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Field Models

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SPONTANEOUS SYMMETRY BREAKING IN ANNEALED
AND QUENCHED GAUGE FIELD MODELS

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Abstract

The structure of annealed and quenched models with local $U(1)$ gauge invariance is studied in terms of the Helmholtz free energy. The first non-trivial, or one-loop, account of fluctuations in the annealed model suggests that spontaneous symmetry breaking occurs in two and three dimensions, through a first-order phase transition. Within the same approximation scheme, the quenched model displays a continuous phase transition. A more complete account of the fluctuations in the annealed model changes the nature of the transition to a continuous one, whereas spontaneous symmetry breaking is then absent with quenched disorder.

The basic structure of the disordered models studied here was proposed by Toulouse^[1]. Two component spins are represented by a complex scalar field, ψ and their interactions are described in terms of a gauge field, A . An essential feature of the models is that they are invariant under a local gauge transformation. The gauge group is a one-dimensional Abelian continuous group, $U(1)$. Our continuum version can be obtained as a naive limit of a lattice model^[2]. In this latter case, matter fields are placed on the sites of a hypercubic lattice, and gauge fields are associated with each lattice link defined by a lattice site and a bond direction. The gauge invariant energy density, or action, comprises two parts; a sum over links of the interaction between the matter fields and their contiguous gauge fields, and the sum over all plaquettes of the product of the gauge fields on their perimeter. For this model the local gauge symmetry simultaneously rotates a matter field and its contiguous link gauge fields. In the absence of matter fields local gauge invariance forbids the occurrence of a spontaneous breaking of a local symmetry, and non-local order parameters describe the different phases.

The annealed and quenched models differ in the interpretation given to the gauge field. In the annealed disorder model A is regarded as a thermodynamic variable, just like the scalar spin field. Our annealed model is, in fact, identical with massive scalar electrodynamics or, equivalently, the Landau-Ginzburg model of superconductivity^[3]. Gauge invariance in this model expresses conservation of local charge. Quenched disorder arises when A is regarded as an external field. The Helmholtz free energy is then obtained by averaging the free energy calculated for a particular configuration of gauge fields.

We shall investigate the possibility of spontaneous symmetry breaking in the annealed and quenched disorder models by studying the Helmholtz free energy. To this end the scalar fields are replaced by a constant plus a new field that is taken to represent fluctuations. The fluctuation induced corrections to the free energy are calculated using two approximations, and for both approximations the annealed model displays a phase transition in

two and three dimensions. The quenched model appears to have more subtle features. A first-order treatment of the fluctuations suggests that the quenched model possesses a continuous phase transition, whereas spontaneous symmetry breaking is absent in a more sophisticated approximation. The meaning of these results for spin glass systems can be inferred from Binder's^[4] proof that a long-range order implies a non-zero Edwards-Anderson order parameter.

Our calculations of the Helmholtz free energy utilise the loop expansion, which is not an expansion order by order in the coupling constants. The expansion parameter multiplies all terms in the energy density, and therefore local symmetry is preserved at every order^[5]. The first corrections to the free energy, determined by one loop terms, are described in sections 3 and 4. These terms are simply the first fluctuation induced corrections to the mean field result. Results from more sophisticated approximations are described in sections 5 and 6. Although these approximations are presented in the form of a variational method, they can also be viewed as the summation of an infinite class of loop diagrams. For example, it is shown in the appendix that applied to the $O(N)$ model of N real scalar fields with $O(N)$ -symmetric quartic self-interactions the variational method leads to the result obtained by summing all cactus, or prickly-pear, diagrams, and the result for the free energy is correct to leading order in $1/N$.

For renormalisable models, like scalar electrodynamics, loop expansion results for the free energy can be improved by using the renormalisation group equation. This procedure does only improve the approximation, and it does not introduce new structure or features^[6]. Clearly, the application of the renormalisation group method is only valid for renormalisable models. In fact, we find for the quenched model that our one-loop result does not agree with findings based on a leading order Wilson type renormalisation group calculation^[7]. It is interesting to note that computer simulation studies of spin-glass systems show that there is no special dependence of the Edwards-Anderson order parameter on the space dimension, d , for $d = 4, 5$ or 6 ^[4].

The next section defines the annealed and quenched models, and the corresponding free energies.

2 Definitions

The energy density associated with the two component scalar field

$$\psi = (\psi_1 - i\psi_2)/\sqrt{2} \quad \text{is}$$

$$L_A(\psi) = \frac{1}{2} \int dx \left\{ m_0^2 |\psi|^2 + \frac{1}{6} \lambda_0 (|\psi|^2)^2 + |(\underline{\nabla} - ie_0 \underline{A})\psi|^2 \right\}, \quad (2.1)$$

where m_0^2 , λ_0 and e_0 are real, positive parameters. The non-uniqueness of gauge invariance requires the introduction of a gauge fixing term in the gauge field energy density, and we shall take this to be

$$\frac{1}{2} \int dx \left\{ |\text{curl } \underline{A}|^2 + |\underline{\nabla} \cdot \underline{A}|^2 / \xi \right\} = \frac{1}{2} (A, RA). \quad (2.2)$$

Here ξ is a parameter, and the equality defines a convenient scalar product notation. The explicit expression for the matrix R in (2.2) is,

$$\xi R_{\alpha\beta}(x-y) = \left\{ \xi (\nabla^\alpha \nabla^\beta - \delta_{\alpha\beta} \nabla^2) - \nabla^\alpha \nabla^\beta \right\} \delta(x-y), \quad (2.3)$$

where α and β denote Cartesian co-ordinates.

Being a continuum model, there is an intrinsic wavevector cut-off that we shall denote by Λ . It is useful to record the dimensionality of the fields and parameters. The mass parameter m_0^2 has the dimension Λ^2 , and ψ^2 , $\underline{A}^2$ and λ_0 , e_0^2 have the dimension Λ^{d-2} and Λ^{4-d} , respectively, where d is the number of space dimensions.

