Ref.TH.2853-CERN

DELVING INTO QCD JETS

K. Konishi
CERN - Geneva

A B S T R A C T

We discuss, in an introductory fashion, the latest developments in the study of hadronic jets produced in hard processes, based on perturbative QCD. Emphasis is on jet calculus (and its applications and generalizations), and on the appearance of a parton-like consistent, over-all picture of jet evolution in momentum, colour, and real spacetime.

Ref.TH.2853-CERN

29 April 1980

1. INTRODUCTION

The study of hadronic jets produced in deep inelastic (high-energy and large-momentum transfer) processes has recently attracted much theoretical and experimental interest^{1,2)}, as providing testing grounds for Quantum Chromodynamics (QCD).

The purpose of this paper is to discuss, in an introductory fashion, the latest developments in the study of jets based on perturbative QCD³⁾. Particular emphasis will be put on the parton picture, a simple probabilistic algorithm -- the jet calculus⁴⁾, and a resulting consistent over-all picture of jet evolution in momentum, colour, and real space-time. The fractal nature of QCD jets is also stressed, and examples of uses of this concept are illustrated.

This paper is motivated by several factors. First, the recent experimental results strongly indicate⁵⁾ that the observed hadronic final states follow the patterns expected from QCD, and experimental projects envisaged for the near future make the need for further theoretical advances strongly felt. Hadronic jets expected in QCD are fuzzy objects [they are "fractals"⁶⁾ -- see below] and their quantitative study requires us to take account of complicated parton branchings. The jet calculus is a quantitative description of this branching decay process, and provides a powerful tool for making calculations and for finding quantitative tests of QCD.

Secondly, the approach based on the lowest-order QCD calculations à la Sterman-Weinberg⁷⁾, is often plagued by certain large logarithms near the kinematical boundaries (where the cross-section tends to be large). Together with

the approach by Dokshitzer et al.^{8,9)} (although in complementary situations), the jet calculus provides a simple method of incorporating such logarithmic factors, and thus improving QCD predictions in an essential way.

Thirdly, the analysis of the evolution of jets, based on some generalization of the jet calculus, has recently led to further insight into the colour structure of final states [preconfinement¹⁰⁾, see below] as well as the evolution in real space-time¹¹⁾, which are of considerable interest on their own.

Furthermore, the presence of a simple probabilistic algorithm (the jet calculus) can be shown to persist beyond the leading log approximation, suggesting further developments along these lines.

In Section 2, we discuss the intuitive picture of QCD jets. The relevance of branching process analogy is explained, and the "fractal" nature of these jets is emphasized and illustrated. In Section 3 we briefly recall the rules of the jet calculus; different (although equivalent) formalisms are compared. The main experimental suggestions and comments are found in Sections 4 and 5, and in some parts of Sections 2 and 6. Section 6 is devoted to recent progress on understanding the colour structure of final states and the space-time evolution of jets, with particular emphasis on the appearance of a consistent over-all picture. In Section 7 we show how the jet calculus can be generalized beyond the leading log approximation, and how all uninterpretable contributions can be absorbed into parton subprocess probabilities, leaving a simple picture of the parton branching process.

Details of analyses and calculations are not presented. They are either found in the original papers or easily reconstructed. The aim of the paper is hopefully to arouse both experimental and theoretical interest in further investigations, by an introductory, but nonetheless quite self-contained presentation of the subject.

No effort is made to review innumerable infrared-safe measures¹²⁾, proposed following the suggestion of Sterman and Weinberg⁷⁾. The primary reason is that there already exist excellent reviews³⁾ and many of these measures are by now well known, and also that the idea behind them is essentially the same as in the original suggestion.

For a general updated review on the use of QCD perturbation theory, see, for example, Ref. 3. Interesting applications to processes involving real photons are found in Ref. 13. For a study on jets, see, for example, Ref. 1.

In the rest of this Introduction, we give a brief summary of the algorithm used in applying QCD perturbation theory to hard processes. The property of asymptotic freedom, i.e. that the effective expansion parameter depends on the momentum transfer, as

$$\alpha_s(Q^2) \sim 1/b \log Q^2/\Lambda^2 \quad (1)$$

[$b = (11 N_C - 2 N_f)/12 \pi$, $\Lambda \sim 0.1 \text{ GeV}$ = hadronic scale; Q is the large momentum transfer of the process], does not necessarily imply the applicability of perturbative expansion in $\alpha_s(Q^2)$ for the following reasons. First, the observed particles are hadrons, and not quarks and gluons, the quanta of QCD. It is not *a priori* clear how the results of calculations in terms of quarks and gluons are related to the observed quantities. We make a reasonable assumption that the final stage of interaction -- confinement, resonance formation, and decays (which are beyond the scope of perturbative methods) -- does not substantially modify those properties of the partonic final states which depend only on the short-distance behaviour of QCD. The factorization of mass singularities^{14,15} [see after Eq. (2)] provides an *a posteriori* consistency check for such an assumption¹⁶. The phenomenon of preconfinement¹⁰ to be discussed in Section 6 goes towards providing a sufficient condition for our treatment to be valid.

The second reason is the appearance of certain large logarithmic factors related to collinear or soft (IR) emissions of quanta. In general, any physical cross-section in a hard process is given, in QCD, by a formula

$$\sigma = (1/Q^2) f[\alpha_s(Q^2), Q^2/m^2, x] \quad (2)$$

because of the renormalizability and for a dimensional reason. In Eq. (2), m and x stand, symbolically, for the mass parameters regularizing IR and collinear divergences, and fixed dimensionless ratios of large variables, respectively. Naïve expansion in $\alpha_s(Q^2)$ is invalidated by the appearance of terms $\sim [\alpha_s(Q^2) \log Q^2/m^2]^n$, or even $\sim [\alpha_s(Q^2) \log^2 Q^2/m^2]^m$. In the deep inelastic leptonproduction (and in total e^+e^- annihilation into hadrons) this difficulty was overcome by the combined use of Wilson operator product expansion and the renormalization group.

One method to overcome the difficulty and to make use of the small expansion parameters $\alpha_s(Q^2)$, is to rearrange the perturbative series for f in Eq. (2), as

$$f = f_1 + \alpha_s(Q^2)f_2 + \alpha_s^2(Q^2)f_3 + \dots, \quad (3)$$

where $f_i = f_i[\alpha_s(Q^2) \log Q^2/m^2, x]$ and to retain the first few terms. The approximation of keeping f_1 alone is called the *leading log approximation (LLA)*¹⁵⁾. This approach, to be really useful, should be complemented by the demonstration that f_i 's do not depend on m^2 except through (unknown) normalization. (Otherwise, we would not know which Q^2 -dependence -- i.e. which m^2 -- to believe.) That such factorization of mass singularities indeed occurs in Eq. (3) has recently been demonstrated¹⁶⁾. The outcome is that the parton model holds good in perturbative QCD, with calculable logarithmic corrections to the scaling in the structure and fragmentation functions.

Another approach to make use of expansion in $\alpha_s(Q^2)$ was initiated by Sterman and Weinberg⁷⁾. They observed that, if the appropriate sum is taken over the final states (over the degenerate states -- those indistinguishable in the massless limit), the resultant cross-section is free of singularities at $m \rightarrow 0$ ¹⁷⁾ in Eq. (2), and therefore may be expanded simply in $\alpha_s(Q^2)$. The result would be a formula similar to Eq. (3), but with f_i just functions of x . This suggestion was immediately followed¹⁸⁾ by a proposal of many experimental measures which are calculable in lowest order QCD -- among them, thrust, and sphericity, and has become one of the standard methods in analysing jets.

Further problems arise when certain x approach their kinematic boundaries, where the cross-sections often display singular behaviours, and one is faced with the task of *re-summing* large $\log x$'s. Evolution of jets discussed in this paper is a typical example of such a situation, where the jet calculus compactly expresses the result of summation of terms of the form, $\alpha_s^n x$ maximum logs. Another example is the situation discussed in Refs. 8 and 9, where an effective quark form factor plays an essential role and where terms $\propto (\alpha_s \log^2 Q^2)^n$ are important. The multiplicity and the multiplicity moments, discussed later, furnish still another example of the situation which necessitates the re-summation of doubly logarithmic terms. Many other similar re-summation problems arise in hard processes, and the very fact that the cross-sections tend to be largest in these regions [small x in $d\sigma/dx$ in $e^+e^- \rightarrow h(x) + \dots$, $p_T \ll \sqrt{s}$ in large p_T jet

production in hadron collisions, $p_T \ll \sqrt{s}$ in the Drell-Yan processes, etc.] renders this task of practical importance as well as of theoretical interests.

Finally, a potentially dangerous situation occurs when the coefficients in Eq. (3) happen to be such that $f_2 \gg f_1$, as in the $O(\alpha_s)$ correction¹⁹⁾ in the Drell-Yan process (relative to the deep inelastic leptonproduction). The hope is again first to re-sum²⁰⁾, to all orders in α_s , large terms such as $[(\alpha_s/\pi) \cdot \pi^2]^n$, which would leave a better convergent series in $\alpha_s(Q^2)$. For further efforts to maximize the convergence of the series, see for example Ref. 21.

2. INTUITIVE PICTURE OF JETS

Jet Evolution as a Branching Process

In the leading log approximation of QCD, the description of hard processes is considerably simplified by working in physical gauges^{*)}, the advantage being that the effects of radiative corrections are computed by summing only ladder-type diagrams¹⁶⁾. As a consequence, each diagrammatic contribution acquires physical, parton interpretation.

The same holds for the evolution inside a jet⁴⁾. A highly virtual quark or gluon produced in a hard process, decays into secondaries which are less off-shell and less energetic. Each of these decays *independently* (in the physical gauges!): the subsequent decay chain is a branching process (Fig. 1). This process continues until the virtual mass of

Fig. 1 Jet evolution as a parton branching process

*) These are gauges in which on-shell gluons contain only transverse polarizations (e.g. axial gauges). The use of these gauges is a powerful technique in the general proof of factorization of mass singularities¹⁶⁾.

the quanta becomes of the order of Q_0 (≥ 1 GeV)²²⁾, where Q_0 is such that $\alpha_s(Q_0^2)/\pi$ is just small enough for the perturbative treatment to make sense. The evolution further downstream cannot be followed by a perturbative method, but the one up to Q_0 (fixed as $Q \rightarrow \infty$) would hopefully be sufficient to determine, apart from non-perturbative details, the structure of final states.

The idea that branching processes²³⁾ are relevant in the physics of jets is not new. General consequences such as the long-range correlation and KNO scaling, in the context of renormalizable field theories, have been pointed out by Polyakov²⁴⁾ and Rittenberg and Orfanidis²⁵⁾.

For QCD jets, however, the analogy with the stochastic branching process is of a subtle nature, owing to the presence of soft (infrared) divergences. If $\log(\log Q^2/\Lambda^2)$ is regarded as a "time" variable, the evolution of QCD jets can formally be described as a Markov branching process²⁶⁾, the Markovian nature being a result of scale invariance of QCD. (The quantitative description of this evolution -- the jet calculus -- is discussed next in Section 3.) The assertion is only correct, however, for *inclusive* distribution of *hard* (not arbitrarily wee) particles.

In describing soft aspects such as the average multiplicity and their moments, spectra at $x \rightarrow 0$, etc., the analogy is still useful^{27,28)}, but the evolution can no longer be regarded as Markovian [but is "age-dependent"²³⁾]. The reason is that the infrared singularities induce certain effective Q^2 dependence in the parton decay probability functions.

The analogy is formal also in the sense that elementary probabilities for parton subprocesses are not always less than or equal to unity (some are in fact infinite!) Only for inclusive cross-sections, where soft infrared divergences are believed to cancel^{*)}, does the evolution look as if it were a Markov process.

None of these peculiarities are present in the toy model of ϕ_6^3 , which is free of soft divergences. In this theory, the evolution of jets is a perfect Markovian process; integrated (in x) distributions obey differential equations

*) Whether the familiar Bloch-Nordsieck cancellation (which is known to work in QED) works for QCD is not yet established. See Ref. 29.

identical to those appearing in the population growth, and solvable in a closed form³⁰⁾.

