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EXISTENCE OF PERIODIC SOLUTIONS OF DIFFERENTIAL EQUATIONS
AND ONE-PARAMETER FAMILY OF OPERATORS

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EXISTENCE OF PERIODIC SOLUTIONS OF DIFFERENTIAL EQUATIONS
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Abstract

The existence of periodic solutions of the system

$$(1) \quad y' = f(t, y), \quad y, f \in \mathbb{R}^n, \quad f \text{ periodic in } t \text{ with period } T,$$

is studied using a λ -parameter family of equations

$$(2) \quad y = G_\lambda y, \quad \lambda, y \in \mathbb{R}^n,$$

where G_λ is an operator depending on the parameter λ and is such that for every λ , (i) the solutions of (2) are the T -periodic solutions of (1), and (ii) G_λ satisfies the conditions of Schauder's theorem on D_λ whenever $G_\lambda D_\lambda \subset D_\lambda$. The problem of existence in D_λ is then reduced to that of finding a λ solution of

$$(3) \quad G_\lambda D_\lambda \subset D_\lambda.$$

This method is used to prove the existence of periodic solutions of a system arising from the dynamics of two floating bodies, for which (2) has the form of an integral equation and (3) is a system of inequalities determining values of λ and the corresponding D_λ for which a solution exists.

1. Introduction

Let $f(t,y)$ be a continuous function $f:R \times R^n \rightarrow R^n$, periodic in t with period T . For the norm of $y=(y_1,y_2,\dots,y_n)$ in R^n we take $\sum_{i=1}^n |y_i|$ and use the symbol $|y|$ instead of $\|y\|$ in order to avoid confusion when defining later the norm of y as an n -valued function on $[0,T]$. Let $C[0,T]$ be the vector space of continuous functions $y:[0,T] \rightarrow R^n$, with the norm $\|y\| = \max_{t \in [0,T]} |y|$, and let D be a set in $C[0,T]$. The problem considered here is that of giving a sufficient condition for the existence of periodic solutions in D of the system

$$(1) \quad y' = f(t,y), \quad 0 \leq t, |y| < \infty.$$

A common method to prove the existence is to find a mapping $F:C[0,T] \rightarrow C[0,T]$ with the property that the fixed points of F are the periodic solutions of (1), and then apply an existence theorem to the equation $y = Fy$. Schauder's fixed point theorem, in which we have to show that $FD \subset D$, is an usual existence theorem. The selection of a set D satisfying $FD \subset D$ is a difficult practical problem whose solution would be useful in many situations. For example, the use of iterative schemes, such as generalizations of the Picard method for initial-value [1] and boundary-value problems [2], [3], requires that all iterations are bounded, which will be the case if $FD \subset D$ and the starting point is in D .

In this paper we use the idea of considering, instead of one operator F , a class of operators G_λ , $\lambda \in R^n$, such that for every λ , (i) the fixed points of G_λ are the periodic solutions of (1), and (ii) G_λ satisfies the conditions of Schauder's theorem on D_λ whenever $G_\lambda D_\lambda \subset D_\lambda$. It turns out that the problem of existence of a periodic solution in D_λ is reduced to that of finding some λ such that

$$(2) \quad G_\lambda D_\lambda \subset D_\lambda.$$

In the following section we derive a class G_λ satisfying the foregoing properties (i) (Theorem 1) and (ii) (Theorems 3 and 4), and in section 3. we use this class to prove the existence of periodic solutions of a second order damped system with forcing term. This is done simply by finding some λ for which the relation (2), which reduces to a system of inequalities, is satisfied.

2. The Class G_λ

Let $A(\lambda, t)$ be a nonsingular matrix whose elements are functions of λ and t , $\lambda \in \mathbb{R}^n$, $0 \leq t < \infty$, and let $\frac{dA}{dt}$ be the derivative of the matrix A with respect to t . Assume that A has elements continuous with respect to t , and let

$$\Psi = \frac{dA}{dt}y + Af(t, y),$$

where f is the function occurring in (1).

THEOREM 1. For any fixed λ the integral equation

$$(3) \quad A(\lambda, t)y = \int_0^t \Psi(\lambda, \tau, \bar{y}) d\tau + c, \quad c \in \mathbb{R}^n$$

is equivalent to (1).