The Helmholtz free energy for annealed disorder is obtained in the usual way from a functional integral over the fields,

$$F_a = -\ell_n. \int (D\psi)(D\underline{A}) \exp \left\{ -L_A(\psi) - \frac{1}{2} (A, RA) \right\}. \quad (2.4)$$

The corresponding result for quenched disorder is

$$F = -\Omega \int (D\underline{A}) \exp \left\{ -\frac{1}{2} (A, RA) \right\} \ell_n. \int (D\psi) \exp \left\{ -L_A(\psi) \right\}, \quad (2.5)$$

where the finite constant Ω is determined by the normalisation of the gauge field distribution. In order to study the possibility of spontaneous symmetry breaking we shift the ψ fields by a constant $\bar{\psi}$. General prescriptions for calculating the free energy, or equivalently the vacuum energy density, in a loop expansion are described in references [8,9]. A broken symmetry state exists if the equation

$$(\partial F / \partial \bar{\psi}) = 0, \quad (2.6)$$

can be satisfied by a non-zero $\bar{\psi}$ in the absence of external fields.

3 One-loop: Annealed Disorder

The one-loop correction to the free energy can be obtained by expanding the energy density as a functional Taylor series about the fields $\bar{\psi}$. Terms linear in the fields ψ , the fluctuations about $\bar{\psi}$, are eliminated by requiring that,

$$\left(\delta / \delta \psi \right) L_A(\psi) \Big|_{\bar{\psi}, A=0} = 0. \quad (3.1)$$

The one-loop approximation is then obtained by terminating the expansion of the energy density at second order in ψ and $\underline{A}$. Hence, at this level of approximation

$$F_a = -\ell_n. \int (D\psi)(D\underline{A}) \exp \left\{ -L_0(\bar{\psi}) - \frac{1}{2} (\psi, \Delta \psi) - \frac{1}{2} (A, R' A) - (A, \vec{\nabla} \bar{\psi}) \right\}. \quad (3.2)$$

Here

$$\Delta_{ab}(x-y) = \left\{ (m_0^2 + \frac{1}{6} \lambda_0 \bar{\psi}^2 - \nabla^2) \delta_{ab} + \frac{1}{3} \lambda_0 \bar{\psi}_a \bar{\psi}_b \right\} \delta(x-y), \quad (3.3)$$

where the labels a, b denote the components of ψ , and $\bar{\psi}^2 = \Sigma \bar{\psi}_a^2$. The matrix R' in (3.2) differs from R defined in (2.3) by the addition of the Higg's mass, namely,

$$R'_{\alpha\beta}(x-y) = R_{\alpha\beta}(x-y) + e_0^2 \bar{\psi}^2 \delta_{\alpha\beta} \delta(x-y). \quad (3.4)$$

Finally, the term in $\underline{A}$ and ψ describes an interaction between the gauge field and the current corresponding to phase invariance, and

$$\vec{T}_a^\alpha(x-y) = -e_0 \bar{\psi}_b \epsilon_{ab\alpha} \vec{\nabla} S(x-y), \quad (3.5)$$

where ϵ_{ab} is the two dimensional antisymmetric matrix. Integration over the ψ field in (3.2) is accomplished with the aid of the identity,

$$\int (D\psi) \exp \left\{ -\frac{1}{2} (\psi, \Delta \psi) - (A, \vec{T} \psi) \right\} = (\text{Det. } \Delta)^{-1/2} \cdot \exp \left\{ \frac{1}{2} (A, \vec{T} \Delta^{-1} \vec{T} A) \right\}. \quad (3.6)$$

The determinant in (3.6) is taken in the functional sense. The identity (3.6) can also be applied to the remaining gauge field integration in (3.2), and the final result for F_a at the one-loop level of approximation is

$$F_a = L_0(\bar{\psi}) + \frac{1}{2} \ln. \text{Det. } \Delta + \frac{1}{2} \ln. \text{Det. } R'', \quad (3.7)$$

where

$$R'' = R' - \vec{T} \Delta^{-1} \vec{T}. \quad (3.8)$$

The matrices Δ and R'' are translation-invariant, since $\bar{\psi}$ is a constant independent of position, and they are therefore diagonal in the wavevector representation. We shall also use the identity

$$\ln. \text{Det. } \Delta = \text{Tr. } L \ln. \Delta \quad (3.9)$$

where Tr denotes a trace operation and the logarithm is functional. To evaluate $\text{Det } R''$ we need the identity

$$\text{det.} (g \delta_{\alpha\beta} + h k^\alpha k^\beta) = g^{d-1} (g + h k^2), \quad (3.10)$$

where $k^2 = \Sigma k_\alpha^2$. With the notation,

$$d\Omega_k = d^d \underline{k} / (2\pi)^d,$$

we obtain from (3.7)

$$F_a = \frac{1}{2} m_0^2 \bar{\psi}^2 + \frac{1}{24} \lambda_0 (\bar{\psi}^2)^2 + \frac{1}{2} \int d\Omega_k \ln. (\text{det. } \Delta) + \frac{1}{2} \int d\Omega_k \ln. \{ (k^2 + e_0^2 \bar{\psi}^2)^{d-1} [\xi e_0^2 \bar{\psi}^2 (1 - k^2/B) + k^2] \}, \quad (3.11)$$

where

$$\text{det. } \Delta \equiv B (B + \frac{1}{3} \lambda_0 \bar{\psi}^2), \quad (3.12)$$

with

$$B = k^2 + m_0^2 + \frac{1}{6} \lambda_0 \bar{\psi}^2. \quad (3.13)$$

In arriving at (3.11) we have divided out a factor equal to the integration over all space, and neglected harmless constant additions. The first term in (3.11) are just the mean-field result, or tree approximation, for the

free energy per unit volume. The first correction to this term comes from the one-loop scalar field diagram. The third and, final, correction in (3.11) is the sum of the pure gauge field loop, and a contribution due to the interaction of $\bar{\Delta}$ and the current corresponding to phase invariance. The dependence of the latter correction on the gauge fixing parameter is cancelled by a two-loop term^[10], and we are therefore free to set $\xi = 0$ in (3.11).

