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GENERALIZED ROUGHENING TRANSITION AND
ITS EFFECT ON THE STRING TENSION

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A B S T R A C T

The well-understood roughening transition of an interface in the $d = 3$ Ising model implies an essential singularity in the string tension of the dual Z_2 gauge model. The roughening transition corresponds to the delocalization of the string due to strong long-wavelength fluctuations, and this reformulation can be generalized to other gauge groups and to $d = 4$ also. It is not a deconfining transition - it is expected to occur deep in the confining region - but its presence would raise serious questions about the continuation of strong coupling expansions of the tension beyond this point. In this paper predictions on the roughening transition are confronted with the available information on the string tension for different gauge groups in three dimensions.

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1. INTRODUCTION

Due to its conceptual simplicity the confining force between two heavy quarks, the string tension, could play a role in QCD similar to that played by the hydrogen atom in quantum mechanics. Recent considerable progress in lattice gauge theory raised our hopes to predict this fundamental, non-perturbative quantity from basic principles^{1)-7),8)}. There are several new techniques provided by lattice regularization, but these new results follow mainly from Monte Carlo simulations and strong coupling expansions.

The strong coupling expansion is a well-tested method in describing phase transitions⁹⁾⁻¹²⁾. There are warning signs, however, in applying this method to the string tension in gauge theories. The most notable case is the tension in the three dimensional Z_2 gauge model. In Hamilton formalism¹³⁾ the model is described by the Hamiltonian:

$$H = -\frac{1}{2} \sum_{\text{links}} (\epsilon_l - 1) - \lambda \sum_{\text{plaquettes}} (\epsilon_1 \epsilon_2 \epsilon_3 \epsilon_4) . \quad (1)$$

This model is dual to the Ising model¹⁴⁾. It has a second order phase transition at $\lambda_c \approx 1.56$ ¹⁵⁾⁻¹⁹⁾. The string tension is expected to behave like $\sim (\lambda_c - \lambda)^\mu$ as $\lambda \rightarrow \lambda_c$ with a critical exponent $\mu = 2\nu \approx 1.26$ ²⁰⁾. Using conventional perturbation theory the string tension can be calculated as a power series in λ . The series is available up to the order $\sim \lambda^{10}$ (see Sect. 3²¹⁾), and the different Padé approximants predict a transition at $\lambda^* \approx 0.85$ with a completely bad critical exponent. Of course, it is always an escape to say that the series is too short. However, the Padé table is stable, and this series is longer than the expansions for the magnetization and mass gap derived earlier¹⁵⁾, giving the correct value $\lambda_c \approx 1.5$.

Similar "bad" results have been found recently by N. Kimura using the Lagrange formulation of the Z_2 model²²⁾.

The key of understanding is the observation that the string tension is just the surface tension of a domain wall in the dual Ising model^{23),18)}. The behaviour of this interface has been the subject of intensive investigations in recent years^{20),23)-35)}. It is well established by now, that there is a phase transition (of infinite order) related to the behaviour of this interface. The transition point is definitely different from the critical point of the ordinary second order phase transition in the bulk phase. The transition is characterized by the domain wall becoming infinitely rough and other thermodynamical quantities related to the surface also becoming singular. The surface tension (or the string tension)

does not vanish at the roughening temperature, therefore it is not a deconfining transition. However, at the roughening point the tension is expected to behave like the free energy of the $d = 2$ XY model at the Kosterlitz-Thouless transition point³⁶⁾, i.e., it has an essential singularity there³⁷⁾. The Padé approximants are expected to accumulate poles around the essential singularity (and along the possible cuts starting from there)³⁸⁾, and it is questionable whether a power series expansion can be continued beyond this point.

It will be shown that both in the Hamilton and Lagrange formulations the power series expansions for the string tension in $d = 3$ Z_2 gauge model are in quantitative agreement with this observation. Using the Hamilton language the string state becomes a high superposition of states containing long-wavelength string fluctuations. At the roughening point the string is completely delocalized, but it has a finite intrinsic-width and tension. In the Lagrange formulation the roughening corresponds to long-wavelength ripples of the minimal surface covering the Wilson loop in a strong coupling expansion.