With these caveats kept in mind, our analogy is very powerful. The decay of a parton a with off-shell mass q^2 , into two partons, $b(x_1)$ and $c(x_2)$, is described, in LLA, simply by the probability density³¹⁾:

$$dx_1 dx_2 (dq^2/q^2) \cdot \left\{ \alpha_s(q^2)/2\pi \right\} \hat{P}_{a \rightarrow bc}(x_1) \delta(x_1 + x_2 - 1), \quad (4)$$

where

$$\begin{aligned} \hat{P}_{Q \rightarrow QG}(x) &= \hat{P}_{Q \rightarrow GQ}(1-x) = C_F^2 (1+x^2)/(1-x), \\ \hat{P}_{G \rightarrow QQ}(x) &= \hat{P}_{G \rightarrow \bar{Q}Q}(x) = N_f \left\{ x^2 + (1-x)^2 \right\}, \end{aligned} \quad (5)$$

and

$$\hat{P}_{G \rightarrow GG}(x) = 2C_A \left\{ (1-x)/x + x/(1-x) + x(1-x) \right\}$$

(for applications, see below). The analogy can be pushed even further. First, the analysis can be generalized, with some approximation, so that the jet evolution in colour (as well as momentum) structure is studied¹⁰⁾. The formalism by Bassetto et al.²⁷⁾, using a generating functional, will be mentioned later.

Furthermore, the analogy with stochastic processes, and consequently the parton interpretation of subprocesses, is useful, even beyond the leading log approximation³²⁾, as will be briefly discussed in Section 7.

Finally, the property of the branching process can be used to construct Monte-Carlo type models³³⁾ for jet evolution and hence for event simulation (see, however, notes towards the end of Section 3).

QCD Jets as Fractals^{*)}

A better intuitive picture of QCD jets is provided by the mathematical concept of fractals, which was introduced by Mandelbrot⁶⁾ to describe a wide range of "shapeless" objects in nature. More adequately, it describes objects with an infinitely complicated shape, so that the more jig-saw

*) This was first noted by Feynman³⁴⁾ and then emphasized by Veneziano¹⁾.

structured and/or fragmented they look, the finer the resolution is. Assuming certain randomness and at the same time exact scale invariance, models for coast lines, landscape, excess noises, matter distributions in the universe, etc., have been successfully constructed. These objects are in general characterized by a non-integer Hausdorff-Besikovich (or fractal) dimension⁶⁾.

Hadron jets in QCD are fractals because their shape (defined in various possible ways) depends on the experimental resolution. This is illustrated in Fig. 2. QCD is of course a scale invariant theory, as is required.

Fig. 2 The schematic picture of a quark-jet event seen with a (a) poor, (b) reasonable, and (c) very good resolution. Shaded boxes represent symbolically calorimeters/detectors.

What is the use? Above all, it elucidates the particular aspects of QCD jets, which tend to be overlooked in approaches based on the lowest order (in α_s) Feynman graphs*). The structure of final states reflects the presence of a complicated branching decay process, and inclusion of those effects is required, even when the purely perturbative level of evolution is considered. As a consequence, use of the fractal concept helps us to find out in which direction quantitative tests of QCD can be looked for. It seems to be better to study how the profile of jets (defined in many different ways) changes with the experimental resolution, than to try to estimate the absolute value of a particular measure. The calorimeter-measured "jet energy spread" proposed in Ref. 4a (and discussed in Section 4 below) is a typical quantitative test of this type. In this example, the relevant moment varies as a function of the calorimeter opening angle δ as (i = species of the parent parton)

$$C_n^i(\delta, Q^2) \underset{\delta \ll 1}{\sim} \sum_b (\delta)^{-A_{n,i} \cdot \alpha_s(Q^2)/\pi} \sim (\delta)^{1-d_f}. \quad (6)$$

So the fractal dimension in this case coincides with the conventional anomalous dimension!

That the power of δ (= one minus the fractal dimension) has something to do with the dimension can be seen as follows. If the distribution of final particles is one-dimensional ($|p_T| = 0$), then $C(\delta, Q^2) \sim \delta^0$; if it is disk-like ($p_x = 0$), then $C(\delta, Q^2) \sim (\delta)^{-1}$; finally $C(\delta, Q^2) \sim (\delta)^{-2}$ if the distribution is spherically symmetric. Notice that the fractal dimension, $1 + A_n \alpha_s(Q^2)/\pi$, is less than unity (!) because of the particular definition of our measure and because of the property of off-shell QCD quanta that, when energetic, tend to spread in angle in decay. Notice also that as $Q^2 \rightarrow \infty$, this fractal dimension tends to unity, a result consistent with the Sterman-Weinberg two-jet formula.

As another simple example illustrating the fractal nature of QCD jets, let us consider the average number of *mini-jets* in a jet. Take an e^+e^- annihilation process for definiteness and cover the hemisphere corresponding to one quark jet by small

*) For instance, in the e^+e^- annihilation process, it is sometimes incorrectly stated that only one hemisphere must have $\langle p_T^2 \rangle_{\text{jet}}$ growing as $\sim Q^2$. Of course, the narrower jet is also expected, in QCD, to have $\langle p_T^2 \rangle \sim Q^2$, although with a smaller coefficient in front of Q^2 . There seems to be some experimental support for this³⁵).

calorimeters, each of half opening angle δ . Call a calorimeter a *mini-jet* when it is fired by hadronic energy E_i such that $x_i \equiv E_i/\Sigma_j E_j$ is larger than some fixed value x_0 ($x_0 \ll 1$). The average multiplicity of these mini-jets is a function of the angular resolution, δ , and is given simply as the integral of one-particle inclusive cross-section in the range $x \geq x_0$. The prediction is³⁶⁾ (for $x_0 \ll 1$),

$$\langle n \rangle_{q\text{-jet}} = \frac{C_F}{C_A} \left\{ \langle n_G \rangle_G + \langle n_q \rangle_G \right\} + \langle n_{q(\text{val})} \rangle, \quad (7)$$

$$\langle n \rangle_{G\text{-jet}} = \langle n_G \rangle_G + \langle n_q \rangle_G,$$

where

$$\begin{aligned} \langle n_G \rangle_{G\text{-jet}} &\approx e^{-a\zeta} I_0 \left(2\sqrt{2C_A\zeta} \log 1/x_0 \right), \\ \langle n_q \rangle_{G\text{-jet}} &\approx \left(\frac{N_f}{3C_A} \right) \left(\frac{2C_A\zeta}{\log 1/x_0} \right)^{1/2} e^{-a\zeta} I_1 \left(2\sqrt{2C_A\zeta} \log 1/x_0 \right), \end{aligned} \quad (8)$$

and

$$\langle n_{q(\text{val})} \rangle \approx 1 - x_0 \left(\frac{C_F\zeta}{\log 1/x_0} \right)^{1/2} I_1 \left(2\sqrt{C_F\zeta} \log 1/x_0 \right).$$

In the above, $\zeta \equiv (1/2\pi b) \log (\log Q/\Lambda / \log \delta Q/\Lambda)$, $a \equiv \equiv 11 N_C/6 + N_f/3N_C^2 \approx 5.6$, $C_A = N_C = 3$, $C_F = 4/3$ (colour Casimirs), $12\pi b \equiv 11 N_C - 2 N_f$, have been used, and $I_{0,1}$ are the modified Bessel functions.

The prediction is plotted in Fig. 3, where, at large δ , the limit $\langle n \rangle$ is seen to approach unity as it should (see Fig. 2a!). Such average multiplicity of mini-jets may be studied experimentally.

Notice that the number of mini-jets defined above is not an absolute number but is a function of the experimental angular resolution δ , *even for each e^+e^- event*. This should be compared with the concept of "number of jets" which became familiar after the work of Ref. 7, and which puts emphasis on the Q^2 dependence/or absolute magnitude of cross-sections for a given number of jets (with a rather tight definition of "n-jet events").

Fig. 3 Average multiplicity of mini-jets defined in the text.

Two mini-jet correlations, or general multiplicity distributions, can easily be studied using the distribution $d\sigma/dx_1, \dots, dx_m$ computed with the rules of Section 3, or more directly using the integrodifferential (Kolmogorov) equation satisfied by the generating functionals²⁶. Contrary to the first example [Eq. (6)], the fractal dimension associated with the number (or correlation) of mini-jets is a complicated function of Q and x_0 .

There are many other kinds of measures characterizing jets which can be studied as a function of experimental resolution, besides the above two examples. Further exploration in this direction will be useful in the near future for experiment-theory confrontation.

3. JET CALCULUS -- EVOLUTION OF JETS IN LLA

In this section we briefly review the simple rules of calculation (jet calculus)⁴ which allow one to take account of full radiative corrections in the leading log approximation (LLA) of QCD. The presence of various equivalent formalisms and the associated parton picture are emphasized.

Fig. 4
Parton evolution function
 $D_{a,i}$ computed from ladder
diagrams in axial gauges.

Description in (x,y) Variables -- Parton Picture

For the moment, let us deal with the jet evolution at the partonic level (see below for inclusion of hadrons). Single-parton spectra are given by a sum over ladder-type diagrams, and the results are (see Fig. 4)¹⁶⁾ ($i, a = q$ or g):

$$(1/\sigma)(d\sigma/dx)^{i \rightarrow a+X} \equiv D_{a,i}(x,Y), \quad (9)$$

with

$$\int_0^1 dx x^n D_{a,i}(x,Q^2) = (e^{A_n Y})_{a,i} \quad (10)$$

where

$$Y \equiv (1/2\pi b) \log \left[\alpha_s / \alpha_s(Q^2) \right]; \quad 12\pi b = 11 N_C - 2 N_f;$$

and $(A_n)_{ba}$ are the anomalous dimension matrices (Table 1).

TABLE 1 The anomalous matrices A_n for the flavour singlet component

$$A_n = \begin{pmatrix} A_n^{qq} & A_n^{qg} \\ A_n^{gq} & A_n^{gg} \end{pmatrix}$$

b	a	A_n^{ba}
q	q	$C_F \left[-\frac{1}{2} + 1/\{(n+1)(n+2)\} - 2 \sum_{j=2}^{n+1} (1/j) \right]$
g	q	$C_F (n^2 + 3n + 4) / \{n(n+1)(n+2)\}$
q	g	$N_f (n^2 + 3n + 4) / \{(n+1)(n+2)(n+3)\}$
g	g	$C_A \left[-\frac{1}{6} + 2/\{n(n+1)\} + 2/\{(n+2)(n+3)\} - 2 \sum_{j=2}^{n+1} (1/j) \right] - \frac{1}{3} N_f$

$C_F = (N_c^2 - 1)/2N_c, \quad C_A = N_c.$

Fig. 5 Two parton spectra for $i \rightarrow a_1 + a_2 + X$ and the triple ladder diagrams contributing to it in LLA. The right-hand side represents the expression in Eq. (11).

Two-parton distribution can be computed from the triple ladder graph of Fig. 5 (it is easy to fix two of the final partons of Fig. 1 and rearrange this figure in the form of Fig. 5), and is given by⁴⁾

$$\begin{aligned}
 (1/\sigma)(d\sigma/dx_1 dx_2)^{i \rightarrow a_1 a_2 X} &\equiv D_{a_1 a_2, i}(x_1, x_2; Y) = \\
 &= \sum_{a_1 a_2, j} \int_0^Y dy \int dx dz dw_1 dw_2 D_{a_1 b_1}(w_1, y) D_{a_2, b_2}(w_2, y) \times \\
 &\times \hat{P}_{j \rightarrow b_1 b_2}(z) \times D_{j, i}(x, Y-y) \delta(x_1 - xzw_1) \delta[x_2 - x(1-z)w_2] .
 \end{aligned}
 \tag{11}$$

This expression has a simple probability interpretation: $\hat{P}_{j \rightarrow b_1 b_2}(z)$ [Eq. (5)] is an elementary decay probability for $j \rightarrow b_1 b_2$; $D_{b, a}(x, Y)$ is the inclusive probability of finding a parton $b(x)$ inside a parton a of off-shell mass integrated up to Q . The intermediate variable $y \propto \log \log$ (virtual mass of parton j) is determined by the relative transverse momenta of partons a_1 and a_2 . This connection allows us to compute the cross-section for $i \rightarrow a_1 + a_2 + X$, differential with respect to $(p_T)_{a_1 a_2}$ as well as to x_1, x_2 , as will be discussed in Section 4 below.

The generalization to n parton spectra is immediate: One has the elementary decay probabilities $\hat{P}$ and the evolution

function $D_{b,a}(x, y_1 - y_2)$ between "time" y_2 and y_1 . One must just sum over all possible subprocesses: summing over all different tree graphs (as in Fig. 5b; for $n \geq 3$ there is more than one graph), over internal time variables and parton indices, with the conservation of longitudinal momenta guaranteed by delta functions as in Eq. (11).