Renormalisation of the result (3.11) is conveniently accomplished by forming the mass function ($\partial F_a / \partial \bar{\psi}^2$) and expanding the result in λ_0 ,

$$\begin{aligned} 2(\partial F_a / \partial \bar{\psi}^2) &= m_0^2 + \frac{1}{6} \lambda_0 \bar{\psi}^2 + \frac{1}{6} \lambda_0 \int d\Omega_k \{B^{-1} + 3/(B + \frac{1}{3} \lambda_0 \bar{\psi}^2)\} + \\ &\quad + e_0^2 (d-1) \int d\Omega_k (k^2 + e_0^2 \bar{\psi}^2)^{-1} \\ &= m_0^2 + \frac{1}{6} \lambda_0 \bar{\psi}^2 + \frac{2}{3} \lambda_0 \int d\Omega_k (k^2 + m_0^2)^{-1} - \frac{5}{18} \lambda_0^2 \bar{\psi}^2 * \\ &\quad * \int d\Omega_k (k^2 + m_0^2)^{-2} + e_0^2 (d-1) \int d\Omega_k (k^2 + e_0^2 \bar{\psi}^2)^{-1}. \end{aligned} \quad (3.14)$$

The expansion of the one-loop scalar field correction in (3.14) to first-order in λ_0 is consistent with the level of approximation. The two terms formed from this correction are then combined with the mean field contributions. For $d = 3$ the gauge field correction, namely the last term, is made finite by adjusting the mass constant by an amount

$$-2e_0^2 \int d\Omega_k / k^2.$$

The integrals we absorb in the mass renormalisation become infinite in the limit $\Lambda \rightarrow \infty$, so the renormalised mass, m^2 , can take all real values.

However, the integral contained in the renormalised coupling constant, λ , is finite in the limit $\Lambda \rightarrow \infty$, and so λ takes all real and positive values. The final expression for the mass function in three dimensions is,

$$2(\partial F_a / \partial \bar{\psi}^2) = m^2 + \frac{1}{6} \lambda \bar{\psi}^2 - 2e_0^2 b (e_0^2 \bar{\psi}^2)^{1/2}, \quad (3.15)$$

where the constant $b = 1/4\pi$. It is evident at once that (2.6) can be satisfied by a non-zero $\bar{\psi}$. From (3.15) it follows that the Higgs mass leads to a term in F_a equal to

$$-\frac{2}{3} b (e_0^2 \bar{\psi}^2)^{3/2}. \quad (3.16)$$

In consequence, the phase transition is of first-order. This result does not survive in the more sophisticated approximation described in section 5.

We now consider the case where $d = 2$. Here, again, m^2 can take all real values and λ is restricted to positive real values. The gauge field contribution in (3.14) is made finite by adjusting the mass by an amount,

$$-e_0^2 \int d\Omega_k (k^2 + \mu^2)^{-1},$$

where the parameter μ^2 may take any real, positive and finite value. Evaluating the integrals in (3.14) in the limit $\Lambda \rightarrow \infty$, we find

$$2(\partial F_a / \partial \bar{\psi}^2) = m^2 + \frac{1}{6} \lambda \bar{\psi}^2 - e_0^2 b \ln(e_0^2 \bar{\psi}^2 / \mu^2). \quad (3.17)$$

For sufficiently large values of m^2 there is no value of $\bar{\psi}^2$ at which the right-hand side of (3.17) is zero. In this case the minimum of F_a is at the origin. As m^2 is decreased continuously, a value is reached at which the mass function is zero, and since this occurs for a finite $\bar{\psi}^2$ the transition is of first-order. On decreasing m^2 still further the position of the minimum in F_a moves out to larger values of $\bar{\psi}^2$.

4 One-Loop: Quenched Disorder

The one-loop approximation to F can almost be written down by inspection of the results derived in the preceding section. From (3.2) we find, using the result (3.6)

$$\begin{aligned} F &= -\Omega \int (D\Delta) \exp \left\{ -\frac{1}{2} (A, R A) \right\} \ln \{ (D \partial_t \Delta)^{-1/2} * \\ &\quad \exp [-L_0(\bar{\psi}) - \frac{1}{2} e_0^2 \bar{\psi}^2 \int d\mathbf{x} \Delta^2(\mathbf{x}) + \frac{1}{2} (A, \vec{\nabla} \Delta^{-1} \vec{\nabla} A)] \} = \end{aligned} \quad (4.1)$$

$$= L_0(\bar{\psi}) + \frac{1}{2} \ln(\text{Det } \Delta) + \frac{1}{2} \Omega \int (D\bar{A}) \exp \left\{ -\frac{1}{2} (A, RA) \right\} \star$$

$$\left\{ e_0^2 \bar{\psi}^2 \int d\bar{x} \bar{A}^2(x) - (A, \vec{T} \Delta^{-1} \vec{T} A) \right\}. \quad (4.1) \text{ Cont.}$$

The first two terms are the familiar molecular field contribution and one-loop scalar field correction, respectively. In order to evaluate the two gauge field terms in (4.1) we make use of the identity

$$\int (D\bar{A}) \exp \left\{ -\frac{1}{2} (A, RA) \right\} (A, \bar{R}A) = (\text{Det } R)^{-1/2} \text{Tr. } (R^{-1} \bar{R}). \quad (4.2)$$

On application in (4.1) the first term on the right-hand side will be cancelled by the normalisation factor, $\bar{A}$. Taking $\bar{R}$ to be the unit matrix we obtain

$$\Omega \int (D\bar{A}) \exp \left\{ -\frac{1}{2} (A, RA) \right\} \int d\bar{x} \bar{A}^2(x) = \text{Tr. } (R^{-1}) =$$

$$= \left(\int d\bar{x} \right) (d-1 + \xi) \int d\Omega_k / k^2. \quad (4.3)$$

From this result it is clear that the third term in (4.1) does not lead to new structure in F , since the term can be combined with the mass term in $L_0(\bar{\psi})$ for all values of d . The last contribution in (4.1) is also evaluated using (4.2), and the result is

$$\Omega \int (D\bar{A}) \exp \left\{ -\frac{1}{2} (A, RA) \right\} (A, \vec{T} \Delta^{-1} \vec{T} A) =$$

$$= e_0^2 \bar{\psi}^2 \xi \left(\int d\bar{x} \right) \int d\Omega_k (k^2 + m_0^2 + \frac{1}{6} \lambda \bar{\psi}^2)^{-1}. \quad (4.4)$$

Because the model is not invariant under a local gauge transformation for a finite ξ we must take $\xi = 0$, and in this case the contribution (4.4) vanishes. We therefore conclude that, within the approximation we are using, the quenched model cannot be distinguished from the pure (ferromagnetic) model. This result holds for all space dimensions.