The above formulation of the roughening transition in a gauge theory is actually independent of the gauge group. A geometrical definition of this structural transition can be given for higher gauge groups and in $d = 4$ also.

We have estimated the roughening transition points for different three dimensional gauge theories. We have performed a strong coupling expansion directly for the surface width in the $d = 3$, $U(1)$ gauge model in Lagrange formulation. Our series is rather short ($\sim \beta^{12}$). It signals a roughening transition at $\beta_R \approx 1.14$.

In the Z_2 gauge model the roughening transition occurs deep in the confining region, where vacuum fluctuations play a very restricted role²⁴⁾. Accordingly, we expect that the approximation where the fluctuations of the original string are kept only, may be a reasonable one for other gauge groups as well. This approximation predicts a roughening transition for different three dimensional gauge models, for instance $\beta_R \approx 0.83$ for Z_3 , $\beta_R \approx 1.01$ for $U(1)$, $\beta_R \approx 1.05$ for $SU(2)$ in Lagrange formulation, and $\lambda_R \approx 0.85 - 0.90$ for $U(1)$ in Hamilton formulation.

The string tension is expected to feel the roughening singularity as in the Z_2 case. Our analysis of the string tension in $U(1)$ and $SU(2)$ [based on the strong coupling expansions of Refs 39), 40)] and in $U(1)$ (based on our 10th order result in the Hamilton formulation) confirms the picture.

The roughening transition described here for $d = 3$ is very different from the structural transition predicted at large space-time dimension d by Drouffe, Parisi and Sourlas⁴¹⁾. Instead of their tree-like surface configurations, long-wavelength surface ripples play the dominant role here and it is predicted to be important for low dimensional gauge theories. In $d = 4$ there are two transversal directions, and only a detailed analysis could decide which picture is the dominant one. Presumably, a direct strong coupling expansion for the surface width is the most powerful method here. It is an exciting question, whether the non-trivial structure of the tension observed in $d = 4$ ^{1),3),4),6)} is related to the roughening transition. But it is even more important to understand whether a simple continuation of the strong coupling expansion towards the continuum limit is possible at all. In $d = 3$ the answer seems to be no.

All over the paper we used both the Lagrange and the Hamilton formulation because it increased the available information.

Our paper is organized as follows. In Section 1 the relevant points on the roughening transition in the $d = 3$ Ising model are summarized. The transition is analysed in terms of the Z_2 gauge model, and the available information on the tension is discussed in Section 2. In Section 3 the Hamilton formulation is described and our 10th order result for the Z_2 tension is analyzed. Section 4 is devoted to the generalization to other gauge groups, and our predictions on the roughening points in different gauge models are presented here. In Section 5 we collect and analyze the existing strong coupling expansions for the string tension in $d = 3$ gauge theories, including our 10th order result for $U(1)$ in Hamilton formulation. The generalization to $d = 4$ is discussed in Section 6, while Section 7 is a summary.

1. ROUGHENING TRANSITION IN THE $d = 3$ Ising model interface

Consider a three dimensional spin configuration at $T = 0$, where for $Z < 0$ and $Z > 0$ all the spins are pointing upwards and downwards respectively (Fig. 1). It is a stable configuration since a decay would require an infinite number of spin slips. At $T \neq 0$ the interface begins to fluctuate. The excess free energy per unit area of the interface is called the surface tension σ ²⁰⁾. The surface tension is gradually decreasing with increasing temperature, and it becomes zero at the Curie point T_c , where the spontaneous magnetization of the two separate bulk regions disappears:

$$\sigma \sim (T_c - T)^\mu, \quad (2)$$

where the exponent $\mu = 2\nu \approx 1.26$. Here ν is the critical exponent of the correlation length. The scaling law (2) follows essentially from the fact that σ has a dimension of $(\text{mass})^2$ (20).

Though the surface tension vanishes only at $T = T_c$, there is a structural phase transition related to the presence of the interface at a much lower temperature $T = T_R$ (24)-(26). This fact is less surprising if we notice that the spins along the surface feel no vertical stabilizing magnetic field from the rest of the lattice. Therefore the boundary plane is expected to behave like a two dimensional Ising model with large fluctuations at $T_c^{d=2} \approx 0.5 T_c^{d=3}$.