Hadronic distributions can be constructed from parton spectra in a simple way^{4a)}. As long as the invariant mass among observed hadrons is large ($\gg \Lambda$), the two (or more) hadron distributions are absolutely predicted (magnitude as well as the Q^2 dependence) in terms of the single-hadron distribution (fixed at some reference Q). Otherwise, a set of non-perturbative fragmentation functions $a \rightarrow h_1 + h_2 + \dots + X$ should be introduced.

Analogy with the Markov Branching Process -- Kolmogorov Equations

The formal analogy with the Markov process allows the following rederivation of the jet calculus rules²⁶⁾. Suppressing parton indices and summation over them for simplicity, the parton decay probabilities at an infinitesimal time interval Δy is given by (u = total initial energy)

$\Delta y \hat{P}(x/u) : \text{decay} ;$

$\delta(x-u) \left[1 - \Delta y \frac{1}{2} \int dz \hat{P}(z) \right] : \text{growth (survival)} .$

(Notice that the total probability is unity.) The generating functional $\Psi(\phi, \Delta y)$ is easily constructed, with $\phi(x)$ the arbitrary sink function. It is the property of the branching process that the evolution is iterative, that is,

$$\Psi(\phi, y + \Delta y) = \Psi[\Psi(\phi, y), \Delta y] , \quad (12)$$

where the right-hand side is known. It is then easy to get the Kolmogorov equation²³⁾ for the generating function at arbitrary time Y ²⁶⁾,

$$\begin{aligned} (\partial/\partial Y)\Psi(u, \phi, Y) = \frac{1}{2} \int dx \hat{P}(x) [& \Psi(xu, \phi, Y)\Psi(u(1-x), \phi, Y) - \\ & - \Psi(u, \phi, Y)] . \end{aligned} \quad (13)$$

Taking functional derivatives with respect to $\phi(x)$ and setting $\phi(x) = 1$, $u = 1$, and using

$$\left\{ \delta^n / \delta \phi(x_1 u) \dots \delta \phi(x_n u) \right\} \Psi \Big|_{\phi=1} = (1/u^n) D(x_1 \dots x_n; Y) \quad (14)$$

one finds a set of integrodifferential equations whose formal solution is just that given by the jet calculus rules. The case $n = 1$ reduces to the equation for $D_{ba}(x, Y)$ given by Lipatov³⁷⁾ and Altarelli-Parisi³¹⁾. Notice that the above derivation automatically gives regularized kernels [due to the second term of Eq. (13)],

$$[P_{b,a}(x)]_+ = \hat{P}_{a \rightarrow bc}(x) - \frac{1}{2} \delta_{ab} \delta(x-1) \sum_{b',c'} \int dz \hat{P}_{a \rightarrow b',c'}(z) \quad (15)$$

describing the evolution function D .

Some practical use of these differential equations is suggested in Ref. 26. A rederivation of the jet calculus in the above form was also given by Kirschner²⁶⁾ following the method of Lipatov³⁷⁾.

An equation similar to Eq. (12), but applied to the case of exclusive and/or integrated distributions, has been exploited in Refs. 27 and 28, where the branching process can no longer be treated as Markovian (see Sections 5 and 6 for related results).

Formalism in the Moments -- Old-Fashioned Perturbation Theory^{4c)}

By taking moments, $D_{\{n_i\}}(Y) \equiv \int \pi dx_i x_i^{n_i} D(x_i, Y)$, of the spectra given above, and then going to the E ("energy") variable by a Laplace transform

$$D_{\{n_i\}}(E) \equiv \int_0^\infty dY e^{-EY} D_{\{n_i\}}(Y) ,$$

the formalism simplifies further. The p -parton intermediate state has a propagator

$$\left(E - \sum_{i=1}^p A_{n_i} \right)^{-1} , \quad (16)$$

and vertex $\hat{P}(x)$ is replaced by

$$\hat{P}_{n_1, n_2}^{j \rightarrow b_1 b_2} \equiv \int_0^1 dx x^{n_1} (1-x)^{n_2} \hat{P}_{j \rightarrow b_1 b_2}(x) . \quad (17)$$

Conservation of x variables is replaced by that of the moment indices at each vertex. Otherwise, all the rules remain as before.

This formalism is particularly suited⁴⁾ for analytic calculation in various limits, and also for formal proof of inclusive sum rules.

Jet Field Theory

Still another, equivalent formalism is to write the jet calculus rules in the form of a two-dimensional, non-local, field theory. Its Lagrangian is given by ($\xi \equiv \log x$)

$$\mathcal{L}_{\text{kin}} = \psi_{\alpha}^{\dagger}(\xi', y) \left\{ \delta_{\alpha\beta} \overleftrightarrow{\partial}_y \delta(\xi' - \xi) - P_{\alpha, \beta}(\xi' - \xi) \right\} \psi_{\beta}(\xi, y) \quad (18)$$

and

$$\begin{aligned} \mathcal{L}_{\text{int}} = & \psi_{\beta}^{\dagger}(\xi_2, y) \psi_{\gamma}^{\dagger}(\xi_3, y) \hat{P}_{\alpha \rightarrow \beta \gamma}(\xi_2 - \xi_1) \psi_{\alpha}(\xi_1, y) \times \\ & \times \delta(e^{\xi_1} - e^{\xi_2} - e^{\xi_3}) , \end{aligned} \quad (19)$$

and $\psi, \psi^{\dagger}$ are destruction and creation operators of partons. It is easily checked that the two point function $D_{ba}(x, Y) = \langle b | \psi^{\dagger} \psi | a \rangle$ coincide with Eq. (18) and n-point functions with those given before.

In discussing "initial parton jets" in Section 4, it will become clear that the Green function of the type

$$G[x_1 = 1, y_1 \approx 0 ; (x_2, y_2) , (x_3, y_3 = Y)]$$

has also direct physical application in deep inelastic processes, as well as $G[x_1 = 1, y_1 = Y; (x_i, y_i)]$ which is relevant to final jets, such as in e^+e^- collisions.

Although non-local, the theory is two-dimensional and should be solved only at the tree level. It would be interesting to see if further simplification is possible.

The relevance of loop corrections has recently been suggested³⁸⁾ in connection with higher (non-leading log) corrections. Although formally attractive, the consistency of the scheme (such as double counting of original Feynman graphs) is not obvious. In Section 8 we mention an alternative approach³²⁾ to higher-order corrections to the jet calculus, which by construction exhausts (without double counting) all the original QCD contributions, in a simple stochastic branching-process-type algorithm.

Notes on Inclusive Sum Rules and a Comment on Monte Carlo Simulation Models

Multiparticle distributions computed using the jet calculus have been shown⁴⁾ to satisfy all the inclusive (energy and charge) sum rules, providing a check that it is a self-contained algorithm. Probabilities are correctly normalized.

In a Monte Carlo model³³⁾ for the parton branching, however, it is necessary to include all the phase space, including the region outside the strongly ordered mass configurations (LLA gets contributions only from the latter). By continuing to use Eqs. (4) and (5), this procedure necessarily brings in the violation of inclusive sum rules, and the probability conservation must be restored by hand³³⁾. A proper treatment of parton branching beyond LLA requires the inclusion of other contributions (such as interference-type graphs), as outlined in Section 7.

It is not clear as yet how good an approximation it is to use Eqs. (4) and (5) and to impose the probability conservation by hand. The discrepancy from (or agreement with) analytic calculations would nonetheless be an indication of how good LLA could be numerically at presently available energies.

4. MEASURING QUARKS AND GLUONS

The simple formalism of the previous section should now be applied to discuss various quantitative tests of QCD.

Observation of a single hadron in e^+e^- annihilation or in deep inelastic leptonproduction, amounts to measuring the fragmentation function, Eq. (7), and thus A_n (Table 1). This is by now familiar, and the data in neutrino processes are not in disagreement with the prediction³⁹⁾. The analysis can be extended beyond the leading log approximation⁴⁰⁾, as in the case of deep inelastic structure functions.

Two-Particle Energy Correlation at Fixed Relative Angle (or p_T)

A more detailed structure of jets is revealed by observing two hadrons with energy fractions x_1 and x_2 of the jet, and their relative angle 2δ (or relative p_T). By fixing δ with $x_1, x_2 \neq 0$, one fixes the virtual mass of the parton j in Fig. 5 [and Eq. (11)], as can easily be seen by the strong

ordering of k_T of successively emitted partons, and by kinematics. The differential cross-section $(1/\sigma) \cdot (d\sigma/dx_1 dx_2 dp_T^2)$ can be read off from Eq. (11) by undoing y -integration and using the relation, $dy = (dp_T^2/p_T^2) \cdot \alpha_s(p_T^2)/2\pi$. [This procedure has recently been justified by a more careful analysis by Bassetto et al.²⁷⁾, and by Kirschner⁴¹⁾.]

A particularly convenient way of making use of such a formula is to take the first moment with respect to x_1 and x_2 , and to sum over the parton species*). Using completeness (or the energy inclusive sum rule),

$$\sum_{h_1} \int dz z d^{a_1 \rightarrow h_1}(z) = 1 ; \quad \sum_a \int dw w D_{a,b}(w,y) = 1 , \quad (20)$$

we find a simple formula⁴⁾ ($p_T^2 \approx \delta^2 Q^2 \approx Q^2 \sin^2 \delta$)

$$\begin{aligned} \sum_{h_1 h_2} \int dx_1 dx_2 x_1 x_2 (1/\sigma) (d\sigma/dx_1 dx_2 dp_T^2)^{i \rightarrow h_1 h_2 + X} = \\ = \frac{\partial}{\partial p_T^2} \sum_b \left[\frac{\alpha_s(p_T^2)}{\alpha_s(Q^2)} \right]_{b,i}^{A_2/2\pi b} = \frac{24}{p_T^2 \log 4p_T^2/\Lambda^2} \left\{ A\eta^{0.6086} + B\eta^{1.386} \right\} , \end{aligned} \quad (21)$$

where $A = 2.955$ (0.920), $B = -0.9548$ (3.680) for $i =$ quark (gluon) jet, and $\eta = \log(2p_T/\Lambda)/\log(Q/\Lambda)$. The same quantity integrated up to certain $p_T^2 = Q^2 \sin^2 \delta^2$, exhibits a simple scaling law in η , and is also of experimental interest.

For a quark jet, this formula has recently been compared with data at PETRA⁴³⁾. The data at low energies show a much wider distribution in δ ; at high energies ($Q \gtrsim 20$ GeV), they rapidly approach the theoretical curve (Fig. 6). Data at higher energies will be needed to see if the agreement keeps improving.

Gluon jets are predicted to be substantially wider than quark jets as seen in Fig. 7⁴⁾. This is in agreement with the results of Shizuya and Tye⁴⁴⁾ where the Sterman-Weinberg formula for gluon jets is compared to that for a quark. In Ref. 44 it was found that $\delta_q \approx (\delta_g) C_A/C_F$, and the integrated version of Eq. (19) agrees with this quantitatively.

*) Further detailed application has been considered elsewhere⁴²⁾.

Fig. 6 Data on two-particle (near forward) correlation and fits with Eq. (21) (taken from PLUTO Collab., Ref. 5).

Fig. 7 Comparison between quark (full curve) and gluon (dashed curve) jets, at i) $Q^2/\Lambda^2 = 200$; ii) $Q^2/\Lambda^2 = 1000$, and iii) $Q^2/\Lambda^2 = 5000$.

The use of formula Eq. (19) is restricted to the region

$$\Lambda^2/Q^2 \ll \delta^2 \ll 1.$$

The first inequality is due to the requirement $\alpha_s(\delta^2 Q^2)/\pi \ll 1$; the second is due to the leading log re-summation of $(\alpha_s \log \delta)^m$'s and to the neglect of interference terms. In the region $\delta^2 \approx 1$, the approach à la Sterman-Weinberg is

Fig. 8 Comparison between Eq. (21) (full curves) and the formula from Ref. (45) (dashed curves). The DDT formula (Ref. 8) are also shown (dotted curves).

more adequate. Computing $O[\alpha_s(Q^2)]$ terms, but with all interference contributions fully included, Basham et al.⁴⁵⁾ find the prediction shown in Fig. 8 (they also obtained the dependence on other angles).

