H containing nearest-neighbour interactions and a field h ,

$$H = K \sum_{n.n.} s_i s_j + h \sum_i s_i \quad . \quad (3)$$

A spin up ($s_i = +1$) is identified with an occupied site, spin down with an empty site.

For an isolated cell the average density of occupied sites is easily found to be

$$p = \frac{e^{4K+4h} + 3e^{2h+2} + e^{-4K} + e^{-2h}}{e^{4K+4h} + 4e^{2h+4} + 2e^{-4K} + 4e^{-2h} + e^{-4K-4h}} \quad (4)$$

and the probability p' that the cell is percolating is given by

$$p' = \frac{e^{4K+4h} + 4e^{2h+2}}{e^{4K+4h} + 4e^{2h+4} + 2e^{-4K} + 4e^{-2h} + e^{-4K-4h}} \quad (5)$$

Considering p and K as our basic variables and h as an auxiliary quantity to be eliminated, equations (4) and (5) constitute an expression for $p' = f_1(p, K)$.

To obtain an expression for the renormalized coupling strength K' a simple two-cell finite lattice approximation is used. The renormalized hamiltonian for two nearest-neighbour cell (Fig.2b) is determined and the renormalized nearest-neighbour interaction K' is extracted from it in the standard way.

Eliminating the field h once more, we obtain in this way $K' = f_2(p, K)$. This equation together with the previous relation $p' = f_1(p, K)$ constitute our renormalization transformation equations.

Fig.2. (a) Percolating cell configurations. (b) Two-cell lattice used in the calculation of K' . An average between the slightly different horizontal and vertical renormalized pair interactions is taken.

The transformation has two non-trivial fixed points $F = (p^*, K^*)$. The first one $F_1 = (0.618, 0)$ corresponds to random percolation, and is an attractive fixed point within a one-dimensional critical subspace, shown in Fig.3. The other fixed point $F_2 = (0.535, 0.568)$ marks the endpoint of the critical line, and corresponds to the critical point of the ferromagnetic Ising

Fig.3. Approximate renormalization result for the critical density for correlated percolation. The asymptotic value $p_c(-\infty)$ is shown by a dashed line.

transition, known to occur at $K_c=0.441$. Thus the critical behaviour for correlated site percolation with $T_c < T \leq \infty$ equals that for random percolation. In contradistinction to the triangular lattice [5], however, the value of p_c on the quadratic lattice (Fig.3) is nonuniversal.

Two kinds of uncertainties is connected with the method used here to determine the function $p_c(K)$. Firstly, it is unclear what constitutes proper definitions of when a cell percolates [10]. In connection with thermal correlations, as here, it is particularly so [5]. Secondly, to set up the renormalization equations we have employed extremely simple approximations. Very accurate results are therefore not to be expected. The qualitative features that emerge are nevertheless interesting. First of all, the fixed-point structure implies that correlated percolation is for $T > T_c$ in the same universality class as random percolation. Secondly, the critical probability is a monotonically decreasing function of the interaction energy (inverse temperature), as one would intuitively expect. The numerical results $p_c(-\infty)=0.68$, $p_c(0)=0.62$, $p_c(K_c)=0.54$ should be compared with $p_c(-\infty)=0.70-0.71$ from the first part of this article, with the series estimate $0.562 < p_c(0) < 0.600$ [14] and with $p_c(K_c)=0.50$, which should be exact [1,3].

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