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Non-universal Spin Systems on Union-Jack Lattices

by

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The universality hypothesis in the theory of phase transitions implies that with respect to critical properties all systems can be divided into a few classes, characterized by the spatial and spin dimensionalities, the symmetry of the ordered phase and whether the range of interparticle interactions is finite or infinite. It is well known that universality does not hold in the presence of some many-spin interactions; the eight-vertex (Baxter 1971, Wu 1971) model, in which the critical exponents vary continuously with the interaction strengths, is the canonical example.

In the present note we remark that certain Ising spin systems on a union-jack lattice with pair interactions, alternatively with three-spin interactions, exhibit non-universal behaviour.

The union-jack lattice, shown in Fig.1a, is composed of elementary cells, in which a central spin s_0 interacts with four neighbouring spins s_1, s_2, s_3 and s_4 (See Fig.1b)

Fig.1 a) The union-jack lattice . b) An elementary cell.

Let us first consider pair interactions and let the contribution to the lattice hamiltonian from the elementary cell be

$$H(s_0, s_1, s_2, s_3, s_4) = -I_1(s_0s_1 + s_0s_2 + s_0s_3 + s_0s_4) - \frac{1}{2}I_2(s_1s_2 + s_2s_3 + s_3s_4 + s_4s_1) - I_3(s_1s_3 + s_2s_4) . \quad (1)$$

The factor $\frac{1}{2}$ in the second term takes into account that each of the corresponding interactions appears twice, in two adjacent cells. Special cases are well known. In particular $I_1=I_3=0$ and also $I_2=I_3=0$ corresponds to the nearest-neighbour Ising model on the square lattice. Moreover, when $I_3=0$ and both I_1 and I_2 are nonvanishing, Vaks et al. (1965) have obtained the exact solution, which again exhibits the same critical behaviour as the n.n. Ising model. Vaks et al. remarked that one "cannot exclude the possibility that when other interactions in the lattice are taken into account, for which the problem cannot be solved exactly, the singularities will be of a different form, but this possibility seems to us physically improbable." We will, however, argue that inclusion of some additional next-nearest-neighbour coupling I_3 , the value of which is determined by I_1 and I_2 , will in fact change the nature of the singularities. This will not be a case of competing interactions.

By a dedecoration transformation (star-square transformation, Pais 1963) the partition function is easily reduced to the partition function of the Baxter model. In the contribution of one cell to the partition function we eliminate s_0 by summation and obtain

$$\sum_{s_0=\pm 1} e^{-\beta H} = \exp[k + (k_2 - \frac{1}{2}\beta I_2)(s_1s_2 + s_2s_3 + s_3s_4 + s_4s_1) + (k_3 - \beta I_3)(s_1s_3 + s_2s_4) + k_4s_1s_2s_3s_4] , \quad (2)$$

with $\beta=1/kT$ and

$$k_2=k_3=\frac{1}{8}\ln \cosh (4\beta I_1) ; k_4=-\frac{1}{8}\ln \frac{(\cosh 2\beta I_1)^4}{\cosh 4\beta I_1} . \quad (3)$$

The right-hand side of (2) describes the Baxter model if the nearest-neighbour coupling vanishes, i.e. when

$$\exp(4\beta I_2) = \cosh(4\beta I_1) . \quad (4)$$

For given interactions $0 < I_2 < I_1$ this is always fulfilled at some temperature. The Baxter model with a four-spin coupling k_4 and a next-nearest neighbour coupling $K_3 = k_3 - \beta I_3$ has a critical temperature determined (Wu 1971) by

$$e^{2k_4} |\sinh 2K_3| = 1$$

which by (3) may be rewritten as

$$\exp(-2\beta I_3) = -\cosh(2\beta I_1) + [\cosh^2(2\beta I_1) + \cosh^2(4\beta I_1)]^{\frac{1}{2}} . \quad (5)$$

We now choose the interaction I_3 so that (4) is fulfilled precisely at this critical temperature. Elimination of β between (4) and (5) yields the required relation between I_1, I_2 and I_3 . Then the partition function of the original model equals the Baxter partition function at criticality. At temperature close to the critical temperature the presence of small nearest-neighbour terms can be taken into account by perturbation theory (see Wegner 1971). Noting also that the Baxter model couplings are regular functions of the original couplings one finds that the specific heat critical index α has the Baxter model non-universal behaviour.

Another example of non-universal behaviour on the same lattice arises from interactions that are odd under spin reversal. Consider a general set of three-spin interactions together with a magnetic field acting on the central spin s_0 , i.e. a cell hamiltonian

$$\begin{aligned}
 -\beta H = & p_1 s_0 s_1 s_2 + p_2 s_0 s_2 s_3 + p_3 s_0 s_3 s_4 + p_4 s_0 s_4 s_1 + p_5 s_0 s_1 s_3 + \\
 & p_6 s_0 s_2 s_4 + h s_0 .
 \end{aligned} \tag{6}$$

Elimination of s_0 yields

$$\sum_{s_0} e^{-\beta H} = \exp[q_1 + q_2 s_1 s_3 + q_3 s_2 s_4 + q_4 s_1 s_2 s_3 s_4] , \tag{7}$$

with no nearest-neighbour interaction in the following three cases:

(i) $p_5 = p_6 = h = 0$, (ii) $p_1 = p_2 = p_3 = p_4 = 0$, as well as (iii) $p_1 = p_2 = p_3 = p_4$ and $p_5 + p_6 + h = 0$. In all cases the new coupling constants may be written in the form

$$\begin{aligned}
 \exp(4q_1) &= 16 \, abcd ; \\
 \exp(4q_2) &= ab/cd ; \\
 \exp(4q_3) &= ad/bc ; \\
 \exp(4q_4) &= ac/bd ,
 \end{aligned} \tag{8}$$

where a, b, c and d are some combinations of the original coupling constants in (6), given in Table 1. In terms of these parameters

	a	b	c	d
(i) $p_5 = p_6 = h = 0$	$\cosh(p_1 + p_2 + p_3 + p_4)$	$\cosh(p_1 + p_2 - p_3 - p_4)$	$\cosh(p_1 - p_2 + p_3 - p_4)$	$\cosh(p_1 - p_2 - p_3 + p_4)$
(ii) $p_1 = p_2 = p_3 = p_4 = 0$	$\cosh(p_5 + p_6 + h)$	$\cosh(p_5 - p_6 + h)$	$\cosh(p_5 + p_6 - h)$	$\cosh(p_5 - p_6 - h)$
(iii) $p_1 = p_2 = p_3 = p_4 = p$ $p_5 + p_6 + h = 0$	$\cosh 4p$	$\cosh 2p_6$	$\cosh(2p_5 + 2p_6)$	$\cosh 2p_5$

Table 1. The functions a, b, c, d that enter Eq.(8) defining the connection between the three-spin interactions and the Baxter model interactions. Cases (i) and (ii) have been considered by Hintermann and Merlini (1972), and Jüngling and Obermair (1975), respectively.

the condition for criticality in the Baxter model (7) can also be written very simply as

$$b + d = |a - c| . \quad (9)$$

In conclusion we have shown that on the union-jack lattice models with pair interactions only, and also models with three-spin interactions, may exhibit non-universal behaviour.

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