PROOF. Let $\bar{y}$ be any solution of (1). We find from (3)

$$\frac{d}{dt} \{A(\lambda, t)\bar{y} - \int_0^t \Psi(\lambda, \tau, \bar{y}) d\tau\} = 0,$$

$$\text{i.e.} \quad A\bar{y} - \int_0^t \Psi d\tau = \text{constant}.$$

Conversely, if $\bar{y}$ is a continuous solution of (3) we have

$$\frac{dA}{dt}\bar{y} + A(\lambda, t)\bar{y}' = \frac{dA}{dt}\bar{y} + A(\lambda, t)f(t, \bar{y})$$

and the theorem is proved.

EXAMPLE. Let (1) be a system of the form

$$y' = S(t)y + F(t, y), \quad F, y \in \mathbb{R}^n.$$

with S an $n \times n$ matrix.

If Y is a fundamental matrix of $y' = Sy$, i.e. one whose columns are linearly independent solutions of $y' = Sy$, we may take $A(\lambda, t) = Y^{-1}(t)$ and with some manipulations we get for (3)

$$y = \int_0^t Y(t)Y^{-1}(\tau)F(\tau, y) d\tau + Y(t)c,$$

which is the classical transformation of the foregoing system into a system of integral equations.

If we restrict ourselves to T -periodic solutions of (1), i.e. solutions of the system

$$(4) \quad \begin{aligned} y' &= f(t, y), \quad 0 \leq t \leq T, \\ y(0) &= y(T), \end{aligned}$$

and assume that $A(\lambda, 0) \neq A(\lambda, T)$, we get from (3) and (4)

$$(5) \quad y = A^{-1}(\lambda, t) \int_0^t \Psi(\lambda, \tau, y) d\tau + A^{-1}(\lambda, t) B^{-1} \int_0^T \Psi(\lambda, \tau, y) d\tau$$

where

$$B = A(\lambda, T)A^{-1}(\lambda, 0) - I.$$

The right-hand side of (5) defines an operator $G_\lambda: C[0, T] \rightarrow C[0, T]$, (depending on λ) and it is easy to see that (5), which may be written as

$$y = G_\lambda y,$$

has the same continuous solutions as (4) for each λ .

Before showing (Theorems 3 and 4 below) that G_λ satisfies the conditions of Schauder's theorem, we recall the definition of relative compactness and the statement of Schauder's theorem [4].

DEFINITION. A subset S of a metric space is called relatively compact if every sequence in S contains a convergent subsequence (i.e. if $\bar{S}$ is compact).

THEOREM 2. (Schauder). Let S be a convex closed subset of a normed space. Let H be a continuous mapping of S into a relatively compact subset of S . Then H has a fixed point in S .

THEOREM 3. Let G_λ be the operator defined by (5), Λ a set in $\mathbb{R}^n$, and let $S \subset C[0, T]$ be a bounded set such that for any $\lambda \in \Lambda$, $G_\lambda S \subset S$. Then $G_\lambda S$ is relatively compact for any $\lambda \in \Lambda$.

PROOF. Since $\Psi(\lambda, t, x)$ is a continuous vector function of t and x , $0 \leq t \leq T, x \in \mathbb{R}^n$, there exists $M > 0$ (depending on λ) such that $|\Psi| < M$ when $x \in S$. From (5) the equation $Ay = AG_\lambda x$ takes the form

$$(6) \quad A(\lambda, t) = \int_0^t \Psi(\lambda, \tau, x) d\tau + B^{-1} \int_0^T \Psi(\lambda, \tau, x) d\tau$$

We have (for any $y \in G_\lambda S$)

$$A(\lambda, t+h)y(t+h) - A(\lambda, t)y(t) = \int_t^{t+h} \Psi(\lambda, \tau, x) d\tau, \quad t+h \in [0, T],$$

or equivalently

$$A(\lambda, t+h)\{y(t+h) - y(t)\} + \{A(\lambda, t+h) - A(\lambda, t)\}y(t) = \int_t^{t+h} \Psi(\lambda, \tau, x) d\tau.$$

Then, for every $\lambda \in \Lambda$,

$$|y(t+h) - y(t)| \leq |A^{-1}(\lambda, t+h) \cdot \frac{dA(\lambda, t+h)}{dt} hy(t)| + |A^{-1}(\lambda, t+h) \int_t^{t+h} \Psi(\lambda, \tau, x) d\tau|$$

where $0 < \theta < 1$. Recalling that $\|\cdot\|$ is the ℓ_1 -norm in R^n and using for the norm $\|\cdot\|$ of a matrix the corresponding $\|\cdot\|_1$ norm, we have

$$|y(t+h) - y(t)| \leq h(\max_{t \in [0, T]} \|A^{-1}\|)(\max_{t \in [0, T]} \|\frac{dA}{dt}\| N + M), \quad y \in G_\lambda S,$$

where N is a bound on y , (from $G_\lambda S \subset S$, S bounded). Hence $G_\lambda S$ is equicontinuous on $[0, T]$, and since for each t in $[0, T]$ the set $\{y(t) : y \in S\}$ is relatively compact in R^n (it is bounded), it follows from a generalization of Ascoli's theorem (see [5] p. 168) that $G_\lambda S$ is relatively compact.