A non-zero ξ effectively biases the gauge field distribution toward a type of long-range order. Let us pursue this alternative viewpoint, and replace

the gauge fixing term in R , equation (2.3), by a mass term,

$$\frac{1}{2} M^2 \delta_{\alpha\beta} S(x-y)$$

The addition of this term biases the distribution toward a ferromagnetic configuration. From (3.10) it is evident that for the new R matrix $\text{Det } R$, and consequently the normalisation Ω , is finite. The gauge field contribution to F , in the one-loop approximation, is now proportional to

$$e_0^2 \bar{\psi}^2 \int d\Omega_k \left\{ (d-1)/(k^2 + M^2) + (M^2)^{-1} - \frac{(k^2/M^2)}{(k^2 + m_0^2 + \frac{1}{6} \lambda \bar{\psi}^2)} \right\} =$$

$$= e_0^2 \bar{\psi}^2 \int d\Omega_k \left\{ (d-1)/(k^2 + M^2) + (M^2)^{-1} - \frac{(k^2/M^2)}{(k^2 + m_0^2)} + \frac{1}{6} \lambda \bar{\psi}^2 \frac{(k^2/M^2)}{(k^2 + m_0^2)^2} \right\},$$

where the expansion in λ_0 is again taken to first-order. All these terms can be grouped with similar terms in the mean field contribution and one-loop scalar field correction ie, there is no new structure in F .

Returning to the original model, when $\xi = 0$ the free energy satisfies

$$2(\partial F / \partial \bar{\psi}^2) = m^2 + \frac{1}{6} \lambda \bar{\psi}^2, \quad (4.5)$$

where m^2 and λ are, respectively, the renormalised mass parameter and coupling constant. Because negative values of m^2 are admitted, spontaneous symmetry breaking can occur, as is well known, and the transition to the broken symmetry state is a continuous one.

5 Variational Calculation: Annealed Disorder

At the one-loop level we find that spontaneous symmetry breaking occurs for both the annealed and quenched models and the transitions are, respectively, of first-order and a continuous nature. We shall now investigate whether these results are artefacts of the low level approximation. Clearly, they might be modified or negated by a fuller account of the fluctuations. A natural step in this direction, within the loop expansion, would be to evaluate all two-loop corrections. While this would represent systematic progress the approximation is unlikely to sample in full the non-linear

features of a model. An approximation capable of doing this probably requires the summation of an entire class of loop diagrams. The variational method used here, and in the next section, is applied in the appendix to the $O(N)$ model of N real scalar fields with quartic interactions, and it is found to yield a result for the free energy that is identical with the one obtained by summing all cactus diagrams. A more general discussion of the variational method is given in reference [11].

The method uses the following inequality for the free energy [12]

$$F \leq F_0 + \langle L - L_0 \rangle_0, \quad (5.1)$$

where L_0 is a part of the energy density that is convenient to handle, F_0 is the corresponding free energy, and the second term is the average of the difference between the full energy density and L_0 calculated with the distribution $\exp(-L_0)$.

The first step in constructing the variational method is to shift the scalar fields by a constant amount $\bar{\psi}$. The self interaction in the resulting energy density contains some terms that are cubic and quartic in the fluctuations ψ . These fluctuations are, by construction, zero on average. Let us assume that they are Gaussian distributed, to a good approximation, and denote the mean-square fluctuation by $\langle \psi^2 \rangle$. We can then make the replacements

$$\psi^3 \rightarrow 3\psi \langle \psi^2 \rangle$$

and

$$\psi^4 \rightarrow 6\psi^2 \langle \psi^2 \rangle - 3 \langle \psi^2 \rangle^2$$

which reduce L to quadratic form. The final step is to define L_0 as the sum of all even products of ψ 's, so that it is diagonal in the spin component space. This last step makes the second term on the right-hand side of (5.1) zero, and F_0 is an upper bound to the free energy. The quantities $\bar{\psi}$ and $\langle \psi^2 \rangle$ are determined by minimizing F_0 .

The method just described is now applied to the model of annealed disorder. It is particularly convenient to introduce a new scalar variable, with the dimension of mass squared,

$$f = m_0^2 + \frac{1}{3} \lambda_0 (\bar{\psi}^2 + \langle \psi^2 \rangle). \quad (5.2)$$

Our upper bound to the free energy is a function of $\bar{\psi}$ and f , both of which are taken to be constants independent of position. We shall elect to use the condition

$$(\partial F / \partial f) = 0 \quad (5.3)$$

to determine f as a function of $\bar{\psi}$.

