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SCALING AT A BIFURCATION POINT *)

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There exists a class of models in statistical physics which exhibit phase transitions with critical exponents which vary with coupling constants, in apparent conflict with universality. A well-known example of this behaviour is the eight vertex model which was solved exactly by Baxter ¹⁾ in 1971. Kadanoff and Wegner ²⁾ applied the scaling theory of critical phenomena to show that the special property of this model was due to the existence of a marginal operator leading to a line of critical behaviour. In some examples this line ends, signalling the appearance of another marginal operator. For further details we refer the reader to the lecture on this subject by

L. Kadanoff in this volume. In this lecture we will show how one obtains the singularities of thermodynamic functions at the end point or bifurcation point of such a critical line ³⁾⁻⁵⁾. For concreteness we will illustrate our approach by a detailed application to the lattice gas q state Potts model. We will present also some recent results obtained in collaboration with John Cardy to include the effects of an ordering field in this model ⁴⁾.

In order to understand the critical behaviour at a bifurcation point we will assume the existence of a scaling field ψ which becomes marginal at this point. In practice an essential problem is to identify this field in the model under consideration, but we shall not concern ourselves with this question here, except to note that the model may have to be extended in a non-trivial way. For example in the case of the q state Potts model, Baxter ⁶⁾ has shown by an exact calculation at the critical temperature that at $q=4$ the phase transition changes from second to first order, but RG calculation failed to obtain this bifurcation until the work of Nienhuis et al. ^{7),8)} who treated the extended lattice gas q state Potts model by a generalization of the majority rule. Subsequently we carried out an approximate Migdal-Kadanoff transformation ³⁾ which revealed the existence of a critical exponent which vanished at the bifurcation point, and we could identify an associated marginal scaling field ψ with dilution in this model. Another well-known example is the x - y model where the RG calculations of Kosterlitz and Thouless ^{9),10)} and Jose et al. ¹¹⁾ included the effects of vortex concentration.

To illustrate how the scaling field ψ becomes marginal we consider a finite scaling transformation which generates a non-linear mapping

$$\psi' = \mathcal{R}(\psi, \epsilon) \tag{1}$$

where ϵ parametrizes the critical line, e.g., for the q state Potts model $\epsilon = 4 - q$. An essential property of this mapping is that it be analytic in ψ and ϵ at the bifurcation point $\epsilon = 0$. It is represented geometrically for the cases $\epsilon > 0$ and $\epsilon = 0$ in Figs. 1 and 2, respectively. For $\epsilon > 0$ this mapping has an attractive (irrelevant) fixed point ψ_- and a repulsive (relevant) fixed point ψ_+ which coalesce when $\epsilon = 0$ at $\psi = \psi_c$. It can be readily verified geometrically that in this case the points $\psi < \psi_c$ map towards $\psi = \psi_c$ while the points $\psi > \psi_c$ map away from $\psi = \psi_c$ corresponding to a marginal fixed point.

Since the mapping has been assumed to be analytic in ψ and ϵ , we can expand Eq. (1) for small values of ϵ and $\psi - \psi_c$. To lowest order we have

$$\psi' = (1 + p\epsilon)\psi + a\psi^2 - r\epsilon \quad (2)$$

where the constants $a, r > 0$, the constant p is arbitrary, and we have chosen for simplicity $\psi_c = 0$. For $\epsilon > 0$ the fixed points $\psi' = \psi$ are approximately at

$$\psi_{\pm} \approx \pm \sqrt{\frac{r}{a}\epsilon} \quad (3)$$

with corresponding eigenvalues $\lambda_{\pm}$

$$\lambda_{\pm} = \left. \frac{d\psi'}{d\psi} \right|_{\psi_{\pm}} = 1 + p\epsilon \pm 2\sqrt{ar\epsilon} \quad (4)$$

At $\epsilon = 0$ we see again analytically that this corresponds to a marginal case where $\lambda_+ = \lambda_- = 1$

For $\epsilon < 0$, there are no fixed points until $\epsilon < (-4ar/p^2)$. We shall consider here in further detail only the case $p = 0$ which corresponds, as we shall see later on, to the lattice gas Potts model. It is of interest to note that this mapping has also been invoked recently by Pomeau and Manneville¹²⁾ to explain the phenomena of intermittency in turbulence in numerical calculations of the Lorenz model.

To solve the non-linear mapping equation it is simplest to consider an infinitesimal scaling transformation. Setting $a \rightarrow a dl$ and $\psi' = \psi + d\psi$ in Eq. (2) and choosing ϵ so that $r=a$, this mapping becomes a differential equation

$$\frac{d\psi}{dl} = a(\psi^2 - \epsilon) \quad (5)$$

This equation corresponds to a reduced form of the equations introduced by Anderson and Yuval¹³⁾ for the Kondo problem and by Kosterlitz¹⁰⁾ for the x-y model. For $\epsilon > 0$ the solution near the fixed points $\psi = \pm \sqrt{\epsilon}$ can be written approximately in the form

$$\psi(l) = \pm \sqrt{\epsilon} + (\psi(0) \mp \sqrt{\epsilon}) e^{y_{\pm}' l} \quad (6)$$

where $y_{\pm}' = \pm 2a \sqrt{\epsilon}$ are the critical exponents.

