ALTERNATING PHASE FOCUSING

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It is known from existing theories of linear accelerators that radial and phase stability cannot be attained simultaneously without auxiliary focusing devices. In this connection various focusing methods have been suggested and developed, employing metal foils and grids, a longitudinal magnetic field, the $\varphi$ component of the magnetic field, magnetron lenses, the method of centrifugal focusing, and, lately, the method of alternating focusing and defocusing of transverse magnetic and electric fields.

The last method appears to be the most effective of all, since the magnetic fields required for focusing are not large (1,500-2000 gauss).

In evaluating the efficiency of focusing methods in linear accelerators for heavy particles, the intensity of the accelerating electric field is ordinary of the order of 20-30 kV/cm, while the wave length $\lambda \sim 1$-2 meters. Naturally, linear acceleration methods will develop further along the lines of increasing the accelerating field intensity to hundreds of kV/cm.

It seems highly probable that it would be convenient to reduce the wave length $\lambda$.

In this case the electric or magnetic field intensities needed for focusing will increase greatly. In the case of focusing by a longitudinal magnetic field the necessary magnetic field $H \sim \sqrt{E/\beta \lambda}$. If the focusing is accomplished by alternating focusing and defocusing lenses with a transverse magnetic field, $H \sim E/\beta \lambda$. Thus, although the method of magnetic focusing (especially by transverse magnetic fields) is effective today, these methods may prove to be of little efficiency with an increase in the intensity of the accelerating field, a shortening of the wave length and a decrease in the energy of the particles injected into the accelerator.

Therefore, it seems suitable to analyse the basic assumptions of the theory which leads to the conclusion about the need for focusing devices; similarly, it seems convenient to ascertain what acceleration conditions are required to render special focusing devices unnecessary.

Let us consider the movement of a particle in an accelerating axially symmetrical high-frequency electric field. The equation describing the movement of the particle along the tube (the z axis) will be:

$$\ddot{z} = E_m E_z e^{i\omega t}$$  \hspace{1cm} (1)

For a synchronously moving particle we obtain:

$$\ddot{z} = E_m E_z (z_0) e^{i\omega t}$$  \hspace{1cm} (2)

We shall assume the longitudinal deviation of the particle coordinate $q = z - z_0$ to be small and resolve $E_z(z)$ in a q series:

$$E_z(z) = E_z (z_0 + q) = E_z (z_0) + q \frac{\partial E_z}{\partial z} \bigg|_{z=z_0}$$

If we subtract equation (2) from equation (1), we obtain

$$\ddot{q} = \frac{\partial E_z}{\partial z} (z_0) e^{i\omega t} q$$  \hspace{1cm} (3)

The equation describing radial motion is

$$\ddot{r} = \frac{e}{m} J_m E_r e^{i\omega t}$$  \hspace{1cm} (3a)

If there are no space charges, foils or grids in the field,

$$\nabla \cdot E = 0.$$  \hspace{1cm} (4)

From condition (4) it follows that in the region adjacent to the acceleration axis ($r \sim 0$) $E_r$ is connected with $E_z$ by the following relationship:

$$E_r = \frac{r}{2} \frac{\partial E_z}{\partial z}$$  \hspace{1cm} (5)

Substituting (5) in (3a), we obtain

$$\ddot{r} = -\frac{e}{2m} J_m \frac{\partial E_z}{\partial z} (z_0) e^{i\omega t}$$  \hspace{1cm} (6)

We shall put

$$f(t) = -\frac{e}{m} J_m \frac{\partial E_z}{\partial z} (z_0) e^{i\omega t}$$

In that case, equations (3) and (6), describing the longitudinal and radial motion of the particle, may be written

$$\ddot{q} + f(t) q = 0$$  \hspace{1cm} (7)
\[
\ddot{r} - \frac{1}{2} f(t) r = 0 \tag{8}
\]

In a travelling-wave accelerator

\[
E_x(z) = E_0 e^{i\omega t / \gamma} \int \frac{dz}{V(z_x)}
\]

and, consequently,

\[
f(t) = -\frac{2\pi e E_0}{m \beta} \cos(\omega t - \phi) \int \frac{dz}{V(z_x)} + \phi_x \tag{9}
\]

If, as this was customarily done, the law of phase velocity change is chosen so that for a synchronous particle

\[
t - \int \frac{dz}{V(z_x)} + \phi_x \omega = \text{Const}
\]

and equations (7) and (8) will assume the form (Ia. B. Fainberg, 1947)

\[
\dddot{q} + c_q q = 0 \tag{10}
\]

\[
\ddot{r} - \frac{1}{2} c_r r = 0 \tag{11}
\]

Equations (10) and (11) point to the impossibility of achieving simultaneous radial and phase stability without special focusing devices in travelling-wave accelerators. In a standing-wave accelerator the condition of radial and phase stability is that the highfrequency oscillations \( \Omega^2_q \) and \( \Omega^2_r \) averaged over the period must be positive (A.I. Akhiezer and G. Liubarski, 1948)

\[
\overline{\Omega^2_q} > 0, \quad \overline{\Omega^2_r} > 0
\]

But since, as it can easily be shown

\[
\overline{\Omega^2_q} = -2\overline{\Omega^2_r}
\]

simultaneous radial and phase stability is impossible in this case too.