At the roughening temperature the surface becomes delocalized, the surface width and other interfacial thermodynamical properties diverge. Strong coupling expansions predict $T_R \approx 0.57 T_c$ (24), (25) in nice agreement with the naive physical picture above. However, there were early signs indicating that the nature of the transition is different from that of the $d = 2$ Ising model (24), (26). This can be understood by studying simplified models describing essentially the same physical situation.

The roughening temperature T_R is rather low and the bulk magnetization is practically the same at $T = T_R$ as at $T = 0$. It is expected that the suppression of these bulk excitations will not alter the physical picture. This assumption leads to a simplified model, to the SOS ("solid on solid") model. (In this approximation certain other configurations, the so-called overhang configurations are also suppressed, see Fig. 2.) In this model a configuration is characterized by giving the heights of the surface points measured from the $T = 0$ interface plane. The interaction energy is given by

$$H^{\text{SOS}} = 2J \sum_{i,j} \left(|h_{i,j} - h_{i+1,j}| + |h_{i,j} - h_{i,j+1}| \right), \quad (3)$$

where (i,j) are the points of a two dimensional lattice, $h_{i,j} = 0, \pm 1, \pm 2, \dots$ and J is the coupling between neighbouring spins in the original $d = 3$ Ising model:

$$H^{\text{Ising}} = -J \sum_{\text{links}} S_{\underline{n}} S_{\underline{n}'}, \quad \underline{n} \text{ --- } \underline{n}', \quad S_{\underline{n}} = \pm 1. \quad (4)$$

Apart from an irrelevant constant, the free energy of the SOS model is just the surface tension in the given approximation.

It has been shown by Chui and Weeks²⁷⁾ that a similar model, the so-called DG (Discrete Gaussian) model, which is defined by the Hamiltonian

$$H^{DG} = 2J \sum_{i,j} \left((h_{i,j} - h_{i+1,j})^2 + (h_{i,j} - h_{i,j+1})^2 \right) \quad (5)$$

is related to the model of the two dimensional Coulomb gas. This observation has been extended by Knops²⁸⁾ showing that both models can be mapped to a generalized $d = 2$, XY model, where next to the usual interaction term $\sim \cos(\phi_i - \phi_j)$ also appear higher harmonics. In this language the roughening transition corresponds to the Kosterlitz-Thouless topological phase transition in the XY model³⁶⁾. At the phase transition point the free energy - and therefore the surface tension - is expected to have an essential singularity, $\sigma \sim \exp(-(c/(T_R - T)^{\frac{1}{2}})) + \text{regular}$ ³⁷⁾.

We note that there is a large number of Monte Carlo simulations supporting the theoretical findings related to the roughening transition³¹⁾⁻³⁵⁾. They confirm also that the different simplified models (SOS,DG) describe essentially the same physics - even quantitatively as the original three dimensional formulation.

2. THE STRING TENSION IN THE Z_2 GAUGE MODEL

The reformulation of the roughening transition in terms of the Z_2 gauge model suggests a trivial generalization, therefore it is worth to repeat a few points in this language.

The partition function of the Z_2 gauge model is given by

$$Z = \sum_{\{A_\ell = \pm 1\}} e^{\beta^g \sum_P F_P} \quad (6)$$

where

$$F_P = A_{\ell_1} A_{\ell_2} A_{\ell_3} A_{\ell_4}, \quad P: \begin{array}{c} \ell_3 \\ \square \\ \ell_1 \end{array} \begin{array}{c} \ell_4 \\ \ell_2 \end{array} \quad (7)$$

Consider the expectation value of a planar Wilson loop of size $N \times N$, $N \rightarrow \infty$:

$$\begin{aligned} \langle W(C) \rangle &= \frac{1}{Z} \sum_{\{A_\ell = \pm 1\}} e^{\beta^g \sum_P F_P} \left(\prod_{\ell \in C} A_\ell \right) \\ &\sim \frac{1}{Z} \sum_{\{A_\ell = \pm 1\}} \prod_P (1 + \text{th} \beta^g F_P) \left(\prod_{\ell \in C} A_\ell \right). \end{aligned} \quad (8)$$

