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DUALITY AND FINITE SIZE EFFECTS IN SIX VERTEX MODELS

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I. Introduction: The Gaussian Model

In this lecture I shall describe some new insight into the properties of six vertex models¹ gained by studying finite size effects, Giles, McLerran, and I were interested in using these models to incorporate internal symmetry in the relativistic string model.² For this application, we required the continuum limit for these models keeping the shape of the system fixed. In the range of parameters where the model exhibits critical behavior, we found a non-trivial dependence of the partition function on the shape of the system.

In order to explain the type of question we were interested in, let's first consider the Gaussian model for a two dimensional lattice consisting of M rows of sites and N columns. To each site (i,j) of the lattice assign a variable ϕ_{ij} , and, for simplicity, choose periodic boundary conditions in both directions. The partition function for the Gaussian model is then

$$Z_G(M,N) = \int \prod_{ij} d\phi_{ij} e^{-\frac{1}{2} \sum_{l,k} (\Delta\phi)_l^2} \quad (1)$$

where $(\Delta\phi)_l$ is the difference of ϕ 's at the ends of the link, l . We are interested in the continuum limit, $M,N \rightarrow \infty$ with $\frac{N}{M}$ fixed.

The integrals in (1) can be easily performed by going to normal coordinates³:

$$Z_G(M,N) = (\text{constant}) \prod_{(m,n) \in (0,0)} \frac{1}{\sqrt{\sin^2 \frac{m\pi}{M} + \sin^2 \frac{n\pi}{N}}} \quad (2)$$

where the constant actually contains a factor of the (infinite) volume of ϕ space. The product ranges over all values of (n, m) in the range $0 \leq n \leq N-1$, $0 \leq m \leq M-1$, excluding the point $(0, 0)$. In the following, we shall set the multiplicative constant in (2) equal to unity. The product over n can be done³:

$$Z_G(M, N) = \sqrt{\frac{M}{N}} \prod_{m=1}^{M-1} \frac{1}{2S \sin h N \left(\sin h^{-1} \sin \frac{m\pi}{M} \right)}, \quad (3)$$

and the limit $M, N \rightarrow \infty$ may then be taken:

$$Z_G(M, N) \rightarrow \sqrt{\frac{M}{N}} e^{-\left[\frac{2\pi i M}{\pi} - \frac{\pi N}{6M} \right]} \prod_{m=1}^{\infty} \frac{1}{(1 - e^{-2\pi i \frac{m}{M}})^2}, \quad (4)$$

where

$$G \equiv \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} \approx 0.9159656$$

is Catalan's constant.

Our result for the F-model will be very similar to Eq. (4), and so we conclude this introduction with some comments on some interesting features of Eq. (4).

First, the nontrivial dependence on $\frac{N}{M}$ is a reflection of the fact that the infinite volume Gaussian model exhibits critical behavior. One way to see this is to recognize that $Z(M, N)$ may be represented in terms of a transfer matrix,

$$\{\phi_i'\} | T(M) | \{\phi_i\} \equiv e^{-\frac{1}{2} \sum_{i=1}^M (\phi_i' - \phi_i)^2 - \frac{1}{4} \sum_{i=1}^M (\phi_{i+1} - \phi_i)^2 - \frac{1}{4} \sum_{i=1}^M (\phi_{i+1}' - \phi_i')^2},$$

as

$$Z_G(M, N) = T_r [T(M)^N] \\ = \sqrt{\frac{M}{N}} \sum_r t_r(M)^N, \quad (5)$$

where we have explicitly exhibited the contribution, $\sqrt{\frac{M}{N}}$, of the continuous spectrum due to translational invariance $\phi \rightarrow \phi + a$.

Comparing with Eq. (4), we see that the maximum eigenvalue of $T_0(M)$, is $e^{-\frac{2MG}{\pi} + \frac{\pi}{6M}}$ and that there are nearby eigenvalues

$$t_m(M) = t_0(M) e^{-\frac{2m\pi}{M}}.$$

Using Eq. (5), a correlation function can be represented $(\Delta_j \equiv |j-j'|)$,

$$\langle O_i; O_{i+j} \rangle - \langle O_i \rangle \langle O_{i+j} \rangle = \frac{T_r [T(M)^{N-\Delta_j} O_i \dots O_{i+j} T(M)^{\Delta_j}]}{T_r [T(M)^N]} - \langle O_i \rangle \langle O_{i+j} \rangle \\ \xrightarrow{N \rightarrow \infty} \sum_{r > 0} \langle O | O_i | r \rangle \left(\frac{t_r}{t_0} \right)^{\Delta_j} \langle r | O_{i+j} | 0 \rangle \\ \xrightarrow{\Delta_j \rightarrow \infty} e^{-\frac{2\pi \Delta_j}{M}} \langle O | O_i | 1 \rangle \langle 1 | O_{i+j} | 0 \rangle.$$

Thus the correlation length $\xi_{M \rightarrow \infty}$ for the infinite system.