After making the various steps outlined above the energy density for annealed disorder becomes,

$$L_A(\psi) = \frac{1}{2} (A, R'A) + \left(\int dx \right) \left\{ -\frac{3}{4\lambda_0} f^2 + \left(\frac{3m_0^2}{2\lambda_0} + \frac{1}{2} \bar{\psi}^2 \right) f \right\} + \frac{1}{2} (\psi, \Delta' \psi). \quad (5.4)$$

Here, R' is defined in equation (3.4), and

$$\Delta'_{ab}(x, y) = \delta_{ab} \{ f + e^2 \bar{A}^2(x) - \nabla^2 \} \delta(x - y). \quad (5.5)$$

The term in $\bar{A}^2$ represents a process that arises in lowest-order in two-loop corrections. From (5.4) we find, after completing the ψ field integrations,

$$F_a = \left(\int dx \right) \left\{ -\frac{3}{4\lambda_0} f^2 + \left(\frac{3m_0^2}{2\lambda_0} + \frac{1}{2} \bar{\psi}^2 \right) f \right\} - \ln \int (d\Delta) (D \det \Delta')^{-1/2} \exp \left\{ -\frac{1}{2} (A, R'A) \right\}. \quad (5.6)$$

To proceed in the gauge field integration we write

$$\Delta' = \Delta \cdot (I + S) \quad (5.7)$$

where I is the unit matrix, and

$$S = e_o^2 \int d\vec{y} \sum_{\vec{b}} \delta_{a\vec{b}} \underline{A}^2(x) S(x-\vec{y}) \Delta_{\vec{b}}^{-1}(\vec{y}-\gamma). \quad (5.8)$$

Then, using (3.9),

$$\text{Det. } \Delta' = (\text{Det. } \Delta) \exp \text{Tr. } L n. (I + S) \quad (5.9)$$

and,

$$\begin{aligned} F_a = & \left(\int dx \right) \left\{ -\frac{3}{4\lambda_o} f^2 + \left(\frac{3m_o^2}{2\lambda_o} + \frac{1}{2} \bar{\psi}^2 \right) f \right\} + \frac{1}{2} L n. (\text{Det. } \Delta) - \\ & - L n. (\text{Det. } \Delta) \exp \left\{ -\frac{1}{2} (A, R'A) - \frac{1}{2} \text{Tr. } L n. (I + S) \right\}. \end{aligned} \quad (5.10)$$

The matrix Δ in (5.7) is translation-invariant. We therefore define

$$\Delta(k) = k^2 + f, \quad (5.11)$$

and it can then be shown that,

$$\text{Tr. } L n. (I + S) = 2 \int dx \int d\Omega_k L n. (1 + e_o^2 \underline{A}^2(x) / \Delta(k)). \quad (5.12)$$

In order to make further progress we find it necessary to expand the logarithm in (5.12). Keeping just the first term, quadratic in $\underline{A}$, gives a new upper bound for F_a . The final result for the free energy per unit volume, and with $\xi = 0$, is

$$\begin{aligned} F_a = & -\frac{3}{4\lambda_o} f^2 + \left(\frac{3m_o^2}{2\lambda_o} + \frac{1}{2} \bar{\psi}^2 \right) f + \int d\Omega_k L n. (\Delta(k)) + \\ & + \frac{1}{2} (d-1) \int d\Omega_k L n. \left\{ k^2 + 2e_o^2 \left[\frac{1}{2} \bar{\psi}^2 + \int d\Omega_p / \Delta(p) \right] \right\}. \end{aligned} \quad (5.13)$$

Perhaps the most interesting feature of this result is the modification of the Higg's mass by the addition of scalar loops. One way of including additional terms in (5.13) of the expansion of the logarithm in (5.12) is to use a form of cumulant expansion, but we shall not do so here.

A Three Dimensions

Renormalisation of the result (5.13) requires the consideration of two integrals. One arises from the scalar loops, and for this we define,

$$J(f) = \int d\Omega_k L n. (k^2 + f). \quad (5.14)$$

Since, for $d = 3$,

$$(\partial^2 J / \partial f^2) = -\frac{1}{2} b f^{-1/2}, \quad (5.15a)$$

where $b = 1/4\pi$, it follows that

$$(\partial J / \partial f) = -b f^{1/2} + s \quad (5.15b)$$

and

$$J = -\frac{2}{3} b f^{3/2} + s f + t, \quad (5.15c)$$

where s and t are integration constants, and they tend to infinity in the limit $\Lambda \rightarrow \infty$. In a legitimate renormalisation s and t are allowed to go to infinity leaving F_a finite, and may depend on the model parameters but not the fields. Substituting (5.15b) and (5.15c) in (5.13), integrating the gauge field contribution by parts, and taking the limit $s \rightarrow \infty$, we obtain a finite expression for F_a if s tends to infinity in the form

$$s \rightarrow \left(\int d\Omega_k / k^2 \right)^2 / e_o^2 \quad (5.16)$$

and the mass is adjusted as

$$(3m_o^2 / 2\lambda_o) = (3m^2 / 2\lambda_o) - s. \quad (5.17)$$

The second integration constant, t , is chosen to make harmless constant terms add to zero. When this is done we find, for $d = 3$,

$$F_a = -\frac{3}{4\lambda_o} f^2 + \left(\frac{3m_o^2}{2\lambda_o} + \frac{1}{2} \bar{\psi}^2 \right) f - \frac{2}{3} b f^{3/2}, \quad (5.18)$$

in which λ_0 is positive, and m^2 may take any real value. The most striking feature of this result is that it is independent of e_0 . Moreover, it has a structure identical with the corresponding result for the ψ^4 -model discussed in the appendix. Therefore a more sophisticated treatment of the fluctuations in the annealed model renders the first-order phase transition found at the one-loop level a continuous phase transition. From (5.18), we obtain

$$(\partial F / \partial f) = -\frac{3}{2\lambda_0} f + \frac{3m^2}{2\lambda_0} + \frac{1}{2} \bar{\psi}^2 - b f^{1/2} = 0. \quad (5.19)$$

We seek a solution of (5.19) for which $f > 0$, and a short calculation leads to the result,

$$3 f^{1/2} = -\lambda_0 b + \{(\lambda_0 b)^2 + 3(3m^2 + \lambda_0 \bar{\psi}^2)\}^{1/2}. \quad (5.20)$$