Two assumptions were made in considering questions of dynamics in linear accelerators. It was assumed that \( |\Omega_q \tau| \ll 1 \) and that \( \Omega^2_q \) and \( \Omega^2_r \) should not change the sign. In that case, as shown above, simultaneous radial and phase stability are impossible without auxiliary devices.

A method of radial focusing has lately been put forward employing alternating focusing and defocusing lenses with a transverse magnetic or electric field. Such a magnetic (or electric) focusing method is evidently the most effective at the present time. However, with the increase in accelerating field intensity and shortening of the wave length, the magnetic fields required for focusing become quite considerable.

The purpose of the present paper is a generalization of the focusing method employing alternating focusing and defocusing lenses for the case of longitudinal (phase) focusing, so as to achieve simultaneous radial and phase stability without auxiliary devices*. For this purpose it is necessary to alter periodically the synchronous phase in such a way that the accelerated particle should alternately be in the region of phase stability and the region of radial defocusing and vice versa. Simultaneous radial and phase stability will in this case be accomplished**. The oscillations of the synchronous phase are effected by a proper selection of the law of wave phase velocity change. In accelerators with drift tubes the length of the drift tubes is for several periods chosen to be smaller than \( \beta \lambda \) and for subsequent periods larger than \( \beta \lambda \) or the wave phase is changed.

We shall consider the question of radial and phase stability for an accelerator with an arbitrary field pattern. It shall merely be assumed that the field is axially symmetric and that there are no space charges, foils or grids in it. It can then be demonstrated that in linear approximation the equations describing radial and longitudinal motion assume the form

\[
\dddot{q} + f(t) q = 0 \tag{12}
\]

\[
\ddot{r} - \frac{1}{2} f(t) r = 0 \tag{13}
\]

where \( f(t) \) is a periodic function. In this way the question of the possibility of simultaneously achieving phase and radial stability is reduced to the question of the existence of a region of stable solutions of the two differential equations (12) and (13) of the second order with periodic coefficients. We shall consider the question of the existence of stable solutions of equations of the type

\[
d^2y/dt^2 = \mu p(t)y,
\]

where \( \mu \) is a small quantity and \( p(t) \) is a periodic function with a period \( \omega \) satisfying the condition

\[
f^{\prime\prime}\mu p(t) dt = 0
\]

Let \( f(t, \mu) \) and \( \varphi(t, \mu) \) be two linearly independent solutions of equation (14) satisfying the requirements:

\[
f(0, \mu) = 1, \quad f(0, \mu) = 0 \quad \varphi(0, \mu) = 0, \quad \varphi'(0, \mu) = 1 \tag{14a}
\]

Let us resolve the functions \( f(t, \mu) \) and \( \varphi(t, \mu) \) in a \( \mu \) series

\[
f(t, \mu) = f_0(t) + \mu f_1(t) + \mu^2 f_2(t) + \ldots
\]

\[
\varphi(t, \mu) = \varphi_0(t) + \mu \varphi_1(t) + \mu^2 \varphi_2(t) + \ldots
\]

Then, coming back to (14) and assuming the quantities

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* It subsequently became known that similar ideas have been propounded by M. L. Good in the USA and L. B. Mullet et al. in Great Britain.

** As a mechanical analogue for the focusing method in question may serve a pendulum with a vibrating support. The most complete experimental and theoretical investigation of such a pendulum was carried out by P. L. Kapitza.
of the same μ order to be equal, we obtain the following system of differential equations for the functions \( f_n \) and \( \varphi_n \):

\[
\begin{align*}
\frac{df_n}{dt^2} &= 0 \\
\frac{d^2\varphi_n}{dt^2} &= 0 \\
\frac{df_n}{dt} \varphi_{n-1} &= p_{n-1} \\
\frac{d^2\varphi_n}{dt^2} - p_{n-1} &= \varphi_{n-1} 
\end{align*}
\]

Here the functions \( f_n \) and \( \varphi_n \) will satisfy the following initial conditions

\[
\begin{align*}
f_0(t) &= 1 \\
\varphi_0(t) &= 0 \\
f_n(0) &= f_n(0) = \varphi_n(0) = 0 \\
\varphi_n(0) &= 0 
\end{align*}
\]

Integrating (15) and making use of (16), we obtain

\[
\begin{align*}
\int f_0(t) &= 1 \\
\int \varphi_0(t) &= t \\
\int f_n(t) &= \int \varphi_n(t) = t \\
\int f_{n-1}(t) &= \int \varphi_{n-1}(t) = t
\end{align*}
\]

From the theory of differential equations with periodic coefficients it is known that the quantity \( \rho_{12} = e^{\Omega t} \) is determined by the equation

\[
x^2 - 2 A x + 1 = 0,
\]

where

\[
A = 1 + \frac{1}{n} \sum \left[ f_n(\omega) + \varphi_n(\omega) \right] e^\omega
\]

It is easy to see that the series coefficient in (17) becomes 0 for the first degree of \( \mu \) when the function \( P(t) \) satisfies the condition \( \int f_0^0 P dt = 0 \).

