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A THEOREM ON RESUMMING LEADING LOGARITHMS IN NON-ABELIAN
GAUGE THEORIES AT SHORT DISTANCES

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ABSTRACT

Given a power series expansion in two parameters α and t (with, for instance, $\alpha = O(1/10)$ and $t \gg 1$), the sum of the leading asymptotic part in t of each individual term in the series does not in general give the correct asymptotic behaviour of the whole sum of the series. In fact, when such a procedure works, it is the exception rather than the rule. In QCD at short distances a similar procedure known as the "leading logarithm approximation" is nevertheless used. In this note we present a theorem justifying this procedure. The underlying reason is the invariance with respect to the renormalization group (RG) of physical quantities and the covariance of Green functions and vertex amplitudes under this group. Whereas the individual terms of straightforward perturbation expansion for these quantities are not invariant (covariant) with respect to RG, the quantities themselves are invariant (covariant). In order to maintain these invariance properties, the usual perturbation expansion must be recast in a different form, the leading term of which (in the short distance region) is precisely the sum of the leading logarithms.

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It has recently become very fashionable to perform various resummations of selected terms of the straightforward perturbation expansion. One of the favourites is the so-called "leading log approximation" which has revived ¹⁾ the works of Landau ²⁾ on the electromagnetic charge. It is rather ironic that the initiator of such an "approximation" in fact studied a case where it is inconsistent (Landau's fictitious pole) and has therefore demonstrated the danger of performing such resummations without having a sound theoretical basis justifying the procedure ³⁾.

However, things seem to go much better now in studying quantum chromodynamics in the short distance region. Why is that so? Such a question is not of minor importance. Indeed, summing the leading asymptotic part of each individual term in a series expansion does not in general give the right asymptotic behaviour of the whole sum of the series. When such a procedure works it is not the rule but rather the exception. One can briefly illustrate this assertion by looking at one among the plethora of examples which one can easily find. Take, for instance

$$K(\alpha, x) = \frac{1}{1 + \alpha \ln x} + \alpha \ln(1 + \alpha \ln x)$$

Its expansion in α is

$$K(\alpha, x) = 1 - \alpha \ln x + \alpha^2 (\ln^2 x + \ln x) - \alpha^3 (\ln^3 x + \frac{1}{2} \ln^2 x) + \dots + (-1)^n \alpha^n (\ln^n x + \frac{1}{n-1} \ln^{n-1} x) + \dots$$

On resumming the leading logs in each individual terms in the region $\ln x \gg 1$, one would get

$$K(\alpha, x) \sim \frac{1}{1 + \alpha \ln x} \underset{\ln x \rightarrow \infty}{\sim} \frac{1}{\ln x} \rightarrow 0$$

However the true asymptotic behaviour of the whole sum is

$$K(\alpha, x) \underset{\ln x \rightarrow \infty}{\longrightarrow} \alpha \ln(1 + \alpha \ln x) \sim \ln \ln x \longrightarrow \infty$$

This elementary example shows, if it is necessary, the importance of non-leading contributions in the individual terms which, in this case, give the right asymptotic behaviour of the whole sum while the leading ones do not.

The reason for the success of the "leading log approximation" in the QCD large Q^2 region rests on the fact that the theory ^{*)} enjoys an invariance called renormalization group invariance (RG invariance) which, in its simplest form, is the invariance of physical quantities under the group of scale transformation

$$\begin{aligned} \alpha_s &\longrightarrow z_3^{-1} z_2^{-2} z_1^2 \alpha_s \\ D &\longrightarrow z_3 D \\ G &\longrightarrow z_2 G \\ \Gamma &\longrightarrow z_1^{-1} \Gamma \quad \text{etc.} \end{aligned} \quad \begin{array}{l} z_i \text{ finite} \\ \text{and real} > 0 \end{array} \quad (1)$$

affecting simultaneously the coupling and various Green functions ^{**)}.

This RG invariance can also be restated ^{***)} as the invariance of the physical quantities calculated in QCD, under a transformation of the renormalization scale μ

$$\mu \rightarrow \mu \exp\{t\} \quad (\text{RG transformation}) \quad (2)$$

together with

$$g \rightarrow \bar{g}(t) \quad ; \quad t = \frac{1}{2} \log Q^2 / \mu^2 \quad (3)$$

which is a representation of (2). This invariance is controlled by non-perturbative arguments ⁴⁾.