Our second comment is that while $Z_G(M, N)$ clearly is symmetric under $M \leftrightarrow N$ (Eq. 2), this symmetry is far from obvious in the continuum limit (Eq. 4). The symmetry of Z_G is a known result from the theory of elliptic functions. Under the Jacobi Imaginary Transformation, $\tau \rightarrow 1/\tau$,

$$f(\tau) \equiv \frac{1}{\sqrt{\tau}} e^{\frac{1}{\tau}} \prod_{n=1}^{\infty} \frac{1}{(1 - e^{-2\pi n \tau})^2}$$

goes into itself:

$$f(\tau) = f(1/\tau). \quad (6)$$

Brink and Nielsen⁴ emphasized the importance of relations like Eq. (6) in dual resonance models. Here we content ourselves with the remark that from Eq. (5), the limit $\frac{N}{M} \rightarrow \infty$ of Z picks out the largest eigenvalue of $T(M)$ whereas all the eigenvalues of $T(M)$ contribute in the limit $\frac{N}{M} \rightarrow 0$. Eq. (6) relates the highest eigenvalue of T to the distribution of all eigenvalues of T . We call this duality between "low energy" and "high energy" Brink-Nielsen duality. It is an important concept in all two dimensional statistical mechanics models which exhibit critical behavior.

II. The Six Vertex Model

The six vertex model is defined by assigning an arrow to each bond of a two dimensional lattice with the constraint that two arrows enter and two arrows leave each vertex. There are six possible vertices:

and the partition function is defined as

$$Z_V(M, N) = \sum_{\text{Configurations}} \prod_{i=1}^6 v_i^{n_i^C}$$

where the sum is over all allowed configurations of arrows and n_i^C is the number of type i vertices in configuration C . The thermodynamic properties of these models have been worked out,

and I refer the reader to the excellent review article of Lieb and Wu⁵ for a detailed description.

One way to solve these models is to construct a $2^M \times 2^M$ transfer matrix $T[M]$ which is the sum over allowed horizontal bonds between two fixed configurations of adjacent rows of vertical bonds. Then for toroidal boundary conditions,

$$Z_V(M, N) = \text{Tr} [T(M)^N]$$

$T[M]$ can be diagonalized by the Bethe Ansatz. This ansatz leads to a coupled set of non-linear equations identical to the ones analyzed by Yang and Yang⁶ in their discussion of the anisotropic one-dimensional Heisenberg anti-ferromagnet. Giles, McLerran and I² worked out the solutions corresponding to eigenvalues of the transfer matrix t_r , with $\lim_{N \rightarrow \infty} -\ln t_r = 0(\frac{1}{N})$, for the F-model ($v_1=v_2=v_3=v_4=v_5=v_6=1$). Such eigenvalues exist only for $1/2 \leq v < \infty$. For this range of v , we obtained in the limit $M, N \rightarrow \infty$,

$$Z_F(M, N) = [K]^{MN} e^{+\frac{\pi}{6} N} \sum_{n=-\infty}^{\infty} e^{-(\pi n)^2 \frac{2N}{M}} \sum_{\rho=-\infty}^{\infty} e^{-\frac{\pi^2}{2} \rho^2 \frac{2N}{M}} \prod_{m=1}^{\infty} \frac{1}{(1 - e^{-\frac{2\pi m N}{M}})^2}$$

where

$$\cos \mu \equiv \frac{1}{2v^2} - 1$$

and $\bar{K}$ has been worked out by Lieb and Wu⁵.