Indeed

\[
\begin{align*}
f_1(\omega) + \varphi_1(\omega) &= f_0^0 df_0^0 + f_0^0 df_0^0 P dt \\
\int f_0^0 df_0^0 P dt &= t \int f_0^0 df_0^0 = \int f_0^0 df_0^0
\end{align*}
\]

Making use of condition (18) we obtain

\[
f_1(\omega) + \varphi_1(\omega) = 0
\]

It will now be shown that, given condition (18), there must exist stable solutions of equation (14); moreover, for small \( \mu \), the period of stable oscillations of \( \tilde{\Omega} \) is proportional to \( \mu \).

For this purpose it is necessary to find the coefficient of \( \mu^2 \) equal to \( f_2(\omega) + \varphi_2(\omega) \)

According to (14)

\[
f_2(\omega) = f_0^0 df_0^0 f_1(\omega) + \varphi_2(\omega) = f_0^0 df_0^0 \varphi_1(\omega)
\]

Making use of (16)' we obtain

\[
\begin{align*}
f_2(\omega) &= f_0^0 df_0^0 f_0(\omega) + \varphi_2(\omega) = f_0^0 df_0^0 \varphi_1(\omega) \\
\int f_0^0 df_0^0 f_0(\omega) + \varphi_2(\omega) = f_0^0 df_0^0 \varphi_1(\omega)
\end{align*}
\]

We shall designate

\[
\int f_0^0 df_0^0 f_0(\omega) = \varphi(\omega)
\]

Then, according to (18)

\[
\varphi(\omega) = 0
\]

we note that

\[
f_0^0 df_0^0 f_0(\omega) = s = f_0^0 df_0^0 - f_0^0 f_0(\omega) df_0^0
\]

On the basis of (21) we can rewrite (19) thus:

\[
\begin{align*}
f_0^0 df_0^0 f_0(\omega) + \varphi_2(\omega) &= f_0^0 df_0^0 f_0(\omega) + \varphi_2(\omega) \\
\int f_0^0 df_0^0 f_0(\omega) + \varphi_2(\omega) &= f_0^0 df_0^0 f_0(\omega) + \varphi_2(\omega)
\end{align*}
\]

For \( \varphi_2(\omega) \) we obtain

\[
\varphi_2(\omega) = f_0^0 df_0^0 f_0(\omega) + \varphi_2(\omega)
\]

Let us now transform the integrals determining \( f_2(\omega) \) and \( \varphi_2(\omega) \) into single integrals and denote \( f_2(\omega) = I_2 + I_3 \), where \( I_1 \) and \( I_2 \) are the first and second integral respectively in (22).

Integrating \( I_1 \) part by part, we obtain

\[
I_1 = \omega^0 f_0^0 f_0(\omega) s \dot{\varphi}(\omega) ds - f_0^0 f_0(\omega) s^2 \ddot{\varphi}(\omega) ds
\]

Integrating \( I_2 \) part by part, we find

\[
I_2 = - (\omega^0 f_0^0 f_0(\omega) s^2 \ddot{\varphi}(\omega) ds - f_0^0 f_0(\omega) s^3 \dddot{\varphi}(\omega) ds)
\]

By virtue of (21'), \( I_3 = - f_0^0 \dot{\varphi}(\omega) dt f_0^0 \ddot{\varphi}(\omega) ds \)

Hence, \( f_2(\omega) + \varphi_2(\omega) = I_1 + I_2 + I_3 \)

\[
= 2\omega^0 f_0^0 f_0(\omega) s^3 \dddot{\varphi}(\omega) ds - f_0^0 f_0(\omega) s^4 \ddot{\varphi}(\omega) ds - f_0^0 f_0(\omega) ds + f_0^0 s \ddot{\varphi}(\omega) ds
\]

But \( f_0^0 f_0(\omega) s^3 \dddot{\varphi}(\omega) = f_0^0 s \ddot{\varphi}(\omega) ds - f_0^0 \dddot{\varphi}(\omega) ds \)

Noting that

\[
2\omega^0 f_0^0 f_0(\omega) s^3 \dddot{\varphi}(\omega) ds - f_0^0 f_0(\omega) s^4 \dddot{\varphi}(\omega) ds = - f_0^0 f_0(\omega) \dot{\varphi}(\omega) ds
\]

and making use of (21') we obtain

\[
2\omega^0 f_0^0 f_0(\omega) s^4 \dddot{\varphi}(\omega) ds = - f_0^0 f_0(\omega) s^4 \dddot{\varphi}(\omega) ds
\]

Let us now calculate the second integral in (25). For this we shall introduce the notation \( f_0^0 f_0(\omega) ds = \chi(s) \)

Then

\[
\int f_0^0 f_0(\omega) ds = \int f_0^0 \chi(s) ds = \frac{1}{2} \left[ \chi'(\omega) - \chi(\omega) \right] + \chi'(\omega)
\]

Consequently,

\[
f_0^0 f_0(\omega) + \varphi_2(\omega) = I_1 + I_2 + I_3 - \omega f_0^0 \dot{\varphi}(\omega) ds + f_0^0 \dddot{\varphi}(\omega) ds
\]