The correlation length ξ scales according to

$$\xi(l) = \xi(0) e^{-l} \quad (7)$$

which leads to the familiar result that

$$\xi_{\pm} \propto |\psi \mp \sqrt{\epsilon}|^{-1/y_{\pm}'} \quad (8)$$

near the fixed points. However, in the limit $\epsilon = 0$ this approximation is obviously not valid and we must solve Eq. (5) exactly.

In the case $\epsilon = 0$, it is of interest to note that Eq. (5) corresponds also to the lowest order renormalization equation for the coupling constant $g^2 = \psi$ in QED and QCD, and for the conductance of a random system discussed in last week's lectures by Thouless, Abrahams and Lee.

The solution of Eq. (5) for $\epsilon = 0$ has the familiar form

$$\psi(l) = \psi(0) / (1 - a\psi(0)l) \quad (9)$$

and substituting this expression in Eq. (7) we find that the correlation ζ has an essential singularity in ψ ,

$$\zeta \sim e^{1/a\psi} \quad (10)$$

To proceed further we now assume there exists in our model also a thermal scaling field φ which is coupled to the dilution field ψ . According to our analyticity assumption at the bifurcation point, φ satisfies the equation

$$\frac{d\varphi}{d\ell} = (y_T + b\psi)\varphi \quad (11)$$

to lowest order in a series expansion in powers of φ , ψ and ϵ , where y_T and b are constants^{3),4)}. For $\epsilon > 0$ the thermal exponents at the fixed point $\psi_{\pm} = \pm \sqrt{\epsilon}$ are

$$y_{\pm}^T = y_T \pm b\sqrt{\epsilon} \quad (12)$$

To obtain the critical behaviour at the bifurcation point $\epsilon = 0$ we substitute Eq. (9) in Eq. (11) and integrate to obtain

$$\frac{\varphi(\ell)}{\varphi} = \left[\frac{\psi(\ell)}{\psi} \right]^{b/a} e^{y/a \left(\frac{1}{\psi} - \frac{1}{\psi(\ell)} \right)} \quad (13)$$

Fixing $\varphi(\ell) = \text{const.}$ outside the critical region, Eq. (13) determines $\psi(\ell)$ as a function of φ and ψ , and it can be readily seen that as $\varphi \rightarrow \pm 0$ for fixed $\psi < 0$ $\psi(\ell) \rightarrow 0^-$. Solving Eq. (7) for the correlation ζ as a function of φ and ψ in this limit we obtain³⁾

$$\zeta(\varphi, \psi) \sim |\varphi|^{-1/y_T} \left[1 - |\psi| \frac{a}{y_T} \ln |\varphi| \right]^{b/ay_T} \quad (14)$$

This expression shows that there are logarithmic corrections to the familiar power law divergence of the correlation function at a bifurcation point except at the special point $\psi = 0$. Similar results have also been obtained by L. Kadanoff at the bifurcation point of the Ashkin-Teller model⁵⁾.

The results which we have obtained here can be applied directly to the lattice gas Potts model^{3),4)}. The Hamiltonian H for this model is

$$H = -J \sum_{i,j} \delta_{s_i s_j} t_i t_j - \mu \sum_i t_i$$

where $s_i = 1, 2, \dots, q$ are the Potts spin variables and $t_i = 0, 1$ are the lattice gas occupation numbers. The scaling fields φ and ψ are non-linear functions of the coupling constant J and the chemical potential μ . Recently den Nijs¹⁴⁾ has conjectured that the thermal exponent $y_T(q)$ for this model is given by

$$y_{\pm}^T(q) = \frac{3+3x}{2+x} \quad (15)$$

where $x = \pm(2/\pi) \cos^{-1}(\sqrt{q/2})$. Expanding near $q=4$, this expression corresponds to Eq. (12) to lowest order in $4-q$ with $\epsilon = 4-q$, $y_T = \frac{3}{2}$ and $b = 3/4\pi$. To obtain the parameter a we evaluate the dependence of the latent heat L on ϵ for $q > 4$ from Eqs. (5) and (11) and compare the results with Baxter's exact calculation⁶⁾. The details of this calculation can be found in Refs. 3) and 4). We find

$$L \sim e^{-\pi/2a\sqrt{q-4}} \quad (16)$$

which has the analytic form of Baxter's solution as $q \rightarrow 4$, and implies $a = 1/\pi$. Substituting these results in Eq. (14) we obtain for the correlation length for the $q=4$ Potts model as $\varphi \rightarrow 0$