Because,

$$2 \left(\frac{dF_a}{d\bar{\psi}^2} \right) = f, \quad (5.21a)$$

it follows from (5.20) that spontaneous symmetry breaking occurs for $3m^2 + \lambda_0 \bar{\psi}^2 = 0$, as expected. The critical temperature is given by $m^2 = 0$ since the susceptibility,

$$\chi = 1 / f(\bar{\psi} = 0). \quad (5.21b)$$

B Two Dimensions

We proceed as in the previous case to make F_a finite by an appropriate choice of integration constants in $J(f)$. For $d = 2$,

$$(\partial^2 J / \partial f^2) = -b/f \quad (5.22a)$$

$$(\partial J / \partial f) = -b \ln f + s. \quad (5.22b)$$

and

$$J = -b \{ f \ln f - f \} + s f + E. \quad (5.22c)$$

we obtain

$$F_a = -\frac{3}{4\lambda_0} f^2 + \left(\frac{3m^2}{2\lambda_0} + \frac{1}{2} \bar{\psi}^2 \right) f + J + b e_0^2 \left(\frac{1}{2} \bar{\psi}^2 - b \ln f \right) * \\ * \{ 1 + \int d(k^2) / (k^2 + E) \} + b e_0^2 s \int d(k^2) / (k^2 + E). \quad (5.23)$$

When $s \rightarrow \infty$, $E \rightarrow 2e_0^2$ and the last term becomes a harmless constant added to t in the integral J . The infinite term in J is absorbed by redefining m_0^2 , as in (5.17), and the remaining infinite term that multiplies the fourth contribution to the right-hand of (5.23) is absorbed in a redefinition of the parameter e_0^2 . If the new charge parameter, which can take all real values, is denoted by e^2 , the final expression for the free energy is,

$$F_a = -\frac{3}{4\lambda_0} f^2 + \left(\frac{3m^2}{2\lambda_0} + \frac{1}{2} \bar{\psi}^2 \right) f - b \{ f \ln f - f \} + \\ + e^2 \left(\frac{1}{2} \bar{\psi}^2 - b \ln f \right). \quad (5.24)$$

From this result we obtain,

$$2 \left(\frac{dF_a}{d\bar{\psi}^2} \right) = f + e^2, \quad (5.25)$$

and

$$(\partial F / \partial f) = -\frac{3}{2\lambda_0} f + \frac{3m^2}{2\lambda_0} + \frac{1}{2} \bar{\psi}^2 - b \ln f - e^2 b / f \\ = \frac{3m^2}{2\lambda_0} + \frac{1}{2} \bar{\psi}^2 - \gamma(f), \quad (5.26)$$

where the second equality in (5.26) defines the function $\gamma(f)$. It follows from (5.25) that, for positive e^2 there is no broken symmetry state since, of course, f is always positive. The function $\gamma(f)$ tends to plus infinity

for both $f \rightarrow 0$ and $f \rightarrow \infty$, and it has a single minimum located at

$$f = \bar{f} = (\lambda_0 b/3) \{-1 + \sqrt{(1 + 6e^2/\lambda_0 b)}\}.$$

Hence, there are no solutions of (5.26) for $(3m^2 + \lambda_0 \bar{\psi}^2)/2\lambda_0$ less than some critical value determined by $y(\bar{f})$. When $(3m^2 + \lambda_0 \bar{\psi}^2)/2\lambda_0$ exceeds this critical value there are two values of f that satisfy (5.26), and both lead to a free energy with a minimum at $\bar{\psi} = 0$.

The situation just described prevails for $e^2 > 0$. However, the renormalised charge parameter can take negative values although the bare charge parameter, e_0^2 , must, of course, be positive in order to have properly defined integrals in the bare theory. In consequence, it is possible to have $f + e^2 = 0$ in (5.25) for a finite $\bar{\psi}$. From (5.26) we find that spontaneous symmetry breaking occurs for $e^2 < 0$ when m^2 falls below the value

$$m_c^2 = -e^2 + (2\lambda_0 b/3) \{\ln(-e^2) - 1\}, \quad (5.27)$$

at which the susceptibility is infinite. We therefore conclude that for $d = 2$ the annealed model displays a continuous phase transition when the mass parameter $m^2 = m_c^2$ given in (5.27), whereas in the one-loop approximation the transition is of first-order.

6 Variational Calculation: Quenched Disorder

The initial steps in the calculation of an upper bound for the free energy for quenched disorder parallel those of the preceding section on annealed disorder. It is straightforward to establish that the analogue of (5.10) is,

$$F = (\int dx) \left\{ \frac{-3}{4\lambda_0} f^2 + \left(\frac{3m_0^2}{2\lambda_0} + \frac{1}{2} \bar{\psi}^2 \right) f \right\} + \frac{1}{2} \Omega \int (DA) * \exp \left\{ -\frac{1}{2} (A, RA) \right\} [e_0^2 \bar{\psi}^2 \int dx A^2(x) + \ln(\text{Det } \Delta')]. \quad (6.1)$$

The first gauge field contribution to this expression is calculated in section 4. The remaining gauge field contribution is evaluated using (5.9)

and (5.12); we find

$$\begin{aligned} \ln(\text{Det } \Delta') &= 2 \left(\int dx \right) \int d\Omega_k \ln \{ \Delta(k) \} + \\ &+ 2 \int dx \int d\Omega_k \ln \left\{ 1 + e_0^2 A^2(x) / \Delta(k) \right\}, \end{aligned} \quad (6.2)$$

where $\Delta(k)$ is defined by (5.11). To accomplish the integration of the logarithm in (6.2) against $\exp\{-\frac{1}{2}(A, RA)\}$, as required in (6.1), we make use of the identity valid for integer n ,

$$\int (DA) \exp \left\{ -\frac{1}{2} (A, RA) \right\} \int dx \bar{A}^{2n}(x) = (2n-1)!! \text{Tr}(R^{-n}) \quad (6.3)$$

$$= d (2n-1)!! \left(\int dx \right) r^n$$

where

$$r = (d-1 + \xi) \int d\Omega_k / dk^2. \quad (6.4)$$

For $n = 1$ we recover (4.3). It can then be shown that,

$$\begin{aligned} \int (DA) \exp \left\{ -\frac{1}{2} (A, RA) \right\} \int dx \ln \{ 1 + e_0^2 A^2(x) / \Delta(k) \} &= \\ = d \left(\int dx \right) (2\pi)^{-1/2} \int_{-\infty}^{\infty} ds \exp(-s^2/2) \ln(1 + e_0^2 r s^2 / \Delta(k)). \end{aligned} \quad (6.5)$$