It will be remembered that the general solution of a differential equation of the second order with periodic coefficients is

\[
y = e^{\mu t} u(t) + e^{-\mu t} u(-t)
\]

where \( u(t) \) is a periodic function with the period \( P(t) \); the stable solutions correspond to real \( \mu \).
Making use of the Schwartz-Bunyakovsky relationship, we note that the coefficient of $\mu^2$

$$f_s(\omega) + \varphi'(\omega) < 0$$

Thus, if the function $p(t)$ satisfies the condition $\int_0^{\omega} p(t) dt = 0$, then for small $\mu$ there always exist stable solutions of equation (30), with the frequency of oscillations of these stable solutions $\tilde{\omega}$ being proportional to the first degree of $\mu$. Besides, for small $\mu$ the region of stable solutions and the frequency of oscillations $\tilde{\omega}$ do not depend on the sign of $\mu$. Therefore, both the radial and the phase movement are stable simultaneously. The formula makes it possible to determine the frequency of stable oscillations for the arbitrary function $f(t)$ if

$$\int_0^{\omega} p(t) dt = 0$$

and $\mu$ is small.

The question naturally arises of choosing such a function $p(t)$ which would ensure the maximum frequency of oscillations and the biggest width of the stability region.

It can be shown that the frequency of oscillations is maximum for a sinusoidal function.

To settle the question of the capture width in accelerators employing this method of focusing it is necessary to solve a corresponding non-linear problem (see below).

A generalization of the solution obtained in linear approximation for the relativistic case encounters no difficulties. The matrix characterizing the transition from one period of the focusing system to another is

$$M = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix},$$

where

$$a_{11} = 1 + [\mu (-\int_0^{\omega} p(t) dt) + \mu^2 (-\omega \int_0^{\omega} \varphi' ds + \int_0^{\omega} \varphi'^2 ds + \frac{1}{2} [I/\int_0^{\omega} \varphi ds]^2]$$

$$a_{12} = \omega + \mu [\omega \int_0^{\omega} p(t) dt - \int_0^{\omega} \varphi' dt];$$

$$a_{21} = -\mu^2 \int_0^{\omega} \varphi ds;$$

$$a_{22} = 1 + \mu \int_0^{\omega} p(t) dt + \mu^2 (-\int_0^{\omega} s \varphi' ds + \frac{1}{2} [I/\int_0^{\omega} \varphi ds]^2]$$

The amplitude of oscillations is determined by the relationship:

$$Y_n = Y_0 \left( \frac{\sin \alpha (n + 1) - a_{22} \sin \alpha n}{\sin \alpha} \right) + Y_0' a_{12} \sin \alpha n \frac{\sin \alpha}{\sin \alpha};$$

$$\cos \alpha = \frac{a_{11} + a_{22}}{2}$$

In the general case when $\mu$ is large the amplitude is determined from the matrix obtained by multiplying the matrices for the half periods $\omega$ where the sign of $p(t)$ is constant.

Now let us consider focusing in travelling-wave accelerators. In this case

$$E(z) = E_0 e^{-i\omega f} dz/V_\varphi(z)$$

We shall introduce the notation $z_0$

$$f(t) = \Omega q^2 = \frac{2\pi eE_0}{m^2 \beta_\varphi(z_0) \lambda} \cos \left[ \omega \left( t - \int_0^{z_0} \frac{d z}{V_\varphi(z_0)} \right) \right]$$

(29)

$$\Omega_t^2 = -f(t)/2 = -\Omega_t^2/2$$

(30)

The equations describing radial and longitudinal motion will be

$$\ddot{q} + f(t)q = 0$$

(31)

$$\dot{r} - f(t)r/2 = 0$$

(32)

As distinct from (10) and (11), the coefficients of $q$ and $r$ are functions of time. If $f(t)$ is a periodic function of $t$, we obtain equations of the Mathieu or Hill type. In this case there are regions of parameter changes characterizing the function $f(t)$ for which both equations (31) and (32) are stable simultaneously, i.e., there is simultaneous radial and phase stability. A specific case of equations (31) and (32) is the equation in which $f(t)$ is a discontinuous function of the type

$$f(t) = n^2_2$$

(33)

$$f(t) = -n^2_2$$

(34)

with a period $\tau_1 + \tau_2$.

This case corresponds to focusing in an accelerator with drift tubes.

Passing on to a solution of equations (31) and (32), let us consider the case in which the synchronous phase experiences harmonic oscillations with the frequency $\Omega$, and $\Omega$ is less than the frequency of the high-frequency field

$$\Omega = \omega/n, \quad n > 1$$

(35)

In this case

$$f(t) = 2\pi eE_0/m^2 \beta_\varphi(z_0) \lambda \cdot \sin \Omega t = A \cos \Omega t$$

(36)

where $\alpha$ is the oscillation amplitude of the synchronous phase and eqs. (31) and (32) become Mathieu equations

$$\ddot{q} + A \cos \Omega t \cdot q = 0$$

(37)