In the region $|\delta - \pi| \ll 1$, both of the approximation schemes are inadequate, and certain double log corrections must be included. This was first done by Dokshitzer et al.⁶⁾, followed by Parisi et al.⁷⁾ (see below). It is seen from Fig. 8 (Ref. 2) that the three different predictions, good in three complementary regions, connect smoothly with each

other; it is only the approximation method that changes, the physics being one.

Jet Energy Spread -- New Measurement of QCD Anomalous Dimensions

Another quantitative way to test QCD is to use a calorimeter of opening angle 2δ , and to measure the total hadronic energy E deposited in it. Actually the full solid angle corresponding to a jet should be covered by such calorimeters, and one measures a moment ($x_j \equiv E_j / \sum_k E_k$),

$$C_n^i(\delta^2, Q^2) \equiv \left\langle \sum_k x_k^n \right\rangle. \quad (22)$$

(This is equivalent to the somewhat different form given in Ref. 4a.) This quantity is just the moment of function*) $D(x, Y-y)$ describing the evolution from $Y \approx \log \log Q^2$ to $y \approx \log \log \delta^2 Q^2$, and the prediction is simply^{4a)}

$$\begin{aligned} C_n^i(\delta^2, Q^2) &= \sum_b \left[\alpha_s(\delta^2 Q^2) / \alpha_s(Q^2) \right]_{b,i}^{A_n/2\pi b} \\ &= \alpha_n^i \eta_+^{\lambda_n^i/2\pi b} + \beta_n^i \eta_-^{\lambda_n^i/2\pi b}, \end{aligned} \quad (23)$$

with $\eta \equiv \log Q / \log \delta Q$, and $\alpha_n^i, \beta_n^i, \lambda_{\pm}^n$ are given in Table 2.

TABLE 2

$$C_n^i(\delta, Q^2) = \alpha_n^i \left(\frac{\alpha_s(\delta^2 Q^2)}{\alpha_s(Q^2)} \right)^{\lambda_+^n/2\pi b} + \beta_n^i \left(\frac{\alpha_s(\delta^2 Q^2)}{\alpha_s(Q^2)} \right)^{\lambda_-^n/2\pi b}$$

n	α_n^q	β_n^q	α_n^g	β_n^g	$\lambda_+^n/2\pi b$	$\lambda_-^n/2\pi b$
2	1.1653	-0.1653	0.3628	0.6372	-0.6085	-1.386
3	1.0937	-0.0937	0.1896	0.8104	-0.8170	-1.852
4	1.0610	-0.0610	0.1271	0.8729	-0.9604	-2.192
5	1.0440	-0.0440	0.0951	0.9049	-1.074	-2.460
6	1.0338	-0.0338	0.0756	0.9244	-1.170	-2.683
7	1.0272	-0.0272	0.0626	0.9374	-1.252	-2.875
8	1.0226	-0.0226	0.0532	0.9468	-1.325	-3.043

$N_f = 4$

*) See Ref. 4a. The evolution function $D(x, Y-y)$ has also been studied in Ref. 4b, in a somewhat restricted range of x for the purpose of improving the Sterman-Weinberg formula. Recently, Duncan et al.⁴⁷⁾ has shown how to improve the Sterman-Weinberg jet formula systematically in a renormalization group approach.

The best way to use this formula seems to be to make $\log C_n$ versus $\log C_m$ plots, as has been done in analysing the neutrino scattering data. For a quark jet (e^+e^- annihilation; virtual photon fragmentation in the leptonproduction), such a plot measures (to a good approximation) the ratio λ_+^n/λ_+^m directly, while for the gluon jets one predicts still straight lines for above-mentioned plot, with calculable effective slopes (see Fig. 9).

Fig. 9a $C_n^i(\delta)$ versus δ at $Q^2/\Lambda^2 = 2000$.

Fig. 9b Plot of $\log C_n^g$ versus $\log C_m^g$. $a_{m,n}$ is the effective slope, defined by $\log C_m^g \approx \log C_n^g + \text{const.}$

It may be said that the calorimeter exploited in this way is a good "partonometer"⁸⁾: the calorimeter *) sees directly a parton of mass $\sim \delta Q$ [assuming that the effect of the soft quanta necessary for neutralizing the colour can be neglected in the positive moments of Eq. (22)]. This is essentially analogous to the deep inelastic lepton production, where the size of the struck parton is changed by varying Q^2 : here this can be done by varying δ at fixed Q^2 , which should be easier for colliding beam experiments.

The approximation, $\log \delta \ll \log Q/\mu$ in Eq. (23), readily leads to the form mentioned in Section 2.

*) During the preparation of Ref. 4a, Mrs. Lloyd first typed this word as colorimeter, everywhere. Regretfully, we have corrected this.

Words of Caution on $\langle p_T^2 \rangle$ Inside a Jet

It is often stated that the average transverse momentum of hadrons inside a jet should increase as

$$\langle p_T^2 \rangle \sim Q^2 / \log Q^2 / \Lambda^2$$

with Q^2 . This is an *incorrect* statement in QCD if all hadrons are included in defining $\langle p_T^2 \rangle$ ^{*}). Soft ($x \approx 0$) particles spoil the simple expectation because

- i) their average multiplicity is large (see Section 5),

$$\langle n \rangle_{\text{soft}} \sim \exp C \sqrt{\log Q^2},$$

and

- ii) they yield small p_T .

Thus the $\sim Q^2 / \log Q^2 / \Lambda^2$ effects due to a hard-gluon emission are smeared out. A Monte Carlo model of Odorico³³⁾ indeed confirms this, where one finds

$$\langle p_T^2 \rangle \sim Q \text{ (not } Q^2 \text{)}$$

for a wide energy range.

Nevertheless it is correct to expect $\langle p_T^2 \rangle \sim Q^2 / \log Q^2$ if a certain selection of final states is made to suppress the soft hadrons [e.g. by restricting to hadrons with $x > x_0$ (fixed), or by weighting with energy fractions as in Eqs. (21) and (22)]. This should be kept in mind whenever we mention $\langle p_T^2 \rangle$ behaving as $Q^2 / \log Q^2$.

All this is a reflection of soft divergences of QCD and of the parton branching. Precisely because of these effects, *inclusive* quantities such as relative p_T distribution or $\langle p_T^2 \rangle$ [Eq. (21)] of two hadrons, or the jet energy spread Eq. (22), are simpler to compute than more exclusive measures such as p_T (or θ) distribution around the jet axis. The latter type of measure is a more complicated quantity to compute (its simplicity is only apparent!), and theory/experimental comparison is less direct there.

^{*}) See K. Konishi (Ref. 1) and D. Amati (Ref. 1). Very recently, related remarks have also been made in Ref. 69.

Initial Parton Jets ("Spectator" Jets)

In QCD, there exist other kinds of jets, i.e. those associated with partons which enter the hard vertex (sometimes called spectator jets). Examples are hadron jets in the direction of the nucleon in lepton productions, and forward and backward jets in large p_T hadronic collisions. These initial parton jets contain the spectator partons whose spectra are not calculable perturbatively. (However, see next subsection.) However, the latter have limited p_T -- they can be treated as free during the short time and short distance interaction -- and as far as the Q^2 -dependent effects such as the increase of $\langle p_T^2 \rangle$ are concerned, they are calculable. The origin of the initial parton jets is nothing but the bremsstrahlung which is responsible for the scaling violation of structure functions.

Because of the difference in the kinematics, the initial parton jets are expected to be more narrow (in the sense of energy angular spread) than the usual final-parton jets. As illustrated in Fig. 10, the p_T -ordering to get the large cross-section and the energy-ordering caused by the emission of quanta, result in compensating effects for the initial

Fig. 10 Kinematical difference between a) initial-parton and b) final-parton (usual) jets. Dominant ordering in p_T and E is illustrated. The line with momentum Q represents a hard momentum transfer (e.g. virtual photon).

parton jets but enhancing ones for the usual, final-parton jets. Typically, $\langle p_T^2 \rangle$ inside a former type of jet can be shown to behave as

$$\langle p_T^2 \rangle \sim (Q^2 / \log Q^2 / \Lambda^2) \cdot (\log Q^2 / \Lambda^2)^a, \quad (24)$$

where a is some negative number related to anomalous dimensions. Not all kinds of $\langle p_T^2 \rangle$ grow as $\propto Q^2 \cdot \alpha_s(Q^2) = Q^2 / \log Q^2 / \Lambda^2$ in QCD! The last factor in the above formula can be understood as an effect of scaling violation.

Fortunately the jet calculus rules are directly applicable in this new situation, by simply modifying the flow of y . The y variable now increases along the space-like parton up to $Y \propto \log \log |Q^2|$. The simplest formula is obtained in the case of lepton production where, apart from the final lepton, a hadron $h(x)$ is measured in the direction of the nucleon. By taking the zero moment of a non-singlet cross-section with respect to x_B (Bjorken variable), and the first moment with respect to x and summing over h , one finds^{4a)} at fixed p_T^2 (p_T of h with respect to the direction of the nucleon),

$$\left(\frac{d\sigma}{dp_T^2} \right)_{0,1} = \left(-A_1^{NS} \right) \frac{\alpha_s(4p_T^2)}{2\pi p_T^2} \cdot F_h^{val}(1, y_0) \left\{ \frac{\alpha_s(\mu^2)}{\alpha_s(4p_T^2)} \right\}^{A_1^{NS}/2\pi b} \quad (25)$$

($A_1^{NS} = 16/9$; F_h^{val} is the first moment of the non-singlet structure function at a reference mass μ). The last factor shows that for jets of this type, the radiative corrections make them more narrow than the naive spectra, while opposite is true for the usual, final jets. Qualitatively a similar difference between the two kinds of jets has been found, in a Stermann-Weinberg-type analysis, by Stevenson^{4b)}.

Effective Quark Form Factors and Distribution at $0 \leq p_T \leq Q^2$

Interesting situations in which an effective quark (or gluon) form factor manifests itself, have been studied first by Dokshitzer et al.⁸⁾, and followed by Parisi and others⁹⁾. Examples are p_T^2 distribution of the μ pair in the Drell-Yan process at $\Lambda^2 \ll p_T^2 \ll Q^2$, near-backward two-particle energy correlation in e^+e^- annihilation processes, semi-inclusive one-hadron lepton production, etc. In all the cases considered, the selection of particular final states forces the off-shell mass of quarks at the hard vertex (with Q^2) to be much less than Q^2 . As a consequence, the cross-sections contain an effective form factor of these quarks.

In the leading double logarithmic approximation, a particularly simple formula ("DDT" formula) exists, first obtained by Dokshitzer et al.⁸⁾ and corrected by Parisi and others⁹⁾. It reads for the Drell-Yan μ pair production,

$$\frac{d\sigma}{dQ^2 dp_T^2 dy} = \frac{4\pi\alpha^2}{9sQ^2} \frac{\partial}{\partial p_T^2} \left\{ D(x_1, p_T^2) D(x_2, p_T^2) T^2 \left(\frac{Q^2}{p_T^2} \right) \right\}, \quad (26)$$

where

$$T^2 \left(Q^2/p_T^2 \right) = \exp \left\{ - \frac{\alpha_s}{2\pi} C_F \log^2 \frac{Q^2}{p_T^2} \right\}; \quad (27)$$

the D's are the leading log evolution functions, and $x_1 x_2 = \exp 2y$. As pointed out by Dokshitzer and D'yakonov^{4,3)}, the form factor effect goes into the right direction in explaining the data at $p_T^2 \ll Q^2$. (In particular, the use of a large non-perturbative smearing employed earlier to explain the data, can be avoided.)

An analogous formula for near-backward two-particle energy correlation in the e^+e^- collision process has recently been compared with experiments at PETRA^{4,3)}. Data are qualitatively consistent with the prediction.

An interesting observation made by Parisi and Petronzio⁹⁾ is that at very high energies, even the distributions at $p_T \approx 0$ might be computable in perturbative QCD. This is understood because at extremely high energies, observed $|\vec{p}_T| \approx 0$ (say of a Drell-Yan pair) arises dominantly from two large $\vec{q}_T$'s (of opposite direction) of the annihilating quarks (whose effects are calculable), and not from the processes with no real emission (with $|\vec{p}_T| > 0$), which are suppressed.

Finally, it was suggested^{5,0)} that even analogous effects related to the *gluon* form factor may be measurable, in particular in $C = \text{even} - \chi(3.5)$ production at $p_T^2 \ll M^2$.