THEOREM 4. The mapping G_λ defined by (5) is continuous on $C[0, T]$.

PROOF. Let $x_0 \in C[0, T]$ and let m be a positive number. In the closed region $[0, T] \times V$, where V is the closed ball $\{x \in R^n : |x - x_0| \leq m\}$, Ψ is uniformly continuous (for any fixed λ). Then, given $\sigma_1 > 0$ there is a $\delta \leq m$ such that $|\Psi(\lambda, \tau, x) - \Psi(\lambda, \tau, x_0)| < \sigma_1$ whenever $|x - x_0| < \delta$. From (6), taking $x \in C[0, T]$ such that $|x(t) - x_0(t)| < \delta$, we have (if $y_0 = G_\lambda x_0$)

$$|y(t) - y_0(t)| \leq |A^{-1}(\lambda, t) \int_0^t (\Psi(\lambda, \tau, x) - \Psi(\lambda, \tau, x_0)) d\tau| + |A^{-1}(\lambda, t) B^{-1} \int_0^T (\Psi(\lambda, \tau, x) - \Psi(\lambda, \tau, x_0)) d\tau| ,$$

which, using the maximum norm in $C[0, T]$ becomes

$$\|y - y_0\| \leq \sigma_1 T (1 + \|B^{-1}\|) \max_{t \in [0, T]} \|A^{-1}\|$$

Given any σ , it is sufficient to take a δ corresponding to

$$\sigma_1 < \sigma / (T(1 + \|B^{-1}\|) \max_{t \in [0, T]} \|A^{-1}\|) \text{ in order to get}$$

$$\|y - y_0\| < \sigma \quad \text{when} \quad \|x - x_0\| < \delta ,$$

which proves that G_λ is continuous at x_0 .

3. Application to a second order system

The problem of the dynamic behaviour of two close floating bodies has arisen at CERN. The corresponding mathematical system is

$$\begin{aligned} (7) \quad & u'' + b_1 |u'| u' + c_1 u + d(u-v)' = F \sin(\omega t + \phi) \\ & v'' + b_2 |v'| v' + c_2 v - d(u-v)' = E \sin(\omega t) \end{aligned}$$

where $b_1, c_1, d, b_2, c_2, F, E, \omega, \phi$ are positive coefficients. We shall determine a set of values of the coefficients such that, for each value in the set, there exists a periodic solution of (7) with period $T = 2\pi/\omega$.

Putting $u = y_1, u' = y_2, v = y_3, v' = y_4$ (7) takes the form (1)

$$\begin{aligned} (8) \quad & y_1' = y_2 \\ & y_2' = -c_1 y_1 - d y_2 + d y_4 - b_1 |y_2| y_2 + F \sin(\omega t + \phi) \\ & y_3' = y_4 \\ & y_4' = d y_2 - c_2 y_3 - d y_4 - b_2 |y_4| y_4 + E \sin(\omega t) . \end{aligned}$$

We take for $A(\lambda, t)$ the following matrix, in which for simplicity $\lambda \in \mathbb{R}$ (instead of $\lambda \in \mathbb{R}^n$), $\lambda > 0$,

$$A(\lambda, t) = \begin{pmatrix} e^{\lambda t} & e^{\lambda t} & 0 & 0 \\ 0 & e^{(\lambda+1)t} & 0 & 0 \\ 0 & 0 & e^{\lambda t} & e^{\lambda t} \\ 0 & 0 & 0 & e^{(\lambda+1)t} \end{pmatrix}.$$