* As pointed out by M. G. Krein, the equations of the second order in the case of an odd sign function were studied by Liapunoff (Comptes rendus, 1896). The condition of stability was formulated for arbitrary $\mu$. A more general case was recently intensively studied by M. G. Krein.
Designating

\[ A = \Omega_\varphi^2 = -2\pi e \varepsilon_0 / m \beta_0(z_0) \lambda. \]

we obtain

\[ \frac{d^2 q}{d \tau^2} - 4 \Omega_\varphi^2 / \Omega^2 \cos 2\tau \cdot q = 0 \]  
\[ \frac{d^2 r}{d \tau^2} + 2 \Omega_\varphi^2 / \Omega^2 \cos 2\tau \cdot r = 0 \]

Assuming

\[ 4 \Omega_\varphi^2 / \Omega^2 = 2 p, \]

we arrive at this canonical-form of Mathieu’s equation

\[ \frac{d^2 y}{d \tau^2} + 2 \Omega_\varphi^2 \cos 2\tau \cdot y = 0 \]

Equation (41) can be solved on the basis of the general theory of Mathieu equations. However, in the general case of arbitrary \( p \), the obtained solutions are difficult to interpret. Therefore, we shall consider the case of small \( p \). The case of small \( p \) in the Mathieu equation is of particular interest to us since in this case there is simultaneous stability of longitudinal and radial motion. The field of stable motion as a function of \( p \) and \( \lambda \) for the Mathieu equation \( y'' + (2p \cos 2\lambda + \lambda) y = 0 \) has a well-known appearance.

In our case \( \lambda = 0 \). It is easy to see that for values of \( p \ll 1 \) the solutions of eqs. (37) and (38) are simultaneously stable. As has been demonstrated by McLachlan, the quantity \( \mu \) characterizing the periodicity of the solution is in this case determined by the relationship

\[ \mu^2 \approx p^2 / 2(1 - \mu^2), \quad p \ll 1 \]

The same result can be obtained if in the limiting form of Hill’s determinant for small \( p \) we assume \( \lambda = 0 \)

\[ \chi_{\mu} \approx 1 + \left( \frac{\pi \mu}{2} \right)^2 = 1 + 2 \sin^2 \frac{\pi}{2} \sqrt{\lambda} + \frac{\pi \mu^2}{2(1 - \lambda)} \sin \pi \sqrt{\lambda} \]

\[ + 0(p^4) \]

The quantity \( \mu \) determines the frequency of radial and longitudinal oscillations. Making use of (42), we obtain:

\[ \mu_\varphi \approx \frac{\Omega_\varphi}{\sqrt{2}} \approx \sqrt{2} \frac{\Omega_\varphi^2}{\Omega^2} ; \quad \mu \approx \frac{\Omega}{\sqrt{2}} \approx \frac{1}{\sqrt{2}} \frac{\Omega_\varphi^2}{\Omega^2}. \]

Passing from \( \tau \) to \( t \), we obtain

\[ \tilde{\Omega}_\varphi \approx \sqrt{2} (\Omega_\varphi^2 / \Omega) \]

\[ \tilde{\Omega} \approx (1 / \sqrt{2}) (\Omega_\varphi / \Omega) \]

The efficiency of the method of focusing outlined, as may be seen from (43) and (44), increases with a rise in the intensity of the accelerating electrical field, a shortening of the wave length, and a decrease of particle velocity. Therefore, this method may prove highly efficient in accelerators with a high field intensity. (The Veksler method, and others.)

The non-linear approximation of the equation of phase oscillations is

\[ \frac{d^2 q}{d \tau^2} + \frac{e}{m} \frac{E_0}{V_\varphi} \cos \varphi_0 q - \frac{e}{m} \frac{E_0}{\omega^2} \frac{1}{V_\varphi^2} \left( \omega^2 - \frac{d}{V_\varphi^2} \frac{d}{dz} \cos \varphi_0 \right) q^3 = 0 \]

The capture width determined from this equation is

\[ (\Delta \varphi)_{\max} = 1.5 \mu^2 \]

where \( \mu = 2\pi e \varepsilon_0 / m \beta \lambda \) and \( \alpha \) is the amplitude of the synchronous phase oscillations.

The admissible scattering of velocities is

\[ (\Delta V)_{\max} = \mu^2 \alpha^2 / 2.5 \]

The system of equations describing conjugated radial-phase oscillations is

\[ q'' + \alpha \varphi_q + k \varphi - \delta q^3 = 0, \quad \delta = \alpha / \beta \lambda \]

\[ \tau - \alpha \varphi - r + \delta q = 0, \quad \delta = \alpha / \beta \lambda \]

After averaging\(^3\), the system assumes the form\(^*\):

\[ \frac{d^2 q}{d \tau^2} + \frac{\mu^2}{2} q + \frac{\mu \delta_1 q^2}{2} + \frac{\mu \delta_2 t^2}{2} = 0 \]

\[ \frac{d^2 r}{d \tau^2} + \frac{\mu^2}{8} + \mu \delta_3 \varphi = 0 \]

The appendix contains a numerical calculation for variable phase focusing.

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Fig. 1. The synchronous phase \( \varphi_0 \).

\(^*\) M. F. Nekrasov took part in the investigation of the question of the relationship between radial and phase movement.
Appendix

Below we give the results of numerical calculations of the phase movement equation in the case of alternating phase focusing, for the accelerator with drift tubes (fig. 1a, 1b).