Using this result in (6.2) we obtain the following upper bound for the free energy per unit volume for quenched disorder,

$$\begin{aligned} F &= -\frac{3}{4\lambda_0} f^2 + \left(\frac{3m_0^2}{2\lambda_0} + \frac{1}{2} \bar{\psi}^2 \right) f + \frac{1}{2} e_0^2 \bar{\psi}^2 r d + \\ &+ (1-d) \int d\Omega_k \ln(\Delta(k)) + d(2\pi)^{-1/2} \int_{-\infty}^{\infty} ds \exp(-s^2/2) * \\ &* \ln \{ \Delta(k) + e_0^2 r s^2 \}. \end{aligned} \quad (6.6)$$

The physical content of the result (6.6) seems to be extracted most easily by considering its derivative with respect to f , namely,

$$(\partial F / \partial f) = -\frac{3}{2\lambda_0} f + \frac{3m_0^2}{2\lambda_0} + \frac{1}{2} \bar{\psi}^2 + (1-d) \int d\Omega_k / \Delta(k) + d(m_c)^{-1/2} \int_{-\infty}^{\infty} ds \exp(-s^2/2) \int d\Omega_k / \{ \Delta(k) + e_0^2 r s^2 \}. \quad (6.7)$$

For three dimension $r \sim \Lambda$, and in the limit $\Lambda \rightarrow \infty$ the last term in (6.7) vanishes. Hence, for $d = 3$, there is no explicit dependence on e_0^2 which is similar to the result obtained in section 5 for the annealed model. However, unlike the annealed model, the structure of F differs from the ψ^4 model, which corresponds to a pure ferromagnet. If we adjust the mass parameter such that

$$(3m_0^2/2\lambda_0) = (3m^2/2\lambda_0) + 2 \int d\Omega_k / k^2, \quad (6.8)$$

we obtain for $d = 3$,

$$(\partial F / \partial f) = -\frac{3}{2\lambda_0} f + \frac{3m^2}{2\lambda_0} + \frac{1}{2} \bar{\psi}^2 + 2bf'^{1/2} = \frac{3m^2}{2\lambda_0} + \frac{1}{2} \bar{\psi}^2 - \gamma(f). \quad (6.9)$$

The function $\gamma(f)$ tends to zero from below as $f \rightarrow 0$, and goes to infinity with f . A single minimum occurs at,

$$f = \bar{f} = (2\lambda_0 b/3)^2 \quad (6.10)$$

and therefore

$$(\partial F / \partial f)_{\bar{\psi}=0} = 0 \quad (6.11)$$

has a solution for mass parameters that exceed the critical value

$$m_c^2 = \frac{2}{3} \lambda_0 \gamma(\bar{f}) = -(2\lambda_0 b/3)^2. \quad (6.12)$$

In general, there are two values of f which satisfy (6.11) when $m^2 > m_c^2$. The smaller of the two roots is unphysical since it corresponds to $f^{\frac{1}{2}} < 0$.

The physically acceptable solution of $(\partial F / \partial f) = 0$ satisfies,

$$3f'^{1/2} = 2\lambda_0 b + \{ (2\lambda_0 b)^2 + 3(3m^2 + \lambda_0 \bar{\psi}^2) \}^{1/2} \quad (6.13)$$

and for this value of f the free energy is,

$$F = f(3m^2 + \lambda_0 \bar{\psi}^2) / 4\lambda_0 \quad (6.14)$$

from which it is clear that F increases continuously with $\bar{\psi}^2$. When $\{ (2\lambda_0 b/3)^2 + m^2 \} < 0$ the free energy becomes complex for sufficiently small $\bar{\psi}^2$. This effect simply reflects a limitation of the approximation and we do not attach any physical significance to it. From (6.9) we infer that the susceptibility χ satisfies,

$$\chi^{-1} = 2 \left(\frac{dF}{d\bar{\psi}^2} \right)_{\bar{\psi}=0} = f, \quad (6.15)$$

and so χ increases steadily as m^2 approaches m_c^2 where it achieves its maximum value,

$$\chi(m_c) = (3/2\lambda_0 b)^2. \quad (6.16)$$

Although the analysis of (6.6) for $d = 2$ we shall now describe is very different from the analysis for $d = 3$, the final outcome is very similar. In particular, spontaneous symmetry breaking does not take place.

In two dimensions r , defined by (6.4), is dimensionless, and since it always appears in (6.6) multiplied by e_0^2 we absorb r into a new charge parameter e^2 . However, e^2 is restricted to be positive or zero. If we again focus on $(\partial F / \partial f)$, equation (6.7), we obtain a finite express by adjusting the mass in the following way,

$$(3m_c^2/2\lambda_0) = (3m^2/2\lambda_0) - \int d\Omega_k / (k^2 + \mu^2). \quad (6.17)$$

Here μ^2 can take any positive and finite value. From (6.7) and (6.17) we find, for $d = 2$,

$$\begin{aligned}
(\partial F / \partial f) &= -\frac{3}{2\lambda_0} f + \frac{3m^2}{2\lambda_0} + \frac{1}{2} \bar{\psi}^2 + b \ln(f/\mu^2) - \\
&- 2b (\partial \tau)^{-1/2} \int_{-\infty}^{\infty} ds \exp(-s^2/2) \ln \{ (f + e^2 s^2) / \mu^2 \}.
\end{aligned} \quad (6.18)$$