Multiplying (5) by $A(\lambda, t)$ and replacing y by x in the right-hand side we get from (8)

$$(9) \quad \begin{aligned} e^{\lambda t}(y_1+y_2) &= \int_0^t \Psi_1 d\tau + \int_0^T \Psi_1 d\tau / (e^{\lambda T} - 1) \\ e^{(\lambda+1)t}y_2 &= \int_0^t \Psi_2 d\tau + \int_0^T \Psi_2 d\tau / (e^{(\lambda+1)T} - 1) \\ e^{\lambda t}(y_3+y_4) &= \int_0^t \Psi_3 d\tau + \int_0^T \Psi_3 d\tau / (e^{\lambda T} - 1) \\ e^{(\lambda+1)t}y_4 &= \int_0^t \Psi_4 d\tau + \int_0^T \Psi_4 d\tau / (e^{(\lambda+1)T} - 1) \end{aligned}$$

where

$$\begin{aligned} \Psi_1 &= e^{\lambda \tau} \{ (\lambda - c_1)x_1 + (\lambda + 1 - d)x_1 - b_1 |x_2| x_2 + dx_4 + F \sin(\omega \tau + \phi) \} \\ \Psi_2 &= e^{(\lambda+1)\tau} \{ -c_1 x_1 + (\lambda + 1 - d)x_2 - b_1 |x_2| x_2 + dx_4 + F \sin(\omega \tau + \phi) \} \\ \Psi_3 &= e^{\lambda \tau} \{ dx_2 + (\lambda - c_2)x_3 + (\lambda + 1 - d)x_4 - b_2 |x_4| x_4 + E \sin(\omega \tau) \} \\ \Psi_4 &= e^{(\lambda+1)\tau} \{ dx_2 - c_2 x_3 + (\lambda + 1 - d)x_4 - b_2 |x_4| x_4 + E \sin(\omega \tau) \} \end{aligned}$$

In operator form

$$Ay = AG_\lambda x, \quad x, y \in C[0, T].$$

We now determine a closed convex set D_λ such that $G_\lambda D \subset D_\lambda$.

We define D_λ by the inequalities

$$|x_i| \leq M_i, \quad i=1, \dots, 4,$$

and choose M_i together with λ and the coefficients of (7), in such a way that

$$|y_i| \leq M_i, \quad i=1, \dots, 4.$$

Assume

$$(10) \quad \lambda > \max\{c_1, c_2, d-1\}$$

$$(11) \quad M_2 \leq (\lambda+1-d)/b_1$$

$$(12) \quad M_4 \leq (\lambda+1-d)/b_2.$$

Using $|x_i| \leq M_i$ and (10), (11), (12) we get from (9)

$$|y_1+y_2| \leq \frac{\lambda-c_1}{\lambda} M_1 + \frac{(\lambda+1-d)^2}{4b_1\lambda} + \frac{dM_4}{\lambda} + \frac{F}{\lambda}$$

$$|y_2| \leq \frac{c_1}{\lambda+1} M_1 + \frac{(\lambda+1-d)^2}{4b_1(\lambda+1)} + \frac{dM_4}{\lambda+1} + \frac{F}{\lambda+1}$$

$$|y_3+y_4| \leq \frac{d}{\lambda} M_2 + \frac{\lambda-c_2}{\lambda} M_3 + \frac{(\lambda+1-d)^2}{4b_2\lambda} + \frac{E}{\lambda}$$

$$|y_4| \leq \frac{d}{\lambda+1} M_2 + \frac{(\lambda+1-d)^2}{4b_2(\lambda+1)} + \frac{c_2}{\lambda+1} M_3 + \frac{E}{\lambda+1}.$$

Similarly, the condition $|y_i| \leq M_i$ will be satisfied if

$$\frac{\lambda-c_1}{\lambda} M_1 + \frac{(\lambda+1-d)^2}{4b_1\lambda} + \frac{dM_4}{\lambda} + \frac{F}{\lambda} \leq M_1 - M_2$$

$$\frac{c_1}{\lambda+1} M_1 + \frac{(\lambda+1-d)^2}{4b_1(\lambda+1)} + \frac{dM_4}{\lambda+1} + \frac{F}{\lambda+1} \leq M_2$$

$$\frac{d}{\lambda} M_2 + \frac{\lambda-c_2}{\lambda} M_3 + \frac{(\lambda+1-d)^2}{4b_2\lambda} + \frac{E}{\lambda} \leq M_3 - M_4$$

$$\frac{d}{\lambda+1} M_2 + \frac{(\lambda+1-d)^2}{4b_2(\lambda+1)} + \frac{c_2}{\lambda+1} M_3 + \frac{E}{\lambda+1} \leq M_4,$$

a system which takes the form

$$\begin{aligned}
 & \frac{(\lambda+1-d)^2}{4b_1} + dM_4 + F \leq C_1 M - \lambda M_2 \\
 & \frac{(\lambda+1-d)^2}{4b_1} + dM_4 + F \leq (\lambda+1)M_2 - C_1 M_1 \\
 (13) \quad & \frac{(\lambda+1-d)^2}{4b_2} + dM_2 + E \leq C_2 M_3 - \lambda M_4 \\
 & \frac{(\lambda+1-d)^2}{4b_2} + dM_2 + E \leq (\lambda+1)M_4 - C_2 M_3 .
 \end{aligned}$$