It is well known that if $\bar{g}^2(t)$ is written as

$$\frac{\bar{g}^2(t)}{4\pi} = \bar{\alpha}(t) = \alpha d(t, \alpha) \quad \left\{ \begin{array}{l} \text{momentum sub-} \\ \text{traction scheme} \end{array} \right\} \quad (4)$$

the invariance under the transformations (1) or (2)-(3) recasts the usual perturbation expansion of d in terms of α into the form

*) The same is true for any renormalizable field theory.

***) Analogous transformations exist for α_s and D when α_s is defined by means of the three-gluon vertex. The unicity of α_s arises from the Slavnov-Taylor identities of this gauge theory.

***) The group of transformations (2)-(3) is smaller than (1). The transformations (1) can be written in a way which leaves invariant the Lagrangian of the theory.

$$d^{-1}(t, \alpha) = f_0(\alpha t) + \alpha f_1(\alpha t) + \dots + \alpha^n f_n(\alpha t) + \dots \quad (5)$$

where the f_i are functions of the single argument $(\alpha \cdot t)$ with known analytic structure, up to numerical coefficients which are to be extracted from straightforward perturbation theory. This new expansion (5) (improved perturbation expansion) has the property that at each stage [for instance stopping after $\alpha^k f_k(\alpha t)$] it satisfies the RG invariance (1) or (2)-(3). This is manifestly not the case for the straightforward expansion in α . Therefore (1), or (2)-(3), recasts the usual straightforward perturbation expansion for $d^{-1}(t, \alpha)$ in such a way that it takes the form (5). As

$$f_0(\alpha t) = 1 + 2b_0 \alpha t \quad ; \quad t = \frac{1}{2} \log Q^2/\mu^2 \quad (6)$$

the first term in (5) gives

$$d(t, \alpha) = \frac{1}{1 + 2b_0 \alpha t} \quad (7)$$

which is just the resummation of the leading logs $\alpha t = (\alpha/2) \ln Q^2/\mu^2$.

It is clear that in the short distance region $\alpha t \gg 1$, the expansion (7) is well behaved and is a first stage in the explanation of the leading log approximation success. A second stage is that, in the same region, the next to leading logs represented by $\alpha f_1(\alpha t)$ are really non-leading, being of the form

$$c, \alpha \ln(1 + 2b_0 \alpha t) \quad (8)$$

and $\ln t \ll t$ in the region $t \gg 1$. The same is true for the next f_i 's which behave like constants and inverse powers of t . So

$$d \approx \frac{1}{1 + 2b_0 \alpha t + b_1 \alpha \ln t + \alpha b_2 + \alpha^2 b_3/t + \dots} \quad (9)$$

in the region $t \gg 1$, ($t \sim \ln Q^2$).

In this sense, the RG invariance of the theory underwrites the legitimacy of the leading log approximation and one can state the following theorem :

Due to the RG invariance of the theory, a resummation of the leading logarithms in each term of the straightforward perturbation expansion of the QCD effective coupling at short distances gives the correct leading behaviour of this effective coupling and is therefore legitimate.

We have extensively discussed the case of the effective coupling (4)

$$\frac{\bar{g}^2(t)}{4\pi} = \bar{\alpha}(t) = \alpha d(t, \alpha)$$

The reason is that it has in QCD a decisive importance since it decides if the theory is asymptotically free or not. But another reason is that when computing other quantities like, for instance, a Green function or the moments of a structure function in deep inelastic scattering, the function $d(t, \alpha)$ also appears. Indeed, if $X(t, \alpha)$ is such a quantity, its expansion, in order to be in agreement with the RG invariance (2)-(3), must be of the following form :

$$\begin{aligned} \ln X(t, \alpha) = c_0 \ln d(t, \alpha) + \sum_{n=1}^{\infty} \frac{c_n}{n} \times \\ \times \alpha^n \left[(d(t, \alpha))^n - (d(0, \alpha))^n \right] \end{aligned} \quad (10)$$

The c_i 's are given by

$$\sum_{i=0}^{n-1} c_i a_{(n-i), 1} = b_{n, 1} \quad (11)$$

and therefore given by the straightforward perturbation expansion of d and $\ln X$

$$d(t, \alpha) = 1 + \sum_{i,j=1}^{\infty} a_{i,j} \alpha^i t^j \quad (12)$$

$$\ln X(t, \alpha) = \sum_{i,j=1}^{\infty} b_{i,j} \alpha^i t^j ; \quad j \leq i \quad (13)$$