The synchronous phase varied according to the law

$$\varphi_n = \alpha \cos \omega t / n$$

where n satisfies the condition

$$\frac{n^2 (1 - \beta^2)^{3/2}}{\beta} = \frac{1}{A} = \text{Const}$$

$$A = \frac{\pi c E \gamma \lambda}{mc^2} = \frac{\alpha}{\pi} a$$

$$\gamma$$ - is the field factor,

$$E$$ - average accelerating field in a period.

In our case $$n^2 = 4.62 \beta/(1-\beta^2)^{3/2}$$. The calculations given here are only illustrative.

The following parameters were chosen

$$E = 150kV/cm \quad \lambda = 200 \text{ cm} \quad a = 0.03$$

The accelerating system consisted of 45 drift tubes. The velocity in it changed from 0.46 c to 0.84 c.

In these calculations the radial movement was not taken into account.

The above-mentioned numerical calculations were carried out by V. S. Tkallest and B. A. Suprunenko.

LIST OF REFERENCES

5. Bogoliubov, N. N. and Mitropolski, Iu. A. (Asymptotic methods of non-linear mechanics.) Moscow (?)
DISCUSSION

R. Wideröe: The latest proposals by D. W. Kerst and G. K. O’Neil to produce collisions between high-energy particles moving in opposite directions, suffer from the disadvantage that the common straight portion of the orbits of the two particles where the collisions occur (the “target length”) is relatively short, thus resulting in only very restricted experimental possibilities.

I have previously described a layout using only one storage ring which would increase the length of interaction considerably and thus produce many more collisions and provide a much greater experimental field. In this arrangement, the particles rotate in counter directions on practically the same orbit and collisions may then in fact occur along all parts of the orbit*. It might, however, be convenient to tilt the orbits of the clockwise and anticlockwise rotating particles relative to each other by means of special magnetic fields, so that the collisions would only occur at the intersection portion of the two orbits. In this way (or by similar methods for displacing the two orbits) it would, in principle, be possible to choose the intersecting part of the two orbits at will and thus create the best conditions for research.

There are two possible methods for using counter rotating particles. One is to use positively and negatively charged particles rotating in a common magnetic storage ring**, while the other is to use particles of the same charge (e.g. protons) rotating in opposite directions in an electrostatic guiding field.

The first proposals using, for instance, protons and negatively charged hydrogen ions, I already considered as early as in 1943. At that time, however, I thought that it would be very difficult to maintain the negative charge of the hydrogen ions during the acceleration period. The two electrons are relatively weakly bound to the proton.

Accordingly, the probability that they would be stripped off by collisions with the gas molecules in the vacuum chamber seemed overwhelming. Today the situation appears to have changed. M. G. White and G. K. O’Neil seriously consider the acceleration of negative hydrogen ions in the new 3 Gev accelerator at Princeton, using a vacuum of about 10^{-10} mm. Hg. It seems possible that such a high vacuum could be maintained even in a very long vacuum vessel using a double-walled metal tube and the new evap-ion pumps working with titanium getters.

However, the second suggestion of using an electrostatic guide field should also receive more attention than has so far occurred. In this case the electrostatic field could be shaped to produce strong focusing effects. However, this would increase the electro-static field strength between the electrodes to more than twice the field strength necessary for bending the particle orbit. I therefore think that it would be better to use an electrostatic bending field of nearly constant strength in the radial direction.

The stabilizing forces required for guiding the particles could easily be produced by means of special magnetic lenses.

Without doubt, the most important question regarding this proposal is: How high a value of electrostatic field strength can be attained between the electrodes?

In the modern electron microscope with electrostatic lenses, field strengths of up to 150 kV/cm are used; but this should not be regarded as the ultimate limit possible. In the electron guns used in betatrons and synchrotrons, the electrostatic fields might for a short time reach values of 1000 kV/cm or even higher. However, experience has shown that such high field strengths can only be maintained with very small inter-electrode distances. When the field strength must be applied over a distance of a few centimetres, the limit might easily drop down to 100 kV/cm or even lower. Here again, the recent improvements in vacuum techniques may change the situation for the better.

It seems to me that a field strength of, for example, 300 kV/cm should not be quite outside the limits of possibility if a certain amount of development work could be concentrated on this problem. This field corresponds to a magnetic guide field of 1000 gauss, which is about 1/10 of the field strength now used in high energy accelerators. Thus the electro-static storage ring required would be about 10 times larger than a storage ring using magnetic guiding fields.

Such large dimensions may certainly lead to difficulties in the mechanical alignment and adjustment of the elec-

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* If the sense of revolution of the particles in the accelerator can be changed, then it might be convenient to locate the accelerator within the storage ring.

** In a FFAG machine, the storage ring could possibly be omitted, both particles being accelerated simultaneously.
trod os. On the other hand, we know that with strong focusing lens systems, deviations from the circular orbit are less dangerous when they are distributed over greater lengths. The particles can be led over "hill and dale" provided the deviation takes place slowly enough.

In the vertical direction the electrodes could have large dimensions relative to the inter-electrode spacing without any great increase in cost. This suggests that the particles might be brought into the storage ring from an inlet passage situated above or below the orbit.