We recall that to each solution of (10), (11), (12) and (13) corresponds a $D_\lambda: |x_i| \leq M_i$ and $G_\lambda D_\lambda: |y_i| \leq M_i$ such that $G_\lambda D_\lambda \subset D_\lambda$.

We try to solve the system of inequalities (10)-(13) (where the unknowns are the coefficients in (7), together with λ and M_i) by taking

$$\begin{aligned}
 M_2 &= \frac{\lambda+1-d}{b_1} \\
 M_4 &= \frac{\lambda+1-d}{b_2} \\
 (14) \quad C_1 M_1 &= \lambda M_2 + \frac{1}{2} M_2 \\
 C_2 M_3 &= \lambda M_4 + \frac{1}{2} M_4
 \end{aligned}$$

which, substituted in (13), gives

$$\begin{aligned}
 & b_1 M_2^2 + 4dM_4 + 4F \leq 2M_2 \\
 (15) \quad & b_2 M_4^2 + 4dM_2 + 4E \leq 2M_4 .
 \end{aligned}$$

The solutions of the system formed by the seven inequalities (10), (14) and (15) are evidently also solutions of (10) to (13), (but not vice-versa). Using the following theorem which gives solutions of (15), we obtain a set of solutions of (10), (14) and (15) .

THEOREM 5. A set of solutions of (15) is given by

$$4F < \frac{1}{b_1}, \quad 4E < \frac{1}{b_2}, \quad M_2 = \frac{1}{b_1}, \quad M_4 = \frac{1}{b_2},$$

$$4d \leq \min\{b_2(\frac{1}{b_1} - 4F), b_1(\frac{1}{b_2} - 4E)\}.$$

PROOF. Assume $4F < \frac{1}{b_1}, 4E < \frac{1}{b_2}$. It is easy to see that for

$M_2 = \frac{1}{b_1}, M_4 = \frac{1}{b_2}$ the expressions $2M_2 - b_1 M_2^2 - 4F$ and $2M_4 - b_2 M_4^2 - 4E$ take their maximum value at $\frac{1}{b_1} - 4F > 0$ and $\frac{1}{b_2} - 4E > 0$ respectively.

Substituting values taken from the set $4F < \frac{1}{b_1}, 4E < \frac{1}{b_2}, M_2 = \frac{1}{b_1}, M_4 = \frac{1}{b_2}$

into (15) it is evident that the two inequalities are solved if

$$4dM_4 \leq \frac{1}{b_1} - 4F \quad \text{and} \quad 4dM_2 \leq \frac{1}{b_2} - 4E.$$

We then have the following solutions of (10), (14), (15)

$$\lambda = d, d \geq \max\{c_1, c_2\}, d \leq \min\{\frac{b_2}{4}(\frac{1}{b_1} - 4F), \frac{b_1}{4}(\frac{1}{b_2} - 4E)\}$$

$$M_1 = (d + \frac{1}{2})/c_1 b_1, M_2 = \frac{1}{b_1}, M_3 = (d + \frac{1}{2})/c_2 b_2, M_4 = \frac{1}{b_2},$$

$$4F < \frac{1}{b_1}, \quad 4E < \frac{1}{b_2}.$$

In conclusion, for values which satisfy the above inequalities and for any ω and ϕ , there exists a $2\pi/\omega$ -periodic solution of (7). For example, the following values suggested by physical requirements are found to satisfy our inequalities :

$$b_1 = 0.1, b_2 = 0.1, c_1 = 0.3, c_2 = 0.2,$$

$$d = 0.3, F = 1, E = 0.5, \phi = 0.5, \omega = 2,$$

which yields $\lambda = 0.3$ and the solution bounds $M_1 = 27, M_3 = 40$, i.e.

$$-27. \leq u \leq 27.$$

$$-40. \leq v \leq 40.$$

We have also computed some solutions (one with the above data) using the iterative scheme $y_{n+1} = G_\lambda y_n$. In all cases the iterates

converged numerically to curves which were verified to be periodic solutions using a variable stepsize extrapolation method of integration.

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14