5. MULTIPLICITY DISTRIBUTIONS IN QCD

With some care, the preceding analysis can be pushed further to the multiplicity distributions. The first question which comes to one's mind, however, is: are they calculable? Indeed, a naive approach in LLA would give

$$\langle n \rangle_{\text{naive}} = \int dx D(x, Q^2) = e^{A_0 Y} \rightarrow \infty$$

as A_0 contains an infinite eigenvalue, reflecting the soft divergences of QCD. So the average number of soft partons is infinite, and one may be tempted to conclude that the multiplicity distributions are out of the scope of a perturbative approach.

In fact this is too pessimistic a viewpoint. It is the distribution of hadrons which interests us really, not one for on-shell partons, and the former is probably more closely related to the multiplicity distribution of *off-shell* partons. As emphasized in Section 2, a parton coming out of a hard process is highly virtual, and they decay through successive branchings. This evolution, as we argued already, can be followed by a perturbative method until the virtual mass of partons gets degraded to order of Q_0 . The evolution below this scale is uncomputable; however, if this final stage of interaction is a smooth one without large momentum transfer, then the final hadron distribution is essentially determined by the one of partons of "size" Q_0^{-1} . (Further discussions follow in Section 6.)

But the average multiplicity of partons (quarks or gluons) with virtual mass Q_0 is a well-defined quantity. (In counting such massive quanta, arbitrary numbers of quanta with virtual mass $\ll Q_0$ should be allowed to be produced also.) All soft and collinear singularities are regularized by Q_0 . On the other hand, the $\langle n \rangle$ of on-shell partons which are divergent, is irrelevant anyway because of confinement.

In computing $\langle n(Q^2, Q_0^2) \rangle$, however, one encounters doubly logarithmic terms of the form $(\alpha_s \log^2 Q^2)^m$, and the simple formalism of Section 3 cannot be used as it is. A more careful treatment, taking into account of the kinematic constraints among ladder integration variables (but still keeping only simple, tree diagrams of LLA), gives^{27,28,51)}

$$\langle n(Q^2, Q_0^2) \rangle \sim \exp C(\sqrt{\log Q^2/\Lambda^2} - \sqrt{\log Q_0^2/\Lambda^2}) \quad (28)$$

with $C = 2\sqrt{C_A/\pi b} \approx 2.4$. In Eq. (28), the logarithmic terms in front of the exponential function, which are sensitive to higher corrections, are neglected.

It is expected that the average multiplicity of hadrons will also obey the asymptotic behaviour of Eq. (28). The increase with Q^2 is very fast, compared with the $\log Q^2$ dependence found at low energies (≤ 10 GeV) in e^+e^- annihilation processes. Nonetheless, such a strong increase may not be a surprise in QCD: it is a scale-invariant theory and the phase space is effectively three dimensional ($\langle p_T^2 \rangle \sim Q^2$ has

the same origin). There would be no reason to expect $\langle n \rangle$ similar to that in multiperipheral processes (which have effectively one-dimensional phase space)*).

A clear upward deviation of $\langle n \rangle_{\text{charged}}$ from the $\log Q$ behaviour has been observed at PETRA at $13 \lesssim Q \lesssim 30 \text{ GeV}$ ⁵²⁾. In Fig. 11a, a tentative fit with Eq. (28) is illustrated, and the same figure in a larger scale (Fig. 11b) shows how dramatic such an effect should be in LEP or $p\bar{p}$ collider energy regions.

Of theoretical interest (and importance) is the factorized dependence of $\langle n(Q^2, Q_0^2) \rangle$ on Q^2 and Q_0^2 , which is analogous to the situation in distributions at finite $x > 0$. This provides a consistency check to our perturbative treatment *a posteriori*, as the answer does not depend on Q_0 apart from a Q^2 -independent normalization. Otherwise, we would not have known which answer (i.e. which Q_0) to choose. Such a factorization also holds for all multiplicity moments^{27,28)}, suggesting that something like the renormalization group may be at work.

One finds also^{4,27,28)},

$$\langle n \rangle_{\text{gluon jet}} \approx \left(C_A/C_F \right) \langle n \rangle_{\text{quark jet}}, \quad (29)$$

$$(C_A/C_F = 9/4),$$

and^{27,28)}

$$\langle n^{\text{quarks}} \rangle_{i\text{-jet}} \approx \left(N_f \beta / 3 N_C \right) \langle n^{\text{gluons}} \rangle_{i\text{-jet}} \quad (30)$$

with $\beta = \sqrt{N_C \alpha_s(Q_0^2)}/\pi$. Equation (29) holds irrespectively of the final parton species** and thus leads to the expectation that the *hadronic* multiplicities in the two kinds of jets are in the ratio⁵³⁾ 9/4.

At the T in colliding beam experiments at PETRA, where three-gluon decay is supposed to be dominant, an increase of $\langle n \rangle_{\text{charged}}$, $\Delta \langle n \rangle \sim +1$, has been observed⁵⁴⁾, compared to the adjacent energy region. This is a much smaller increase than

*) Theories with a non-trivial UV fixed point give²⁴⁾ $\langle n \rangle \sim (Q)^C$ with some positive constant C .

**) An example of "quark-gluon universality"⁴⁾ which was shown to hold in QCD on a wide ground.

Fig. 11 a) Average charged particle multiplicity at PETRA (taken from G. Wolf, Ref. 2). Theoretical curves are (A): $\langle n \rangle = C \exp 2\sqrt{C_A/\pi b} \cdot \sqrt{\log Q^2/\Lambda^2}$ with $\Lambda = 0.5$, $C = 0.0156$; and (B) (taken from G. Wolf, Ref. 2): $\langle n \rangle = n_0 + a \exp d\sqrt{\log Q^2/\Lambda^2}$ with $\Lambda = 0.5$, $n_0 = 2.92$, $d = 2.85$ and $a = 0.0029$. b) The same as a), enlarged to show the rough expectation at higher energies.

that expected from Eq. (29). However, it should not be forgotten that Eq. (29) is an asymptotic prediction and that all other perturbative-QCD signals -- large $\langle p_T^2 \rangle \sim Q^2$ ⁵⁾, fast increase of $\langle n \rangle$ ⁵²⁾, typical two-particle energy correlations⁴³⁾, etc. -- are just beginning to show up at these energies. The fact that we see a clear jump of $\langle n \rangle$ at the T resonance at all, is, in our opinion, quite encouraging, although the data clearly do not exclude other explanations.

The multiplicity moments can also be studied by the use of generating functions and of integro-differential equations satisfied by them, as briefly mentioned in Section 3. We find, for instance^{4,27,28)}, for the Wroblewski ratio $[D \equiv \sqrt{\langle n^2 \rangle - \langle n \rangle^2}]$,

$$D/\langle n \rangle \approx 1/\sqrt{3} \approx 0.58 \text{ (gluon jet)} \quad (31)$$

$$\approx \sqrt{3}/2 \approx 0.87 \text{ (quark jet)} ; \quad (32)$$

so that the long-range correlation (typical of branching processes) is present. The considerable difference of this ratio for quark (in e^+e^- , deep inelastic leptonproduction) and gluon (e^+e^- , Drell-Yan process, etc.) jets suggests the possibility of making a new test of QCD*).

The KNO scaling holds, and the corresponding KNO function has been estimated in Ref. 27.

QCD predictions discussed in this section are based on the approximation of keeping simple tree diagrams of LLA, together with the kinematical constraints among the ladder integration variables.

The q^2 dependence of $\alpha_s(q^2)$ from the one-loop β function, is also taken into account. The sum contains a whole series of non-leading contributions as well as the leading doubly logarithmic terms, and the neglect of other diagrams such as crossed ladder diagrams is not obviously justified, from a purely formal point of view. Although the results obtained are interesting and look reasonable (such as the factorization of Q_0 dependence), more rigorous study is certainly needed. [In Ref. 55, certain higher-order corrections have been studied where the results Eqs. (28), (29) and (30), have been found to be stable.]

*) At presently available energies, however, the kinematical effects may be large, which tend to make $D/\langle n \rangle$ much smaller than the asymptotic values, Eqs. (31), (32), as suggested by the Monte Carlo model of Odorico³³⁾.

A better-defined statement of the problem of multiplicity distributions in QCD has recently been proposed³²⁾, which is based on the reduction of unitarity equations discussed in Section 8, and which allows a systematic improvement of approximation.

6. MOMENTUM, COLOUR, AND SPACE-TIME EVOLUTION OF JETS

Preconfinement¹⁰⁾

So far, only the momenta and parton species (quark or gluon; valence or sea quark) have been specified in the analysis, but not the colour quantum number. The knowledge of colour flow is evidently of great interest, as the QCD quanta are to be converted into hadrons at the end of the jet evolution. Amati and Veneziano¹⁰⁾ suggested that *finite-mass colour singlet* clusters are already formed by the perturbative evolution alone (preconfinement).

Considering again e^+e^- annihilation for definiteness, notice first that only planar tree graphs are important^{*)} in the physical gauges in LLA, so that there is a natural way to group partons into clusters, each of which consists of a $q\bar{q}$ pair and gluons (but no other q or $\bar{q}$'s) (see Fig. 12). All final quanta

Fig. 12 A typical contribution to $e^+e^- \rightarrow \gamma^* \rightarrow$ quarks and gluons in LLA.

*) Strictly speaking, this statement is correct up to vertex correction graphs. In LLA, the latter is a part of contributions giving rise to the running coupling constant.

have virtual mass $\sim Q_0$. The invariant mass of such a cluster has been estimated to obey the distribution^{*)},

$$\frac{1}{\sigma} \frac{d\sigma}{dM} \underset{Q^2 \rightarrow \infty}{\sim} (Q_0/M)^C \cdot (\text{logarithmic function of } M)$$

with

$$C > 1 \text{ (} C = \text{a calculable number) ,} \quad (33)$$

so that $\langle M \rangle \approx \text{finite as } Q^2 \rightarrow \infty$.

In other words, as Q increases with Q_0 fixed, the system prefers producing a larger number of the clusters, rather than increasing the mass of individual clusters.

The suppression of high-mass "primitive" colour singlet clusters ($q\bar{q}$ + gluons, no other q , $\bar{q}$'s) is due to an incomplete cancellation of soft singularities, because only the real (and not virtual) $q\bar{q}$ pairs are forbidden inside a cluster. In this respect, this effect is analogous to the quark (or gluon) form factor, which is due to the suppression of real emission. In fact, in a ϕ_6^3 model (devoid of soft divergences), all clusters have mass increasing as Q^{10} .

As Amati and Veneziano¹⁰⁾ pointed out, the statement is not that *only* colour singlet clusters have finite masses: there are other finite-mass clusters in higher representations. It should, of course, be some non-perturbative mechanism that selects colour singlet clusters and binds them into hadrons.

Preconfinement does, however, imply that hadronic distributions are essentially (apart from details such as resonance decays) computable in perturbative QCD, as the long-range interactions will be only crucial inside a cluster, not between them.

This last statement would be exact in the $N_C \rightarrow \infty$ limit with $g^2 N_C$ and N_f/N_C fixed⁵⁶⁾. All exchange of quanta between two singlet clusters are at least down by $1/N_C$. The relevant

*) An improved treatment of IR soft divergences⁵⁵⁾ has led to a somewhat stronger damping (i.e. faster than any power) without affecting any of the qualitative features of preconfinement.

limit is Veneziano's⁵⁶⁾, and not 't Hooft's $N_C \rightarrow \infty$ limit⁵⁷⁾ (with $g^2 N_C$ and N_f fixed, in which case there is no $q\bar{q}$ pair production at all). The smooth (and very intriguing) connection between the perturbative dynamics -- dominated by planar tree graphs in axial gauges -- and the non-perturbative one (in which non-planar graphs are suppressed by $1/N_C$), is suggestive of further theoretical advance in this direction.

Generalizing the Jet Calculus

Careful analyses of jet evolution with the colour flow (as well as momentum flow) specified, require the knowledge of exact kinematical constraints as well as the higher-order (that is, beyond leading logs) corrections. The usual cancellation of soft singularities among real and virtual quanta does not work here.

In the approximation of keeping only the tree diagrams of a physical gauge, but taking account of exact kinematical constraints, Bassetto, Ciafaloni and Marchesini²⁷⁾ performed a detailed analysis of jet evolution and final states in hard processes. External partons are given on off-shell mass Q_0 to regularize the soft divergences.