$$\xi \sim |\varphi|^{-2/3} [-\ln|\varphi|]^{1/2} \quad (17)$$

We can readily extend the RG Eqs. (5) and (11) to include the effects of a symmetry breaking field h ⁴⁾. Setting

$$\frac{dh}{d\ell} = (y_H + c\psi)h \quad (18)$$

we obtain for $\epsilon > 0$ at the fixed points $\psi_{\pm} = \pm \sqrt{\epsilon}$ that the magnetic exponents $y_{\pm}^H$

$$y_{\pm}^H(q) = y_H \pm c \sqrt{\epsilon} \quad (19)$$

Until recently, it was generally assumed that $y_H(q) = 15/8$ for all q which implies $c = 0$. However, Baxter's ¹⁵⁾ recent solution of the hard hexagon problem indicates that $y_H(3) = 28/15$. The best conjecture which fits all the known data on $y_H(q)$ has been proposed by Pearson ¹⁶⁾, who suggests that

$$y_{\pm}^H(q) = \frac{15 + 8x + x^2}{8 + 4x} \quad (20)$$

with x the same function of q as in den Nijs conjecture, Eq. (15). This implies that $y_H = 15/8$ and $c = 1/16\pi$ in Eq. (19).

Proceeding in a similar manner as before we now find for the spontaneous magnetization of the $q=4$ state Potts model ⁴⁾

$$M \propto |\varphi|^{\beta} [-\ln|\varphi|]^{-(b\beta+c)/a} = |\varphi|^{\frac{1}{12}} [-\ln|\varphi|]^{-\frac{1}{8}} \quad (21)$$

with $\beta = (2 - y_H)/y_T$ while at the critical temperature ⁴⁾ $\varphi = 0$

$$M \propto |h|^{1/8} [-\ln h]^{-2c/ay_H} = [h/(-\ln h)]^{1/15} \quad (22)$$

with $\delta = y_H(2 - y_H)^{-1}$. Also at $\varphi = 0$ the order parameter correlation function $G(r)$ has a logarithmic correction in the distance r and we find ⁴⁾

$$G(r) \propto r^{-\eta} (\ln r)^{-2c/a} = r^{-1/4} (\ln r)^{-1/8} \quad (23)$$

where $\eta = 2(2 - y_H)$. Recently this same $(\ln r)^{-1/8}$ correction was found for the correlation function of the x - y model ¹⁷⁾.

Finally in Figs 3a and 3b, we show the schematic phase diagram for the $q=4$ and $q=3$ state Potts models. There is experimental interest in these diagrams because these models represent approximately the properties of certain monolayers of gases absorbed on graphite. For example, absorbed He^4 and Kr can be represented by $q=3$, while N_2 and Xe may provide representations for $q=4$. In all these cases by varying the pressure, the surface density can be changed so that transition between the in-registry solid phase and a liquid or gas-like surface phase can occur. In particular we find that for $q=4$, the Potts lattice gas exhibits an essential singularity in the density change $\Delta\rho$ of the form

$$\Delta\rho \sim e^{-A/\bar{t}} \quad \bar{t} > 0 \quad (24)$$

where $\bar{t} = (T_t - T)/t$ measures how far the temperature is below the point of phase separation. We do not know A because the parameters relating the scaling fields to the temperature are not determined. The behaviour given in Eq. (24) should be contrasted with the $q=3$ state Potts model in which

$$\Delta\rho \sim |\bar{t}|^\omega \quad (25)$$

where $\omega \approx \pi/4$. Finally the latent heat ΔL is proportional to $\Delta\rho$ while the discontinuity in the order parameter varies as $(\Delta\rho)^{1/4}$.

The renormalization group approach discussed here based on differential equations in which the parameters can be determined from known exact or numerical results extends our understanding of scaling and of logarithmic and essential singularities which occur at a bifurcation point in critical phenomena. In addition it provides essential clues for designing Padé approximations and organizing Monte Carlo calculations to obtain information about this type of critical behaviour. Previously, the only detailed information on logarithmic corrections which arise near an upper critical dimension have been studied in terms of corrections to mean field theory for a three-dimensional uni-axial ferromagnet with strong dipolar couplings¹⁸⁾.

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FIGURE CAPTIONS

Figures 1 and 2 Geometrical illustration of the mapping $\psi \rightarrow \psi'$,
Eqs. (1) and (2), for $\epsilon > 0$ and $\epsilon = 0$.

Figure 3a The phase diagram for a $q=4$ Potts lattice gas indicating
the cusp which arises from the essential singularity in $\Delta\rho$
at the bifurcation point.

Figure 3b The corresponding phase diagram for the $q=3$ state
Potts model.

fig. 1

fig. 2

FIG. 3