However, the electrostatic guide field will certainly also result in certain advantages as compared with the magnetic guide field. The electrode systems should be much cheaper and simpler than the necessary magnets and the excitation of the field would appear to be also simple and inexpensive.

The electric field could be produced by means of a Van de Graaff generator. It may, however, also be possible to charge the electrodes by means of isotopes producing $\beta$-rays of sufficient energy, for example, Rh-106 which produces $\beta$-rays with a maximum energy of 3.5 Mev.

In conclusion I would like to say that it might be advantageous to investigate further the possibilities of the electrostatic guiding field. If only a small part—say about 5%—of the grants now being directed to the development of big new machines could be used for investigations on electrostatic guide fields, this might result in major improvements and open the way to new developments in the accelerator technique.

E. O. Lawrence: Even with fields as low as 10 kV/cm it is difficult to prevent sparks over long periods of time.

G. K. O'Neill: The possibility of accelerating negative H-ions and protons in opposite directions in the same machine was also pointed out by M. G. White. In machines without storage properties it is, however, almost impossible to get large enough interaction rates, since the rates are proportional to the square of the number of pulses stored.

The vacuum needed for the acceleration of H-ions is $10^{-9}$ to $10^{-10}$ mm. Hg.

M. G. White: What is the gas pressure in the plasma?

Ia. B. Fainberg: $10^{-5}$ to $10^{-6}$ mm. Hg.

With respect to means of producing plasma one can distinguish:

1. Plasma in wave guides with axial magnetic field.
2. Plasma without a magnetic field.
3. Gabor's ring electrode arrangement—Gabor's lenses (permits high charges to be accumulated).
4. G. J. Budker's type of electron stabilized beams.
5. Electron beams with and without a magnetic focusing field.

H. Wergeland: 1. How can reversal of Čerenkov effect take place? By conservation laws, transfer of energy from a radiation field to a particle is only possible when the refractive index of medium is greater than 1. Can this take place in the plasma?

2. Referring to radiation pressure as a means for accelerating particles, the formula force = energy-current times cross section, is relevant if the radiation wave length $\lambda$ is short in comparison with the particles' radius as was shown by Debye (1909). For long radiation wavelength a transfer of energy does not seem possible.

V. I. Vekler: As for the answer to 1, if we superpose a magnetic field on the plasma, the refractive index becomes bigger than 1. A. A. Kolomenski made detailed calculations of the Čerenkov effect in such a media (his papers were published).

As to question 2, in order to accelerate electrons by light pressure, it is necessary that light wave lengths obey $\lambda \gg r_0$ where $r_0 = e^2/mc^2$ is the classical radius of the electron. In the case of neutral bunches this condition turns out to be $\lambda \gg a$ where $a$ is the radius of the bunch.

The classical scattering from a sphere as calculated by Debye is given by the following graph, Fig. 2.

At light frequency of course, there is no coherent acceleration, only ordinary Thomson scattering.

T. A. Welton: What does the accelerating system look like for the plasma? Is it a wave guide field, or a wave guide bounded plasma, or are the heavy particles accelerated directly by plasma? How can you force high electric fields into the plasma for acceleration, since plasma conductivity is generally high?

Ia. B. Fainberg: If the plasma has high conductivity it is difficult to force electric field into it. In our scheme, heavy particles are accelerated in a region where the plasma is rarified to the rather small density of $10^6$ particles
per cm³. The conductivity is so low as \( \sigma = 10^4 \) to \( 10^5 \). These data were obtained in the linear approximation.

The plasma phase velocity is less than \( c \) only for the case where the plasma is bounded.

R. Deminirkhanov: Intensities of 100 A/cm² have been practically attained, even more seems possible. Stability and interaction of plasma are now being investigated. There seems to be no experimental obstacle to the production of very low density plasma. Interaction of plasma and travelling waves are now being studied and we hope to report more results at the next occasion.

J. P. Blewett: How are losses avoided in the containers and structures that limit the plasma? Are there periodic structures around the plasma?

Ia. B. Fainberg: The plasma is mostly not in close contact with the walls of the chamber. The electromagnetic field goes down rapidly away from the plasma. The exciting mode is imposed by externally forced vibrations.

R. Wideröe: In 1952 H. Alfven proposed at a meeting of CERN to accelerate ions by plasma. He had a long heater and a wave travelling around it. With very high fields being Accommodated along this heater a linear accelerator would have been possibly built much shorter than with all other known forms.

M. Seidl: Is V. I. Veksler’s coherent accelerating method based on the inverse mechanism of energy transfer than that one which takes place in electron wave-tubes? In these tubes two parallel electron streams are placed side by side and amplification of electro magnetic waves is attained by action of one of them on the other.

V. I. Veksler: My idea dates from 1951 and is not related to H. Alfven’s, where electron bunch and protons have to move together with the same velocities and where accelerating forces do not depend on the number of protons. On the contrary, in the coherent acceleration principle, the electron stream or bunch goes ahead of the accelerating particles and the action of this stream is proportional to the square of the number of acceleration particles.

J. D. Lawson: In Budker’s ring, has the criterion according to which, if \( v_1 > 1 \) one has instability, and stability if \( v_1 < 1 \), a simple physical explanation?