Working in the light-like axial gauge, the evolution equation for the generating functional

$$G_a(k^2, Q_0^2; \{z\}) \equiv 1 + \sum \frac{1}{n!} \int \pi dx_i D_{c_1 \dots c_n, a}(Q^2, Q_0^2) \times \\ \times \prod_i [\phi_{c_i}(x) - 1], \quad (34)$$

is given ($\epsilon = Q_0^2/k^2$; $dY = [\alpha_s(k^2)/2\pi k^2] dk^2$) by,

$$(\partial/\partial Y) G_a[k^2, Q_0^2; \{\phi(x)\}] = -G_a \int_{\epsilon}^{1-\epsilon} dz P_a^a(z) + \\ + \frac{1}{2} \sum_{c_1 c_2} \int \hat{P}_{a \rightarrow c_1 c_2}(z) G_{c_1}(zk^2, Q_0^2; \{\phi(xz)\}) \times \\ \times G_{c_2}((1-z)k^2, Q_0^2; \{\phi[x(1-z)]\}) dz. \quad (35)$$

The presence of $z(1-z)$ in the arguments of G accounts for the kinematical constraints at the real emission vertex, $a \rightarrow c_1 c_2$.

Equation (35) contains all the information for inclusive and exclusive cross-sections when final colour quantum numbers are not specified.

Another ingredient is the introduction of "colour-connecting" evolution functions $\Gamma_a^q(K^2, Q_0^2; z)$ where the colour of the quark (Q_0^2) is connected to the parent parton a (k^2) by the emission of gluons only. They satisfy integro-differential equations analogous to the Altarelli-Parisi equations, which can be solved. The two-parton distribution with q and $\bar{q}$ forming a colour singlet and an invariant mass $\approx k^2$, obeys an evolution²⁷⁾ equation analogous to Eq. (11),

$$\begin{aligned}
 (\partial/\partial Y)(1/\sigma)d\sigma/dk^2 dx_1 dx_2 = & \sum_c \sum_{c_1 c_2} \int (dx/x^2) D_q^c(Q^2, k^2, x) \times \\
 & \times \int dz/[z(1-z)] \cdot \hat{P}_c^{c_1 c_2}(z) \times \Gamma_{c_1}^q(zk^2, Q_0^2, x_1/xz) \times \\
 & \times \Gamma_{c_2}^{\bar{q}}[(1-z)k^2, Q_0^2; x_2/x(1-z)] .
 \end{aligned} \tag{36}$$

Using this and the properties of Γ_C^q , the spectra of colour singlet clusters can be studied.

Their main results are²⁷⁾ that

- i) preconfinement is shown to occur;
- ii) $d\sigma/dMdx$ of the colour singlet cluster is explicitly computed;
- iii) at fixed p_T , the single-parton distribution shows a form-factor damping consistent with Ref. 58;
- iv) multiplicity distributions/KNO function are explicitly computed;
- v) the exact form of suppressions in (i) to (iii) above is sensitive to the higher-order corrections (see Ref. 55);
- vi) $\langle n \rangle$ computed in a light-like axial gauge is shown to be actually gauge independent (in a class of axial gauges).

Many other results can be obtained by the use of their formalism.

Their model has been further explored by Odorico³³⁾ in a Monte Carlo model. His model shows also how fast the various asymptopia are approached, which may be of relevance for present-day experiments such as those at PETRA.

A more careful treatment of infrared-sensitive quantities has recently been given⁵⁵⁾. A simple recipe is proposed which amounts to a modification of the Altarelli-Parisi kernel $[\hat{P}_{Q,Q}(z) = (1+z^2)/(1-z)]$,

$$\alpha_s(q^2) [\hat{P}_{Q,Q}(z)]_+ \rightarrow \left\{ \alpha_s[q^2(1-z)] \hat{P}_{Q,Q}(z) \right\}_+$$

[recall that $[f(x)]_+ \equiv f(x) - \delta(x-1) \int dy f(y)$]. The above replacement is argued to summarize the dominant non-leading effects for $z \approx 1$, and leads to a simplified treatment of quark form factor, $x \approx 1$ behaviour of structure functions, etc.

A Note on the Use of Light-Like Axial Gauges

The use of gauges, $n \cdot A = 0$ with $n^2 = 0$, is proving a convenient way of making calculations^{27,28,32,40,51,55)}, other gauges being either too complicated in diagrammatic structure (covariant gauges) or simply too intricate (axial gauges with $n^2 \neq 0$), beyond LLA. The price is the appearance of divergences in each diagram, which are neither of ultraviolet nor of infrared (soft or collinear) origin, but are a peculiarity of $n^2 = 0$ gauges. To next to leading order, these divergences have been shown³²⁾ to be absent in the gauge-invariant set of Feynman diagrams. (See also Ref. 40.)

Jet Evolution in Real Space-Time

Another relevant question⁵⁹⁾ concerns the way in which jets evolve in real space-time. This question is particularly relevant because of the confinement: the separation of colour (e.g. the one carried by the original $q\bar{q}$ pair in e^+e^- annihilation into hadrons), during the perturbative evolution, should not exceed the order of 1 fm and, in particular, should not increase as $Q \rightarrow \infty$. Otherwise, we should either observe naked colour, or conclude that perturbative calculation does not make much sense.

It is also an interesting question⁵⁹⁾ whether the time evolution in hard processes is similar to high-energy, low- p_T hadron collision processes, where a long time-scale $\sim \sqrt{s}$ (= total c.m. energy) appears⁶⁰⁾.

The problem was studied in Ref. 11, in the case of e^+e^- annihilation. It was found that the average lifetime of a parton with virtual mass $\sim M$ and with energy E , is given, in the c.m. system of e^+e^- , by¹¹⁾

$$\langle t \rangle \sim E/M^2 \quad (37)$$

which coincide with the naive expectation from the uncertainty principle and the Lorentz time dilation. Knowing the structure of the branching decay process in the momentum space, as discussed above [in particular, knowing the number of steps of branchings from the average multiplicity, Eq. (28)], it is easy to estimate the total time required for the jet evolution up to the appearance of partons with virtual mass $\sim Q_0$ (with finite energy fractions of Q). Roughly, one finds¹¹⁾

$$\langle T \rangle \xrightarrow{Q \rightarrow \infty, Q_0 \text{ fixed}} \text{const. } Q. \quad (38)$$

This comes essentially from the last part of the evolution, which lasts for $\Delta t \sim Q/Q_0^2$ as seen from Eq. (37).

Thus the total time increases indefinitely as $Q \rightarrow \infty$, and it is of crucial importance to check that the neglect of non-perturbative effects is not a disastrous one. Caneschi and Schwimmer used the results of Ref. 10 to estimate the time needed for producing the first $q\bar{q}$ pair: they find^{11a)}

$$\langle T \rangle_{1st \ q\bar{q}} \xrightarrow{Q \rightarrow \infty} \text{finite}. \quad (39)$$

Thus such a pair is produced and screens the colour of original quarks, before the latter are separated too far apart. This is illustrated in Fig. 13. The conclusion is that, in spite of the long time-scale of total jet evolution [Eq. (38)], the use of perturbative QCD is justified up to the appearance of quanta with a large but fixed mass $\sim Q_0$.

Monte Carlo models based on a similar approximation (tree graphs only but keeping the kinematical constraints) as in Refs. 11, leads to a qualitatively analogous answer⁶¹⁾.

The appearance of a long time-scale [Eq. (38)] is reminiscent of high-energy, low- p_T , hadronic processes. However, the presence of highly virtual partons and the rapid deceleration of the decay processes due to the degradation of parton (off-shell) masses, are features not shared by the soft processes.

Fig. 13 Space-time evolution of the process $e^+e^- \rightarrow \gamma^* \rightarrow$ mesons.

The picture which emerges is that of the "inside-outside cascade" conjectured by Bjorken⁵⁹⁾, which is valid in the soluble two dimensional Quantum Electrodynamics⁶²⁾. Details of the final states such as the multiplicity of final hadrons (see also below), are however quite different from those in $(QED)_2$ where no physical gluons exist.

"Reversing" the Rapidity Ordering

A closely related subject⁵⁹⁾ is the structure of final states in the rapidity space. The problem is that the naive picture of successive gluon emission from the original $q\bar{q}$ pair apparently gives the final state, which is in contradiction to the picture discussed above. [This indeed happens in the QED-type model discussed earlier⁶³⁾.]

In Fig. 14 *) , the gluon A is more energetic than B, being emitted earlier; therefore the quark a would, on the average, be faster than the quark b (this second part of the argument is false in QCD: see below). Thus the quark a

*) Gluons shown in Fig. 14 are assumed to have some minimum energy. The summation over arbitrarily soft gluons is understood.

Fig. 14

which is closer (in the sense of colour structure, as seen in Fig. 14) to the original quark O , would be further apart in rapidity from (say) c , than b is. This is in contrast to the picture discussed in preceding sections. Furthermore, since the "valence" quark v is known to be the fastest among q 's and $\bar{q}$'s, it would be hard to imagine how a smooth transition can occur from a perturbative to a non-perturbative scheme as envisaged in the preconfinement (Fig. 12).

This apparent problem is resolved by noticing that, in QCD, the gluon A is not only more energetic but also much more *off-mass-shell* than B , and therefore it must distribute its energy among many more quarks than B must*). The high average multiplicity of quarks inside a gluon (of off-shell mass $\sim q^2$),

$$\langle n \rangle \sim \exp C \sqrt{\log q^2 / \Lambda^2} , \quad (28')$$

is sufficient⁶⁴⁾ to reverse the rapidity ordering between a and b , leading to

$$y_a < y_b < y_v .$$

This can be checked by a more quantitative argument⁶⁴⁾, but it is clear that the gluon self-coupling [giving rise to Eq. (28')] is essential for achieving this.

Another way⁶⁵⁾ to get to the same conclusion is to compare the average $\langle x \rangle$ of a generic quark (a or b in Fig. 14) determined by the usual evolution function D with that of the quark o , carrying the original colour of O , as controlled by the colour-connecting evolution function Γ . It can easily be seen that $\langle x \rangle_o \ll \langle x \rangle_a$, as $\Gamma(x)$ is much more peaked at $x \rightarrow 0$ than $D(x)$.

*) This observation grew out of discussions with J. Kalinowski.

Thus all the aspects of jet evolution seem to fit together to form a consistent picture. The original quark (or antiquark) produced by the virtual photon is highly virtual and emits a gluon in a short time. The gluon, being also highly virtual and at the same time relatively non-energetic, decays in a short period and, through a branching, produces the first $q\bar{q}$ pair. This screening of the original colour (which may be seen as the first splitting of the string of colour flux), takes place in a finite time. The original quark, after the first gluon emission, survives for a relatively long time as it is more energetic, until it emits a second gluon, and a similar branching of the gluon follows. According to this picture, the softest mesons are produced earliest in time, in the central region of rapidity in the e^+e^- c.m. system; the more energetic mesons appear at a later time. The whole process continues for a time $\sim Q$, until the fastest meson, often containing the original quark flavour, is produced. In understanding all this, the multiple branching process of QCD quanta (especially of gluons) is crucial. Preconfinement is nothing but the statement that the final partons $\rightarrow$ hadrons transition should occur smoothly.

7. THE JET CALCULUS BEYOND LEADING LOGS³²⁾

Analyses performed up to this point have been based on the leading log approximation, in which the simplicity of the results and the presence of an intuitive parton picture helped our study of jet evolution.

Does a simple parton picture persist beyond the leading log approximation? Does a simple rule of calculation -- the jet calculus -- exist which automatically takes account of corrections of higher orders (of logs)?

That such a simplification is indeed possible beyond the leading log approximation, has recently been shown³²⁾ by the use of a diagrammatic analysis of jet evolution. The simplest way to see how a parton-branching process picture emerges in QCD, is to combine the off-shell unitarity equation of Polyakov²⁴⁾ with the use of an axial gauge.