Let us expand the product $\prod_P (1 + \text{th} \beta^g F_P)$ and, after averaging, order the non-vanishing terms in terms of increasing powers of $\text{th} \beta^g$ (strong coupling expansion). The lowest order contribution is $\sim (\text{th} \beta^g)^{N^2}$, corresponding to choosing "1" along every plaquette except along the plaquettes of the minimal surface covering the loop C where " $\text{th} \beta^g F_P$ " is chosen. In the higher order contributions this minimal surface is deformed and disconnected closed surfaces also occur (vacuum fluctuations). Let us neglect the vacuum fluctuations and discard those covering surfaces also which contain overhangs (Fig. 2). Then the covering surface can be characterized by giving the heights of the columns: $h_{ij} = 0, \pm 1, \pm 2, \dots$, $i, j = 1, 2, \dots, N$, $N \rightarrow \infty$. The contribution from this configuration is

$$(\text{th} \beta^g) \left\{ \sum_{ij} (|h_{ij} - h_{i+1,j}| + |h_{ij} - h_{i,j+1}|) + N^2 \right\} \quad (9)$$

Therefore in the given approximation we obtain

$$\langle W(C) \rangle = \sum_{\{h_{ij}\}} e^{(\ln \text{th} \beta^g) \left\{ \sum_{ij} (|h_{ij} - h_{i+1,j}| + |h_{ij} - h_{i,j+1}|) + N^2 \right\}} \quad (10)$$

The right-hand side is just the partition function of the SOS model defined before. The relation between the coupling constants is given by $(\beta^I = (1/kT)J)$:

$$-\ln \tanh \beta^g = 2 \beta^I. \quad (11)$$

By remembering that the free energy of the SOS model is the interfacial tension σ , we obtain

$$T(\beta^g) = \beta^I \sigma(\beta^I) \Big|_{\beta^I = -\frac{1}{2} \ln \tanh \beta^g}. \quad (12)$$

This relation between the string tension T and surface tension σ is independent of the SOS approximation^{23),18)}. The roughening transition corresponds to the complete delocalization of the covering surface, or equivalently, the confining string between the "quarks" has large fluctuations. These are long-wavelength fluctuations, the surface "moves" rather coherently. According to Monte Carlo studies the average height difference between neighbouring columns is small (~ 0.25) even at the roughening point³³⁾.

As we discussed before, the string tension has an essential singularity at the roughening point. In itself, it does not necessarily imply a conspicuous behaviour at the transition point. There are several evidences, however, suggesting the type of behaviour sketched in Fig. 3, implying an inflection point in the internal energy and a maximum in the specific heat of the SOS or DG model. These evidences include Monte Carlo data³²⁾⁻³⁵⁾, different approximate solutions^{26),42)} and an exact solution for a simplified SOS model⁴³⁾. Therefore we expect a sharp change in the logarithmic derivative of the string tension around the roughening point.

Poles of the Padé approximants have a tendency to cluster about the essential singularity of the function³⁸⁾. An isolated essential singularity would not necessarily prevent the Padé approximant to converge beyond the singular point. In our case, however, we expect a much worse analytic behaviour³⁷⁾, and it is really questionable, whether information beyond this point can be obtained from a power series expansion.

The $d = 3$, Z_2 string tension has been calculated by Kimura²²⁾ [see also Ref. 40)] up to $\sim (\tanh \beta^g)^{14}$. The corresponding Padé table is shown in Table 1. The singularity at $\tanh \beta^{*g} \approx 0.51$ corresponds to a transition in the Ising model at $\beta^{*I} \approx -\frac{1}{2} \ln 0.51 = 0.3366$. The second order phase transition point is $\beta_c^I \approx 0.2217$, therefore

$$1/\beta^* I \approx 0.64 \cdot 1/\beta_c I \quad (13)$$

which should be compared with the strong coupling results on the roughening transition $T_R \approx 0.57 T_c$ ²⁴⁾. Equation (13) and the completely bad value of the critical index (it is off by a factor of 5 !) clearly indicate that this singularity has nothing to do with the second order phase transition of the bulk phase.