We have not succeeded in reducing the integral in (6.18) to a simple closed form. An obvious simplification is to replace the Gaussian in the integrand by a Lorentzian. If we choose the amplitude and area of the Lorentzian to match those of $\exp(-s^2/2)$ we obtain an approximate analytic form that is exact in the limit $e^2 \rightarrow 0$. Our final expression for the upper bound to the free energy for quenched disorder in two dimensions then satisfies

$$(\partial F / \partial f) = \frac{3m^2}{2\lambda_0} + \frac{1}{2} \bar{\psi}^2 - \gamma(f) \quad (6.19)$$

where

$$\gamma(f) = \frac{3}{2\lambda_0} f - b \ln(f/\mu^2) + 4b \ln \{ (e + f^{1/2}) / \mu \} \quad (6.20)$$

When $e = 0$, $\gamma(f)$ ranges from minus infinity to plus infinity. Hence there is never a solution of $(\partial F / \partial f) = 0$ at $f = 0$. For $e \neq 0$, $\gamma(f)$ tends to plus infinity for small f , and it has a single minimum, which occurs at

$$f = \bar{f} = e^2 \left(1 - 6e^2 / \lambda_0 b + \dots \right), \quad (e^2 / \lambda_0) \ll 1. \quad (6.21)$$

In general,

$$3m^2 + \lambda_0 \bar{\psi}^2 = 2\lambda_0 \gamma(f) \quad (6.22)$$

admits two solutions; the smaller one decreases with increasing $\bar{\psi}^2$, whereas the larger one increases. When (6.22) is satisfied,

$$F = b \left\{ (3f^2 / 4\lambda_0 b) + 4e^2 \ln(1 + f^{1/2}/e) + f - 4ef^{1/2} \right\}, \quad (6.23)$$

where we have adjusted the constant of integration that occurs in going from (6.19) to (6.23) so that the free energy is zero for $f = 0$. Of the

two solutions of (6.22) the smallest leads to the lowest free energy and for this solution F is always negative, and tends to zero as $\bar{\psi}^2 \rightarrow \infty$. For

$$3m^2 < 2\lambda_0 \gamma(\bar{f}) \quad (6.24)$$

there is no solution of (6.22) for small $\bar{\psi}^2$. As with the three dimensional case, we interpret this effect as a limitation of the approximation.

7 Conclusions

The first non-trivial approximation to the contribution of fluctuations to the Helmholtz free energy for annealed disorder leads to the conclusion that a broken symmetry state is reached through a first-order phase transition for $d = 2$ and 3. This is consistent with a Wilson-type of renormalisation group calculation carried out to leading order in $\varepsilon = 4 - d$ [13]. The agreement between the two calculations is expected since the one-loop contributions listed in section 3 contain all the information necessary to obtain the multiplicative renormalisation parameter to leading order in λ_0 and e_0^2 . With a more sophisticated treatment of the fluctuations the nature of the transition evolves from being first-order to continuous. Results from the two approximation schemes are compatible in as much that they both predict a phase transition. This is not the case for quenched disorder; a more complete account of fluctuations negates the continuous phase transition found at the one-loop level.

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Appendix

The $O(N)$ model has an energy density

$$L(\psi) = \int \left\{ \frac{1}{2} m^2 \psi^2 + \frac{\lambda}{8N} (\psi^2)^2 + \frac{1}{2} (\nabla \psi)^2 \right\}, \quad (A1)$$

Here we employ the shorthand,

$$\psi^2 = \sum_{\alpha} \psi_{\alpha}^2 \quad (A2)$$

and

$$(\nabla \psi)^2 = \sum_{\alpha} (\nabla \psi_{\alpha})^2, \quad (A3)$$

and the sums run over all N components of the scalar fields. The first step in applying the variational method outlined in section 5 is to shift each field component by a constant amount $\bar{\psi}$. Moreover, we retain only even terms in $L_0(\psi)$. Thus, we replace $(\psi^2)^2$ in (A1) by

$$(\psi^2)^2 \rightarrow (\bar{\psi}^2)^2 + 2\bar{\psi}^2 \psi^2 + (\psi^2)^2, \quad (A4)$$

having dropped a term which is a factor of $1/N$ smaller than those retained. Assuming ψ to be a Gaussian distributed fluctuation, and keeping leading terms in N , we now replace $(\psi^2)^2$ in (A4) by

$$(\psi^2)^2 \rightarrow 2\psi^2 \langle \psi^2 \rangle - \langle \psi^2 \rangle^2, \quad (A5)$$

where $\langle \psi^2 \rangle$ denotes the average of ψ^2 over the distribution of the fluctuations. After these various changes and approximations we obtain

$$L_0(\psi) = \frac{1}{2} (\psi, \Delta \psi) + \int dx \left\{ \frac{1}{2} m^2 \bar{\psi}^2 + \frac{\lambda}{8N} (\bar{\psi}^2)^2 \right\} \quad (A6)$$

where

$$\Delta_{ab} = \delta_{ab} \left\{ -\nabla^2 + m^2 + \frac{\lambda}{2N} (\bar{\psi}^2 + \langle \psi^2 \rangle) \right\} \delta(x-y). \quad (A7)$$

By construction, $(L - L_0)$ contains only terms which are odd multiples of ψ_a 's, or are smaller by a factor $1/N$ than those retained in L_0 . Therefore, in the limit of large N , we obtain from the inequality (5.1)

$$F \leq \left(\int dx \right) \left\{ \frac{1}{2} m^2 \bar{\psi}^2 + \frac{\lambda}{8N} (\bar{\psi}^2)^2 \right\} + \frac{1}{2} \ln(\det \Delta).$$

Defining,

$$\chi = m^2 + \frac{\lambda}{2N} (\bar{\psi}^2 + \langle \psi^2 \rangle), \quad (A8)$$

we find, apart from trivial constants,

$$F \leq \left(\int dx \right) \left\{ -\frac{N}{2\lambda} \chi^2 + \left(\frac{Nm^2}{\lambda} + \frac{1}{2} \bar{\psi}^2 \right) \chi \right\} + \frac{1}{2} N \left(\int dx \right) \int d\Omega_k \ln(k^2 + \chi). \quad (A9)$$

The right-hand side is identical with the large N limit derived by two different methods in reference [9,14].

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