Let us consider, for the purpose of illustration, the e^+e^- annihilation process. A similar analysis can be carried out in any hard process. In the lowest order of the electromagnetic coupling, the total cross-section is given by the discontinuity,

$$\sigma_t \propto \frac{\alpha^2}{(Q^2)^2} \times \text{Disc}_{Q^2} \pi(Q^2) \equiv \frac{\alpha^2}{Q^2} \text{Disc}_{Q^2} \tilde{\pi}(Q^2) . \quad (40)$$

Following Polyakov, but in QCD in an axial gauge, we write $\text{Disc}_0^2 \tilde{\pi}(Q^2)$ as

$$\text{Disc}_{Q^2} \tilde{\pi}(Q^2) = \text{Disc}_{Q^2} \left\{ \text{---} \bigcirc \text{---} + \text{---} \bigcirc \text{---} + \text{---} \bigcirc \text{---} + \dots \right\} \quad (41)$$

where a crossed line is the imaginary part of an exact propagator of a quark or gluon, the dotted line is the virtual photon, and the vertices and (uncut) propagators are also exact ones. This form of unitarity equation can be obtained starting from the ordinary one with on-shell cut lines [each with $\theta(k_0)\delta(k^2)$], first by decomposing the amplitudes in one-particle irreducible vertices, inserting them back into the original equation, and finally recombining to get the discontinuity of exact propagators.

Each of the crossed lines is a jet of particles; in an axial gauge the entire leading log contributions (all two-jet events) comes from the first term of Eq. (41). Three-jet events come from all of the terms shown explicitly (and from some others).

Equation (41) can be further modified by the following steps³²⁾. First, we introduce an arbitrary mass scale M_0 , and for each crossed line having mass larger than M_0 we write ($\sqrt{q^2} > M_0$)

$$\begin{aligned}
 \text{---}x\text{---} &\equiv \text{Disc}_{q^2} G(q^2) \\
 &= \text{---} \bigcirc \text{---} + \text{---} \bigcirc \text{---} + \text{---} \bigcirc \text{---} + \text{---} \bigcirc \text{---} + \dots \quad (42)
 \end{aligned}$$

in terms of the cut exact propagator with smaller masses. Equation (42) is a strict analogy of Eq. (41) for $\text{Disc}_Q^{2\tilde{\pi}}$. The lines with mass less than M_0 are left as they are. We repeat the same operation for each crossed line in Eq. (42). This operation is repeated in all lines of all terms, until all the crossed lines have mass $\leq M_0$ and all uncut lines have mass $\geq M_0$.

The next step is to introduce effective vertices to eliminate (formally) the non-diagonal terms. We repeatedly use the substitution³²⁾ (defining thus the effective vertices) such as

Diagrammatic equation (43) illustrates the replacement of a loop with a self-energy insertion. On the left, a loop with a cross (representing a self-energy insertion) is shown. An arrow points to the right, where the loop is replaced by a sum of diagrams. The first diagram in the sum is a loop with two dots (representing effective vertices). The second diagram is a loop with a cross and a dot. The sum is indicated by a plus sign and an ellipsis. The final diagram on the right is a loop with two dots and a cross.

Notice that for two-particle reducible subgraphs, we have for instance,

Diagrammatic equation (44) shows the reduction of a two-particle reducible subgraph. On the left, a subgraph with two dots and a cross is shown. An arrow points to the right, where the subgraph is replaced by a sum of diagrams. The first diagram in the sum is a subgraph with two dots and a cross. The second diagram is a subgraph with two dots and a cross, with an additional cross on the internal line. The sum is indicated by a plus sign and an ellipsis. The final diagram on the right is a subgraph with two dots and a cross, with an additional cross on the internal line.

The reduction illustrated in Eq. (43) is thus performed only for non-diagonal subgraphs.

A more rigorous definition of these operations is given in Ref. 32. After these two operations have been performed, the right-hand side of Eq. (42) is a sum of diagonal terms such as those shown in Fig. 15, with all crossed lines having mass less than M_0 , all internal (uncut) ones with mass larger than M_0 .

The advantage of these operations is that, in axial gauge QCD, the collinear singularities arise only when there are two Feynman denominators of the same momenta (appearing as a complex conjugate pair in Fig. 15). Each of the effective vertices themselves is free of mass singularities and can be computed with final external lines set to be on-shell and massless. The diagrams such as those in Fig. 15 then clearly show how collinear mass singularities arise (for a more rigorous statement, see Ref. 32).

The physical picture which emerges with the rewriting of the unitarity equation above, is that of the scale-non-invariant parton model⁶⁶⁾. Scaling violations are controlled by collinear mass singularities^{16,17)}, and the density of partons of a given size M^{-1} increases with Q^2 , because the average number of steps to reach M increases in Fig. 15. The formalism being developed may thus be regarded as a quantitative realization of the Kogut-Susskind picture⁶⁶⁾, valid to all orders of α_s and of logs in QCD.

Fig. 15 Rewriting the unitarity equation in terms of effective vertices.

Once the unitarity equation is cast into this form, various differential cross-sections, in terms of hadrons or partons, are easily (at least formally) related to the diagrams obtained above. It is, for example, possible to re-derive the results of Ref. 16 on the factorization theorem of mass singularities.

Of particular interest, however, is the appearance of the parton probabilistic picture and the resulting simple jet calculus rules for the evolution of jets, which contains the rules of Section 3 as a special case (leading log approximation). For instance, the three-parton distribution is given by the effective tree graphs shown in Fig. 16, where the vertices are those introduced in the above-mentioned operations.

Fig. 16 Contribution to $i \rightarrow a_1 + a_2 + a_3 + X$ in the jet calculus beyond leading order.

These vertices $V_{j \rightarrow ab \dots}(q^2, x_i)$ can be computed as a power-series expansion of $\alpha_s(q^2)$: to a given order of α_s , there exist only a finite number of such vertices. The generating functional for multiparton distributions satisfies Kolmogorov's equations (of branching processes), which are slight generalizations of Eq. (13).

The appearance of probabilistic quantities is, of course, a delicate matter beyond the leading log approximation. It is the consequence of the Kinoshita-Lee-Nauenberg theorem¹⁷⁾ and the factorization theorem of mass singularities that one *can* choose the convention such that $V_{j \rightarrow ab \dots}$ have formally the interpretation of elementary probabilities to all orders of $\alpha_s(q^2)$. The aim (and result) of the diagrammatic analysis mentioned above was to show that everything is simply expressed in terms of these probability functions.

All the integrations over internal masses are cut off from below by M_0 , and lead to finite distributions which are functions of $M = \delta Q$ (upper limit of invariant mass, defining a jet), and M_0 .

How the results on jets based on the LLA jet calculus will be modified by inclusion of higher-order corrections is under study³²⁾.

8. CONCLUSION AND OUTLOOK

In this paper, we have reviewed the study, based on QCD, of the properties of hadronic jets in deep inelastic processes, and discussed some new developments. Among others, the intuitive "fractal" concept of jets, the consistent picture of jet evolution in momentum, colour and space-time, and the presence of a parton description with a simple stochastic algorithm (which persists beyond LLA), discussed in this paper, together seem to form a powerful guide for further advancement in our endeavour.

We shall close this paper by contemplating in which direction such further progress may be looked for.

- As emphasized in Ref. 67, "looking into jets" (as discussed here) goes in the direction of bridging the purely hard processes to soft-hadronic processes. With the presence of the parton-branching-process picture valid beyond LLA, is it possible to get closer to a theoretical understanding of soft processes?
- Hadronic exclusive processes (which were not considered here) are another interesting domain⁶⁸⁾ of application of perturbative QCD. Exclusive-inclusive connections are also interesting. Use of the off-shell unitarity equation in the axial gauge QCD will probably prove to be fruitful.
- All the approximation schemes invented thus far rely on the expansion in $1/\log Q^2$, which may not always be very useful. Careful maximization of convergence of the perturbative series²¹⁾, or re-summation of certain large terms, should be studied further. Also, is it possible to find approximation methods which go beyond the logic of LLA (and do not rely on the smallness of $1/\log Q^2$)? A question such as which variable the average multiplicity in lepton productions or large p_T hadronic reactions depends upon -- s , $Q^2(p_T^2)$ or something else -- goes indeed outside the logic of LLA but is nonetheless an interesting problem to understand.
- Although not discussed here, processes involving real (or almost real) photons are another interesting and very relevant domain in which perturbative QCD prove to be powerful and predictive¹³⁾. Whole new series of phenomenology should open with the 2γ processes in high-energy e^+e^- experiments.

Acknowledgements

My sincere gratitude goes to my colleagues of Rutherford Laboratory and CERN, for the numerous discussions I had during the preparation of this work. In particular, I thank G. Veneziano for illuminating discussions on most of subjects presented here, and for reading the manuscript. I profited also from useful comments by J. Kalinowski.

References

1. For reviews, see, for example, Yu.L. Dokshitzer, D.I. D'yakonov and S.I. Troyan, Phys. Reports 58, 271 (1980).
 Yu.L. Dokshitzer and D.I. D'yakonov, 14th Winter School of the LNPI, Leningrad, 1979 (Institute of Nucl. Phys., Leningrad, 1979), p. 27. [English translation: DESY L-Trans-234 (1979)].
 G. Veneziano, Talk given at the 3rd Workshop on Current Problems in High-Energy Particle Theory, Florence, 1979. [and preprint CERN TH.2691 (1979)].
 D. Amati, Proc. 14th Rencontre de Moriond, Les Arcs, 1979 (ed. J. Tran Thanh Van) (Editions Frontières, Dreux, 1979), p. 241.
 M. Ciafaloni, *ibid.*, p. 257.
 K. Konishi, *ibid.*, p. 199.
 D. Sivers, ANL-HEP-CP-79-42 (1979).
 S.D. Ellis, Univ. Washington, RLO-1388-811 (1980).
 G. Curci, M. Greco and Y. Srivastava, preprint CERN TH.2632 (1979) reviews an alternative approach based on coherent states;
 S. Wolfram, Caltech preprint, CALT-68-778 (1980).
 For earlier works, see J. Ellis and C. Sachrajda in Ref. 3.
2. For experimental reviews on jets, see, for example, H. Schopper, DESY 79/79 (1979).
 G. Wiik, DESY 79/84 (1979).
 G. Wolf, DESY 80/13 (1980).
 L. Di Lella, preprint CERN-EP/79-145 (1979).
 Proc. Conf. on Jets and QCD, NBI-Nordita (1978), published in Phys. Scripta 19 (1979).
 Proc. 19th Int. Conf. on High-Energy Physics, Tokyo, 1978 (Physical Society of Japan, Tokyo, 1979), Sections A9 and A10.
 Experimental reviews in Proc. 14th Rencontre de Moriond, Les Arcs, 1979 (ed. J. Tran Thanh Van) (Editions Frontières, Dreux, 1979).
3. For general reviews on perturbative QCD, see for example, C. Llewellyn Smith, Lectures given at the 1979 Boulder Summer Institute [and Oxford Univ. preprint 54/79 (1979)].
 J. Ellis, Invited talk presented at the 1979 Int. Symp. on Lepton and Photon Interactions, Batavia, 1979 [and preprint CERN TH.2744 (1979)].
 J. Ellis and C.T. Sachrajda, CERN TH.2782 (1979).
 H.D. Politzer, CALT 68-691 (1979).

A. De Rújula, Talk given at the EPS Int. Conf. on High-Energy Physics, Geneva, 1979 (CERN, Geneva, 1980), p. 418. For specialized topics on Deep Inelastic Leptonproduction, see for example A. Buras, Rev. Mod. Phys. 52, 199 (1980).

4. a) K. Konishi, A. Ukawa and G. Veneziano, Nucl. Phys. B157, 45 (1979).
 b) K. Konishi, A. Ukawa and G. Veneziano, Phys. Lett. 80B, 259 (1979).
 c) K. Konishi, A. Ukawa and G. Veneziano, Phys. Lett. 78B, 243 (1978).
 K. Konishi, Phys. Scripta 19, 195 (1979), and Proc. 19th Int. Conf. on High-Energy Physics, Tokyo, 1978 (Physical Society of Japan, Tokyo, 1979), p. 241.
5. See for example, JADE Collaboration (1), Phys. Lett. 91B, 142 (1980).
 PLUTO Collaboration, Phys. Lett. 90B, 312 (1980).
 TASSO Collaboration, Phys. Lett. 89B, 418 (1980).
 JADE Collaboration (2), Phys. Lett. 88B, 171 (1979).
 D.P. Barber et al., Phys. Rev. Lett. 43, 830 (1979).
 C. Bromberg et al., CALT-68-738 (1979).
6. B.B. Mandelbrot, Fractals -- Form, Chance and Dimension, (W.H. Freeman and Company, San Francisco, 1977).
7. G. Sterman and S. Weinberg, Phys. Rev. Lett. 39, 1436 (1977).
8. Yu.L. Dokshitzer, D.I. D'yakonov and S.I. Troyan, Proc. 13th Winter School of the LNPI, Leningrad, 1978 (Institute of Nuclear Physics, Leningrad, 1978), Vol. I, p. 1.
 [English translation: SLAL-TRANS-183 (1978).]
9. G. Parisi and R. Petronzio, Nucl. Phys. B154, 427 (1979).
 C.Y. Lo and J.D. Sullivan, Phys. Lett. 86B, 327 (1979).
 See also, N.S. Craigie and H.F. Jones, ICTP preprint IC/79/60 (1979).
10. D. Amati and G. Veneziano, Phys. Lett. 83B, 87 (1979).
 D. Amati in Ref. 1.
 A. Bassetto et al., Ref. 27.
11. a) L. Caneschi and A. Schwimmer, Phys. Lett. 86B, 179 (1979).
 b) C.B. Chiu and J. Swzed, Max Planck Institute preprint, MPI-PAE-PTh 15/79 (1979).
12. See, for example, J. Ellis and C. Sachrajda in Ref. 3.