3. HAMILTON FORMULATION

The Z_2 gauge theory is described by the Hamiltonian in Eq. (1), while for the dual Ising model we have:

$$H^I (d=2+1) = -\frac{1}{2} \sum_{\text{links}} (\epsilon_3 \epsilon_3 - 1) - \lambda \sum_{\text{sites}} \epsilon_1 \quad (14)$$

Perturbative expansions for the magnetization and mass gap¹⁵⁾, and exact diagonalization on a finite lattice¹⁹⁾ predict $\lambda_c \approx 1.56$ for the second order phase transition. This value is consistent with the results of approximate renormalization group transformations¹⁶⁾⁻¹⁸⁾.

The domain wall of the Lagrange formulation corresponds to the state in Fig. 4 at $\lambda = 0$. For $\lambda \neq 0$ this state evolves into a superposition of states with a fluctuating dividing line. In the SOS approximation a state is characterized by a one dimensional sequence of integers. It is easy to see that the corresponding Hamiltonian has the form

$$H^{\text{SOS}} = \sum_i |L_i - L_{i-1}| - \lambda \sum_i (U_i + U_i^+) \quad (15)$$

where $U_i = e^{i\Phi_i}$ and Φ_i, L_i are angle-angular momentum like conjugate variables. The height of the i^{th} column is measured by L_i , and it is increased (decreased) by one unit under the action of U_i (U_i^+). For a system of infinite size this Hamiltonian can be rewritten as

$$H^{SOS} = \sum_i |l_i| - \lambda \sum_i (u_i u_{i+1}^+ + c.c.) , \quad (16)$$

where $l_i = L_i - L_{i-1}$ and $u_i = e^{i\phi_i}$. l_i, ϕ_i are conjugate variables.

The Discrete Gaussian model corresponds to the Hamiltonian:

$$H^{DG} = \sum_i l_i^2 - \lambda \sum_i (u_i u_{i+1}^+ + c.c.) , \quad (17)$$

which is identical to the Hamiltonian of the XY model⁴⁴⁾. We see that the relation between the roughening transition and the XY model is almost trivial in the Hamilton formulation.

Power expansion and other methods predict $\lambda_c^{XY} \approx 0.85 - 0.90$ ^{44), 19)} for the Kosterlitz-Thouless phase transition point. We expect the transition point of the SOS model to be somewhat smaller. This critical point is far from the second order phase transition point of the $d = 2 + 1$ Ising model, therefore we expect the SOS model to be a very good description of the roughening transition:

$$\lambda_{roughening}^{d=2+1} \approx \lambda_c^{SOS} \lesssim \lambda_c^{DG} \equiv \lambda_c^{XY} \approx 0.85 - 0.90 . \quad (18)$$

Let us analyze now the string tension. Using ordinary Rayleigh-Schrödinger type perturbation theory we have calculated the tension up to $\sim \lambda^{10}$ ²¹⁾. The computer gave the results:

$$T = 1 - 0.5 \lambda^2 - 0.2395833 \lambda^4 - 0.1752025 \lambda^6 - 0.143891095 \lambda^8 - 0.058303339 \lambda^{10} \dots \quad (19)$$

The calculation is based on the observation that there is a lattice of minimum size on which an exact tenth order result can be obtained (10×6 in our case). By increasing this lattice the coefficients must not be changed, giving a stringent test on our results. The logarithmic Padé table is shown in Table 2, consistently signaling a singularity around $\lambda^* \approx 0.85$. It is clear again that this singularity is not related to the second order phase transition of the bulk phase. This

can be seen even more explicitly by comparing the series in Eq. (19) to that of the magnetization in the Ising model derived by Pfeuty and Elliott¹⁵⁾:

$$\langle \epsilon'_z \rangle = 1 - 0.5 \left(\frac{\lambda}{2} \right)^2 - 0.255 \left(\frac{\lambda}{2} \right)^4 - 0.1672 \left(\frac{\lambda}{2} \right)^6 \dots \quad (20)$$

The coefficients are almost the same, but the expansion variable is $\lambda/2$ here. Therefore, the predicted Curie point is almost twice as large as λ^* above.