It can easily be seen ⁵⁾ that the coefficients $a_{1,1}, a_{2,1}, \dots$ are the $-2b_0, -2b_1, \dots$ of the β function. The $b_{1,1}, b_{2,1}, \dots$ are associated with the renormalization of $X(t, \alpha)$. In a first approximation, from (10)

$$X(t, \alpha) = (d(t, \alpha))^{c_0} = \left(\bar{\alpha}(t)/\alpha \right)^{c_0} \quad (14.a)$$

with, from (11),

$$c_0 = b_{1,1}/a_{1,1} = -b_{1,1}/2b_0$$

so that if $X(t, \alpha) \equiv M^N(Q^2)$,

$$\frac{M^N(Q^2)}{M^N(Q_0^2)} = \left[\frac{\bar{\alpha}(t)}{\bar{\alpha}(t_0)} \right]^{-b_{1,1}/2b_0} \quad (14.b)$$

which again is the result of the leading log resummation ^{*)}. The expressions (14.a) and (14.b) are RG invariant. [E.g., $\mu(\partial/\partial\mu) [(14.a) \text{ or } (14.b)] = 0$.] We therefore see in a detailed way how RG invariance validates a leading log approximation.

Extrapolating from these results, the temptation can be strong to extend this leading log approximation to cases where the operator product expansion (OPE) is not valid, i.e., to processes which are not light-cone dominated. In fact a tremendous amount of work has been done according to this ansatz. The introduction of the cut-vertex expansion (CVE) ⁶⁾ opens the door to the same argumentation using the RG invariance that we have discussed here ^{**)} .

In conclusion, one can say that the constraints imposed by RG invariance lead to an improved perturbation expansion which coincides with the resummation of leading logarithms, then the next-to-leading logs and so on, and give a justification of this procedure, though it is a priori mathematically not legitimate.

Of course, when confronted with an infinite series of large individual terms in particle physics, one might be tempted to resum this series by brute force, using Dyson - or Bethe-Salpeter-like - equations. This means defining

*) If $X(t, \alpha)$ is the moment $M^N(Q^2)$ it is easy to show that the perturbation coefficient $b_{1,1} = \gamma_0^N$ ⁷⁾.

**) A new and very promising approach based on the factorization property of mass singularities has been developed recently ⁸⁾ which allows one to calculate beyond the leading order the Q^2 dependence of all QCD processes, whether space-like or time-like.

an approximate kernel and seeking the solution of the integral equation obtained this way. One can find numerous examples of such a procedure in the literature. The comment we would make about such practices is that they can be legitimized only if the solutions found admit RG invariance and have a physical basis such as a symmetry or a conservation law, etc. Secondly, the kernels used are always approximations which neglect terms believed to be unimportant or "non-leading" in the approximation considered; it is necessary to have formal proof that those neglected terms are really unimportant. Only then is the resummation physically sensible.

If, as in Ref. 9), the infra-red properties of the theory are at the basis of such a resummation of large logarithms, it is then fully justified since the solution obtained does not violate RG invariance, but can be assimilated by a change of renormalization prescription.

This contrasts with summations of sets of diagrams whose selection is dictated by considerations of simplicity of calculation - see Gribov and Lipatov ¹⁾ for references.

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R E F E R E N C E S

- 1) V. Gribov and I. Lipatov, Sov.J.Nucl.Phys. 15, 438 (1972);
Y. Dokshitzer, D. D'yakonov and S. Troyan, Physics Reports C (in press).
Also SLAC preprint (1978).
- 2) L.D. Landau, A. Abrikosov and I. Khalatnikov, Dok.Akad.Nauk USSR 95, 773,
1177 (1954).
- 3) N.N. Bogoliubov and D. Shirkov, Dok.Akad.Nauk USSR 103, 203, 391 (1955);
ibid. 105, 685 (1955); ibid. JETP 30, 77 (1956); ibid. Nuovo Cimento
3, 845 (1956); ibid. Introduction to the Theory of Quantized Fields,
Interscience, New York (1959). See also :
J. Ellis and C. Sachrajda, CERN preprint TH.2799 (1979), sec.4.2.1; to be
published in the Proc. of the 1979 Cargèse Summer Institute, Plenum (1980).
- 4) V. De Alfaro, S. Fubini and G. Furlan, CERN preprint TH.2799 (1979).
- 5) K.E. Eriksson, Nuovo Cimento 27, 178 (1963).
- 6) A.H. Mueller, Phys.Rev. D18, 3705 (1978),
- 7) D. Amati, R. Petronzio and G. Veneziano, Nuclear Phys. B140, 54 (1978);
B146, 29 (1978).
- 8) G. Curci, W. Furmanski and R. Petronzio, CERN preprint TH.2815 (1980).
- 9) D. Amati et al, CERN preprint TH.2831 (1980).