I now invite you to compare the shape dependence of the F-model partition function to that of the Gaussian model partition function (see Eq. 4). The only difference is that $\sqrt{\frac{M}{N}}$ is

replaced by

$$\sum_{k=-\infty}^{\infty} e^{-\frac{(\pi-\mu)k^2 N}{M}} \sum_{\ell=-\infty}^{\infty} e^{-\frac{\pi^2 N}{\pi-\mu} \ell^2}$$

which approaches $\frac{M}{N}$ as $\mu \rightarrow \pi$ ($\nu \rightarrow \infty$). I would like to close the lecture with a description of a modified Gaussian model whose shape dependence looks like the F model for $\mu \neq \pi$.

The modification I propose is to make ϕ an angular variable, i.e., two configurations for which a whole row or column of ϕ 's differ by an amount $k\pi$ are identified. Technically, we can achieve this by modifying the 1,N and 1,M bonds:

$$\begin{aligned} (\phi_{i,N} - \phi_i)^2 &\rightarrow (\phi_{i,N} - \phi_{i,1} - \ell\pi)^2 \\ (\phi_{N,j} - \phi_j)^2 &\rightarrow (\phi_{N,j} - \phi_{1,j} - k\pi)^2 \end{aligned}$$

and then summing over k, ℓ :

$$Z_{PG}(M, N) = \sum_{k, \ell} Z_{k\ell}$$

where $Z_{k\ell}$ is the modified partition function. To evaluate this, for each (k, ℓ) , redefine ϕ_{ij} :

$$\begin{aligned} \phi_{i,j} &\rightarrow \phi_{i,j} + \frac{i}{M} k\pi + \frac{j}{N} \ell\pi \\ \text{so } (\phi_{i+1,j} - \phi_{i,j})^2 &\rightarrow (\phi_{i+1,j} - \phi_{i,j} + \frac{k\pi}{M})^2 & i \neq M \\ (\phi_{N,j} - \phi_{1,j} - k\pi)^2 &\rightarrow (\phi_{N,j} - \phi_{1,j} - \frac{k\pi}{M})^2 \\ (\phi_{i,j+1} - \phi_{i,j})^2 &\rightarrow (\phi_{i,j+1} - \phi_{i,j} + \frac{\ell\pi}{N})^2 & j \neq N \\ (\phi_{i,N} - \phi_{i,1} - \ell\pi)^2 &\rightarrow (\phi_{i,N} - \phi_{i,1} - \frac{\ell\pi}{N})^2 \end{aligned}$$

In the sum over i , the cross terms cancel and we obtain

$$\sum_{i,j} (\phi_{i+1,j} - \phi_{i,j})^2 \rightarrow \sum_{i,j} (\phi_{i+1,j} - \phi_{i,j})^2 + \frac{k^2 \pi^2}{M} N.$$

A similar thing happens with the j index, with the result

$$Z_{k\ell} = e^{-\frac{k^2 \pi^2}{2M} N - \frac{\ell^2 \pi^2}{2N} M} Z_{00}.$$

Z_{00} is just the original Gaussian partition function so we obtain

$$Z_{PG}(M, N) = \left(\sum_{k, \ell} e^{-\frac{k^2 \pi^2}{2M} N - \frac{\ell^2 \pi^2}{2N} M} \right) Z_G(M, N).$$

Finally, we apply the Poisson resummation formula to write

$$\sum_{\ell=-\infty}^{\infty} e^{-\frac{\ell^2 \pi^2}{2N} M} = \sqrt{\frac{2\pi N}{M \pi^2}} \sum_{m=-\infty}^{\infty} e^{-\frac{2\pi^2 N}{M \pi^2} m^2},$$

so that

$$Z_{PG}(M, N) = \sqrt{2\pi} \sum_{k, \ell} e^{-\frac{k^2 \pi^2}{2} \frac{N}{M} - \frac{2\pi^2 \ell^2 N}{\pi^2} \frac{M}{N}} \left(\frac{1}{\pi} \sqrt{\frac{N}{M}} Z_G(M, N) \right),$$

which has the same shape dependence as the F-model with the identification $(\pi - \mu) \leftrightarrow \frac{2\pi^2}{\pi^2}$.

To conclude, I would like to emphasize once again the power of the constraints put on the form of Z by the $M \leftrightarrow N$ symmetry when there is nontrivial $\frac{M}{N}$ dependence. This is the new insight we have gained, and we believe that this Brink-Nielsen duality is one of the basic reasons a wide class of two dimensional models exhibit essentially identical critical behavior.

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