4. GENERALIZATION OF THE ROUGHENING TRANSITION TO OTHER GAUGE GROUPS

The reformulation of the roughening transition in terms of the Z_2 gauge model as strong long-wavelength string fluctuations (or long-wavelength surface ripples of the surface covering the Wilson loop) offers an immediate generalization to other gauge groups.

Consider first the case of Lagrange formulation. The vacuum functional is given by

$$Z = \prod_l \int dU_l \prod_p e^{\beta^g \chi(U_p)} \quad , \quad (21)$$

where U_p is the product of fields U_l around the plaquette p and χ is the real part of the trace in the fundamental representation. The generalization of the expansion in Eq. (8) is the character expansion of $e^{\beta^g \chi(U_p)}$ (12), 45):

$$e^{\beta^g \chi(U_p)} = \sum_r v_r \beta_r(\beta^g) \cdot \chi_r(U_p) \quad . \quad (22)$$

The sum is over the irreducible representations of the group, v_r is the dimension of the representation r , $\chi_r(U_p)$ is the trace of U_p in this representation.

Consider now the expectation value of an $N \times N$ Wilson loop ($N \rightarrow \infty$) lying in the $Z = 0$ plane. A plaquette will be called "marked" in a configuration if a non-trivial term of the character expansion is assigned to it (i.e., if the plaquette is excited). Let us denote by $\rho_{\parallel}(z)$ the probability of a plaquette lying in the $Z = z$ plane being marked. $\rho_{\perp}(z + \frac{1}{2})$ is the analogous probability for plaquettes orthogonal to the $Z = 0$ plane and having a centre with $Z = z + \frac{1}{2}$.

For $\beta^E = 0$, $\rho_{\perp}(z + \frac{1}{2}) = 0$ and $\rho_{\parallel}(z) = 0$ ($z \neq 0$). The surface is sharp. Surface and vacuum fluctuations imply a non-trivial function ρ . The irrelevant contribution independent of the presence of the Wilson loop is cancelled in the combination:

$$\begin{aligned} \rho_{\perp}(z - \frac{1}{2}) - \rho_{\perp}(z + \frac{1}{2}) , \\ \rho_{\parallel}(z-1) - \rho_{\parallel}(z) \end{aligned}$$

The moments of these quantities are appropriate to describe the surface roughening. Similar moments have been investigated in Ref. 24) for the Ising model interface.

We have derived a strong coupling expansion for the quantity

$$\langle z^2 \rangle = \sum_{z=1}^{\infty} \left(\rho_{\perp}(z - \frac{1}{2}) - \rho_{\perp}(z + \frac{1}{2}) \right) z^2 + \sum_{z=1}^{\infty} \left(\rho_{\parallel}(z) - \rho_{\parallel}(z+1) \right) z^2 \quad (23)$$

up to $\sim \beta^{12}$ in the $d = 3$, $U(1)$ gauge model. Unfortunately, it is a short series, since it starts with $\sim \beta^4$. We present our result in terms of $x = I_1(\beta)/I_0(\beta)$, where I_n is the n^{th} modified Bessel function:

$$\begin{aligned} \langle z^2 \rangle = & 3 x^4 + 5.5 x^6 + 43.5 x^8 + 32.90625 x^{10} + \\ & + 716.3531251 x^{12} , \end{aligned} \quad (24)$$

This series is similar in character to that obtained by J.D. Weeks et al., in the Ising model²⁴⁾. In their case the $[1,2]$ and $[2,2]$ Padé approximants (or the $[1,2]$ logarithmic Padé) predicted a roughening point which was essentially unchanged by the higher Padé's. In our case these are the highest approximants we can obtain from Eq. (24), giving

$$\beta_R \approx 1.14 , \quad d = 3 , \quad U(1) . \quad (25)$$

As we have seen, in the Z_2 gauge model the roughening transition occurs deep in the confining region where vacuum fluctuations play a very restricted role. This suggests an approximation where only the fluctuations of the original string are considered, while vacuum fluctuations and string bits with higher fluxes are neglected. In this approximation the Wilson loop expectation value is a sum over all possible non-intersecting covering surfaces, while in the