A. A. Naumov: Experimental considerations show that, if ordinary betatron beams (\( v \leq 1 \)) are absolutely stable, plasma ring currents (\( v \gg 2 \)) as a rule are not stable. In stabilized beams densities of particles are bigger than in ordinary betatron beams, but lower than in ordinary plasma currents. The stability limit seems to lie in the region of \( v \sim 2 \) which we have not yet realized experimentally. S. Belaiev can give you further theoretical information on beam stability.

S. Belaiev: Stability of dense beams cannot be examined as a complete problem because of the complexity of the subject. We have made investigations of such beams only with respect to certain perturbations along the \( r \) and \( z \) directions and applied self consistent field methods to solve electron and ion motion problems in these beams. The stability condition is

\[ v_1 < \frac{\bar{v}}{\beta} \sim \beta^2 \gamma^2 \]

In the non relativistic region \( \bar{v} \) is small, in the relativistic region \( \bar{v} \) approaches 1. So instability may start at low values of \( v_1 \) but for high \( \gamma \) instability can start from \( \bar{v} = 1 \) onwards. This criterion is true if \( \lambda \ll r \) (\( \lambda \) wave length of perturbation arising in plasma, \( r \) - inner radius of electron beam).

The phase velocity \( v_{ph} \) versus wave length of perturbation \( \lambda \) in plasma begins to rise linearly with small \( \lambda \) because the phase is independent of frequency in unbound plasma. In bounded plasma it is constant with frequency so that the curve approaches a constant value.

At \( v < \bar{v} \) the limit value of \( v_{ph} \) is less than the velocity of electrons at vice versa.

Instabilities with respect to kinking of the beam occurs at low \( k = 1/\lambda \). At higher values of \( k \) the ring is more stable. Long wave instabilities around the ring seem not to be dangerous since they can be dealt with by external means.

I. I. Rabi: Is the acceleration of heavy particles (protons) by electrons in Veksler’s coherent method done by a sort of dynamic impact of the high relativistic electrons on the slow heavy particles or is a more complicated mechanism, as by means of a Cerenkov wave, involved?

V. I. Veksler: In non relativistic cases one can interpret the acceleration mechanism in the following simple way.

Consider a fast electron stream sweeping past a proton. The electron density ahead of the proton will be greater than it was before reaching the proton because the Coulomb attractive forces squeeze the electrons together, so that a net force in the direction of velocity on the proton is obtained and the proton is dragged along by the electrons. The expression for the Coulomb force acting on a proton is \( F = e^2/r^2 \cdot p/r \) where \( r^2 = p^2 + v^2 \), \( p \) is the impact parameter, and \( v \) the relative velocity.
Using simple calculation one can easily obtain the formula for the accelerating force acting on a proton

\[ \frac{dW}{dx} = Z^2 e^4 \nu v^2 \cdot \rho \ln (...) \]

The relation between \( \frac{dW}{dx} \) and \( v \) is shown on the fig. 5.

We work at velocities over the minimum point.

I. I. Rabi: In so highly relativistic conditions, is much energy connected with one passage, or do we need processes of this kind? Is it a dragging along of the particle or some sort of force that is comparable to viscosity?

V. I. Veksler: The viscosity is a force of short range type. Here we work with long range Coulomb forces.

P. B. Moon: How are the particles filling Budker’s chamber brought down to such a small section as Budker’s ring, and how do the betatron oscillations disappear?

A. A. Naumov: Presents slide 3 of G. J. Budker’s report and explains the time sequence of the various phenomena occurring at injection in detail, as can be found in the second part of Budker’s paper, written jointly with himself.

P. B. Moon: How are betatron oscillations that occur when the beam has been injected damped in order to get the small beam diameter you wish?

A. A. Naumov: I would like to point out that the betatron oscillations that occur in practice with our experimental injection scheme are small in both directions. The spiralling inwards of the injected beam takes place with small energy of betatron oscillations. The chamber is gradually filled with electrons that approach their equilibrium orbits as the guide field is fixed. Then the rapidly increasing betatron accelerating magnetic field is switched on.

During the whole injection process the guide field is nearly constant and made suitable for focusing. An n-value of about 0.5 is established throughout the whole region. The energy being increased and the field remaining constant, the particle can only spiral out.

A. Lundby: What is the efficiency of either of these coherent acceleration methods?

V. I. Veksler: The formulae have been worked out first in a linear approximation, where the changes of density are small compared to the density. The consideration of non-linear terms show that the efficiency could be made of the order of unity, say 30%.

D. W. Kerst: Is it only betatron oscillation damping that will bring the beam to its small size?

A. A. Naumov: In our experimental chamber betatron oscillations are very small and particles orbits approach the equilibrium orbit asymptotically.

D. W. Kerst: How big is the beam at the end of operation?

A. A. Naumov: At 3 Mev the beam has 3 cm.

D. W. Kerst: At which energy does the beam collapse to a small string for guiding other particles.

A. A. Naumov: At present only the conditions of production of high currents are studied. As it was pointed out in my report at sufficiently high currents a beam would experience a very strong contraction due to electromagnetic radiation of electrons in self magnetic fields. Then one can begin to accelerate ions. Neutralization of space charge by ions in our experimental betatron has not yet been investigated experimentally.