Hamilton formulation the state is a superposition of states with a fluctuating string. Apart from the (unimportant) overhang configurations this approximation defines an effective SOS model [Eq. (10) and Eq. (15)]. In Eq. (10) $\text{th}\beta$ is replaced by $\beta_1(\beta)/\beta_0(\beta)$, where β_0 and β_1 are the first two coefficients in the character expansion Eq. (22). Similarly, using the Hamilton formulation, λ is replaced by $\lambda \cdot c_2/c_1$ in Eq. (15), where c_1 is the value of the quadratic Casimir operator $E^a E^a$ in the fundamental representation $\{q\}$ [$c_1 = 1$ and $(N^2 - 1)/2N$ for $U(1)$ and $SU(N)$ respectively], while c_2 is the coefficient of the trivial representation in $\{q\} \otimes \{\bar{q}\}$ ($c_2 = 1$ and $1/N$ for $U(1)$ and $SU(N)$, $N > 2$ respectively).

In the SOS approximation of the Z_2 gauge model the roughening point was given by $\text{th}\beta \approx 0.45$. From this value we obtain immediately ($d = 3$):

$$\begin{aligned}
 Z_2 & \quad \text{th } \beta_R \approx 0.45 \quad \rightarrow \quad \beta_R \approx 0.48, \\
 Z_3 & \quad \frac{e^{\beta_R} - e^{-\frac{1}{2}\beta_R}}{e^{\beta_R} + 2e^{-\frac{1}{2}\beta_R}} \approx 0.45 \quad \rightarrow \quad \beta_R \approx 0.83, \\
 U(1) & \quad \frac{I_1(\beta_R)}{I_0(\beta_R)} \approx 0.45 \quad \rightarrow \quad \beta_R \approx 1.01, \\
 SU(2) & \quad \frac{I_2(2\beta_R)}{I_1(2\beta_R)} \approx 0.45 \quad \rightarrow \quad \beta_R \approx 1.05.
 \end{aligned} \tag{26}$$

In the Hamilton formulation $\lambda_R \approx 0.85$ was found for Z_2 . Therefore we predict ($d = 2 + 1$):

$$U(1) \quad \lambda_R \approx 0.85,$$

and

group	$SU(2)$	$SU(3)$	$SU(N)$	$SU(\infty)$	(27)
$(q^2 N)_R$	3.5	2.3	$\frac{2N}{\sqrt{N^2-1}}$ $\frac{1}{\sqrt{0.85}}$	2.17	

where we have used the relation $\lambda = 2/g^4$.

5. THE STRING TENSION

As we discussed before, the string tension is expected to feel the roughening transition. In this section we collect and analyze the available expansions for the string tension in $d = 3$ ($d = 2 + 1$) gauge theories other than Z_2 . As we shall see, in every case there is an unexpected singularity in the logarithmic Padé approximants, which, however, can be understood in terms of the roughening transition.

$$\underline{d = 2 + 1, U(1)}$$

The method was the same as that of used for Z_2 . We obtained

$$\begin{aligned} T = & 1 - 0.333... \lambda^2 - 0.203191 \lambda^4 - 0.082701 \lambda^6 - \\ & - 0.043232 \lambda^8 + 0.055478 \lambda^{10} \dots \end{aligned} \quad (28)$$

The logarithmic Padé table is shown in Table 3. There is a stable singularity at

$$\lambda^* \approx 1.02, \quad (29)$$

which can be compared to the approximate roughening point obtained in Section 4 [Eq. (27)].

$$\underline{d = 3, U(1)}$$

An expansion up to $\sim \beta^{14}$ can be found in Ref. 39)*). The logarithmic Padé approximants signal a singularity at

$$\beta^* \approx 1.17, \quad (30)$$

which can be compared to Eqs (25) and (26).

$$\underline{d = 3, SU(2)}$$

This series has been derived also in Ref. 39) [see also Ref. 40)]. The Padé table is rather unstable with singularities around

$$\beta^* \approx 1.0 - 1.5, \quad (31)$$

which is to be compared to Eq. (26).

*) There are a few misprints in Eq. (5.2) and (5.9) of Ref. 39). We are indebted to C. Lang and A. Duncan for a discussion on this point.

6. A FEW REMARKS ON THE ROUGHENING TRANSITION IN $d = 4$

In four dimensions the surface over the Wilson loop (or the string in the Hamilton formulation) fluctuates in two transversal directions. It has been shown by Drouffe, Parisi and Sourlas⁴¹⁾ that for $d \rightarrow \infty$ chains of three dimensional cubes in tree-like configurations are dominant. The picture of trees growing randomly in the different transversal directions is just opposite to that offered by the roughening transition in three dimensions, where long-wavelength fluctuations dominate. A strong coupling expansion for the surface width would be able to distinguish between these possibilities and could also provide quantitative information on the transition point.

In the approximation where only the fluctuations of the original string are considered we obtain an effective Z_2 problem, as it was discussed in Section 4. In this approximation Eq. (26) [Eq. (27)] would be essentially unchanged, except that 0.45 [0.85] would be replaced by another number. In $d = 4$ there is a first order transition in the Z_2 gauge model at $\text{th}\beta_c = 0.414$ ⁴⁶⁾. A priori, the Padé approximant is expected to continue the function smoothly beyond this point into the metastable region. On the other hand the logarithmic Padé table in Ref. 22) signals a singularity at $\text{th}\beta^* \approx 0.396$. If this singularity is related to a roughening transition then the roughening point in $d = 4$ would be quite close to that in $d = 3$ ($\text{th}\beta_R^{d=3} \approx 0.45$).

It is an open and exciting question whether the "break" in the string tension observed in $d = 4$ in Monte Carlo simulation^{1),3)} and strong coupling expansions⁴⁾⁻⁶⁾ can be understood in terms of the roughening transition.

7. SUMMARY

The rather well understood roughening transition in the $d = 3$ Ising model implies an essential singularity in the string tension of the dual Z_2 gauge model. This singularity is deep in the confining region and prevents a simple-minded continuation of the strong coupling expansion for the tension beyond this point.

The reformulation of the roughening transition in terms of the Z_2 gauge model offers a simple generalization to other gauge groups. An analysis of the available information on the string tension in different gauge theories indicates that the string tension feels a roughening singularity.

Clearly, more work is needed to clarify and confirm this picture. But it is difficult to imagine a mechanism by which the roughening transition would be restricted to the Z_2 gauge theory in three dimensions. The more so, as the roughening occurs at a rather strong coupling where there is not much difference between different gauge groups.

After the completion of this work we have been informed^{*)} that C. Itzykson, M.E. Peskin and J.B. Zuber also observed the relevance of the roughening transition on the string tension, and that their results partly overlap with the results presented here.

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Table 1

The logarithmic Padé approximants for the string tension in $d = 3$, Z_2 gauge theory, Lagrange formulation²²⁾.

$[N,M]$	$[2,1]$	$[2,2]$	$[2,3]$	$[3,1]$	$[3,2]$
$th\beta^*g$	0.511	0.508	0.520	0.509	0.510

Table 2

The logarithmic Padé approximants for the string tension in $d = 2 + 1$, Z_2 gauge theory, Hamilton formulation.

$[M,N]$	$[1,1]$	$[1,2]$	$[2,1]$	$[2,2]$
λ^*	0.85	0.87 ^{*)}	0.87	0.85

^{*)} with a small imaginary part

Table 3

The logarithmic Padé approximants for the string tension in $d = 2 + 1$, $U(1)$ gauge theory, Hamilton formulation.

$[N,M]$	$[1,1]$	$[1,2]$	$[2,1]$	$[2,2]$
λ^*	1.02	1.02	1.02	1.02

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FIGURE CAPTIONS

Fig. 1 : Three dimensional Ising spin system at $T = 0$. For $z > 0$ ($z < 0$) all the spins are pointing upwards (downwards).

Fig. 2 : An overlap configuration which is neglected in the SOS approximation.

Fig. 3 : The behaviour of the string tension in the $d = 3, Z_2$ gauge model around the roughening point.

Fig. 4 : The ground state with a domain wall at $\lambda = 0$ in the $d = 2 + 1$ transverse Ising model. Here $\sigma_3 | + \rangle = | + \rangle$ and $\sigma_3 | - \rangle = - | - \rangle$.

FIG. 1

FIG. 2

FIG. 3

FIG. 4