13. C. Llewellyn Smith, Phys. Lett. 79B, 83 (1978).
 E. Witten, Nucl. Phys. B120, 189 (1977).
 S.J. Brodsky, T. De Grand, J.F. Gurion and J.H. Weiss, Phys. Rev. Lett. 41, 672 (1978).
 H. Fritzsch and P. Minkowski, Phys. Lett. 69B, 316 (1977).
14. H.D. Politzer, Phys. Lett. 70B, 430 (1977).
 D. Amati, R. Petronzio and G. Veneziano, Nucl. Phys. B140, 54 (1978).
15. L.L. Gribov and L.N. Lipatov, Yad. Fiz. 15, 781 (1972)
[Sov. J. Nucl. Phys. 15, 438 (1972)].
16. D. Amati, R. Petronzio and G. Veneziano, Nucl. Phys. B146, 29 (1978).
 R.K. Ellis, H. Georgi, M. Machacek, H.D. Politzer and G.G. Ross, Phys. Lett. 78B, 281 (1978) and Nucl. Phys. B152, 285 (1979).
 J. Frenkel, M.J. Shailer and J.C. Taylor, Nucl. Phys. B148, 228 (1979).
 S.B. Libby and G. Sterman, Phys. Lett. 78B, 618 (1978), and Phys. Rev. D 18, 3252 (1978).
 A.H. Mueller, Phys. Rev. D 18, 3705 (1978).
 For closely related works on diagrammatic derivation of the QCD improved parton model, see, Yu. Dokshitzer, Sov. Phys. JETP 72, 1216 (1977).
 C. Llewellyn Smith, Lectures given at Schladingen, 1978 [and Oxford Univ. preprint 47/78 (1978)].
 W.R. Frazer and J.F. Gunion, Phys. Rev. D 20, 147 (1979).
 W.J. Stirling, Nucl. Phys. B145, 477 (1978).
 Yu. Dokshitzer et al., Ref. 8.
 Y. Kazama and Y.-P. Yao, Phys. Rev. Lett. 41, 611 (1978), and Phys. Rev. D 19, 3121 (1979).
 D.J. Pritchard and W.J. Stirling, Nucl. Phys. B165, 237 (1980).
17. T. Kinoshita, J. Math. Phys. 3, 650 (1962).
 T.D. Lee and M. Nanenber, Phys. Rev. 133, 1549 (1964).
18. See J. Ellis and C.T. Sachrajda in Ref. 3.
19. G. Altarelli, R.K. Ellis and G. Martinelli, Nucl. Phys. B143, 521 (1978) [Erratum B146, 544 (1978)] and Nucl. Phys. B157, 461 (1979).
 J. Kubar-André and F.E. Paige, Phys. Rev. D 19, 221 (1978).
 J. Abad and B. Humpert, Phys. Lett. 78B, 627 (1978).
 K. Harada, T. Kaneko and N. Sakai, Nucl. Phys. B155, 169 (1979) (Erratum to be published).

- J. Ambjørn and N. Sakai, Nordita-80/14 (1980).
 B. Humpert and W.L. van Neerven, Phys. Lett. 85B, 293 (1979).
 G.T. Bodwin, C.Y. Lo, J.D. Stack and J.D. Sullivan, Univ. Illinois ILL-(TH)-80-2 (1980).
 K. Harada and T. Muta, Kyoto University preprint, RIFP-391 (1980).
20. G. Parisi, Phys. Lett. 90B, 295 (1980).
 J. Kripfganz, Comments on Regularization Prescription Dependence of QCD Corrections to the Drell-Yan Formula, McGill University preprint (1979).
21. W.A. Bardeen, A.J. Buras, D.W. Duke and T. Muta, Phys. Rev. D 18, 3998 (1978).
 W. Celmaster and R.J. Gonsalves, Phys. Rev. D 20, 1420 (1979).
 R. Barbieri, L. Caneschi, G. Curci and E. d'Emilio, Phys. Lett. 81B, 207 (1979).
 M.R. Pennington and G.G. Ross, Phys. Lett. 86B, 371 (1979).
 L.F. Abbott, Choosing an expansion parameter for QCD, Brandeis Univ. preprint (1980).
22. See D. Amati, M. Ciafaloni and K. Konishi in Ref. 1; and G. Veneziano in Ref. 1.
23. T.H. Harris, The Theory of Branching Processes (Springer, Berlin, 1963).
 S. Karlin and H.M. Taylor, A First Course in Stochastic Processes (Academic Press, NY, San Francisco, London, 1966).
 M.S. Bartlett, An Introduction to Stochastic Processes (Cambridge Univ. Press, 1978).
 For further references, see for example I. Andreev (Ref. 26) and A. Giovannini (Ref. 26).
24. A.M. Polyakov, Zh. Eksp. Teor. Fiz. 59, 542 (1970).
[Sov. Phys. JETP 32, 296 (1971)], and Sov. Phys. JETP 33, 850 (1971).
25. S.J. Orfanidis and V. Rittenberg, Phys. Rev. D 10, 2892, (1974).
26. P. Cvitanović, P. Hoyer and K. Konishi, Phys. Lett. 85B, 413 (1979).
 I. Andreev, Proc. 14th Rencontre de Moriond, Les Arcs, 1979 (ed. J. Tran Thanh Van) (Editions Frontières, Dreux, 1979), p. 269.

Related ideas are further explored in A. Giovannini, Nucl. Phys. B161, 429 (1979).

See also R. Kirschner, Phys. Lett. 84B, 266 (1979).

27. A. Bassetto, M. Ciafaloni and G. Marchesini, Nucl. Phys. B163, 477 (1980); and Phys. Lett. 83B, 207 (1979).
28. K. Konishi, Rutherford Lab. RL-79-035 (1979).
29. R. Doria, J. Frenkel and J.C. Taylor, Oxford Univ. preprint OUTP 79-73 (1979).
A. Andrasi, M. Day, R. Doria, J. Frenkel and J.C. Taylor, OUTP 80-37 (1980).
30. J.C. Taylor, Phys. Lett. 73B, 85 (1978).
See also Ref. 4a.
31. G. Altarelli and G. Parisi, Nucl. Phys. B126, 298 (1977).
G. Parisi, Proc. 11th Rencontre de Moriond, 1976 (ed. J. Tran Thanh Van).
See also, T. Uematsu, Phys. Lett. 79B, 97 (1978).
J.F. Owens, Phys. Lett. 76B, 85 (1978).
32. J. Kalinowski, K. Konishi and T. Taylor, Jet calculus beyond leading logs, Part I, CERN TH.2902 (1980).
J. Kalinowski, K. Konishi, P. Scharbach and T. Taylor, as above, Part II, in preparation.
33. R. Odorico, Univ. Bologna IFUB 80/5 (1980); ECFA-LEP Note SSG/8/9 (1980).
G.C. Fox and S. Wolfram, CALTECH-68-755 (1979).
K. Kajantie and E. Pietarinen, DESY 80/19 (1980).
Also D. Drijard, private communication.
34. R.P. Feynman, Talk given at the CalTech Workshop on High-Energy Physics (February 1979).
35. H. Schopper and G. Wiik, in Ref. 2.
36. K. Konishi, unpublished. Some related calculations are found in I. Andreev in Ref. 26.
37. L.N. Lipatov, Yad. Fiz. 20, 181 (1974).
38. Yu.L. Dokshitzer et al., in Ref. 1.
39. W. Scott, 14th Rencontre de Moriond, Les Arcs, 1979 (ed. J. Tran Thanh Van) (Editions Frontières, Dreux, 1979), p. 307.

40. G. Curci, W. Furmanski and R. Petronzio, preprint CERN TH-2815 (1980).
E. Floratos et al., Saclay preprint, DPh.T77/80 (1980).
J. Kalinowski et al., see Ref. 32.
41. R. Kirschner, preprint CERN TH.2823 (1980).
42. M.J. Puhala, Some comments on two-particle correlations in the jet calculus of KUV, Iowa State Univ. preprint (1980).
D.E. Willen, Univ. Illinois ILL-(TH)-79-34 (1979).
43. PLUTO Collaboration in Ref. 5.
44. K. Shizuya and S.-H. Tye, Phys. Rev. Lett. 41, 787 (1978).
45. C.L. Basham, L.S. Brown, S.D. Ellis and S.T. Love, Phys. Rev. D 17, 2298 (1978); Phys. Rev. Lett. 41, 1585 (1978); Phys. Rev. D 19, 2018 (1979).
46. R.K. Ellis and R. Petronzio, Phys. Lett. 80B, 249 (1979).
G. Curci and M. Greco, Phys. Lett. 79B, 406 (1979).
47. A. Duncan, S. Gupta and A.H. Mueller, Columbia Univ. CU-TP-179 (1980).
48. P.M. Stevenson, Nucl. Phys. B150, 357 (1979) and B156, 43 (1979).
49. Yu.L. Dokshitzer and D.I. D'yakonov, Phys. Lett. 84B, 234 (1979).
50. Yu.L. Dokshitzer et al. in Ref. 1.
51. W. Furmanski, R. Petronzio and S. Pokorski, Nucl. Phys. B155, 253 (1979).
52. TASSO Collaboration and JADE Collaboration (2) in Ref. 5.
53. This was conjectured earlier in S.J. Brodsky and J.F. Gunion, Phys. Rev. Lett. 37, 402 (1976).
54. See, for example, G. Wolf in Ref. 2.
55. D. Amati, A. Bassetto, M. Ciafaloni, G. Marchesini and G. Veneziano, preprint CERN TH.2831 (1980).
See also G. Curci and M. Greco, preprint CERN TH.2786 (1979).
G. Parisi in Ref. 20.

56. G. Veneziano, Nucl. Phys. B117, 519 (1976).
57. G. 't Hooft, Nucl. Phys. B72, 461 (1974).
58. J.M. Cornwall and G. Tiktopoulos, Phys. Rev. D 13, 3370 (1976).
59. J.D. Bjorken, Proc. 1973 Summer Institute on Particle Physics (SLAC-167, 1973).
60. J. Koplik and A.H. Mueller, Phys. Rev. D 12, 3638 (1975).
T. Jaroszewicz et al., Phys. Lett. 79B, 127 (1978).
61. G.C. Fox et al. in Ref. 33.
62. A. Casher, J. Kogut and L. Susskind, Phys. Rev. D 10, 732 (1974); see also,
H. Minakata, Fermilab-Pub-80/28-THY (1980).
63. S.J. Brodsky and N. Weiss, Phys. Rev. D 16, 2325 (1977).
64. K. Konishi, unpublished.
65. G. Marchesini, private communication.
66. J. Kogut and L. Susskind, Phys. Rev. D 9, 697 (1974)
and D 9, 3391 (1974).
67. G. Veneziano, Proc. 19th Int. Conf. on High-Energy Physics, Tokyo, 1978 (Physical Society of Japan, Tokyo, 1979), p. 725.
68. A.V. Efremov and A.V. Radyushkin, Dubna preprints JINR-E2-11535 (1978) and JINR-E2-11983 (1979).
S.J. Brodsky and G.P. Lepage, Phys. Lett. 87B, 359 (1979); and SLAC-PUB-2294.
G.R. Farrar and D. Jackson, Phys. Rev. Lett. 43, 246 (1979).
G. Parisi, Phys. Lett. 84B, 225 (1979).
A. Duncan and A.H. Mueller, Columbia University preprint, CU-TP-172 (1979).
69. F. Halzen and D.M. Scott, Wisconsin preprint, COO-881-141 (1980); also, B. Webber et al., in preparation.

CONTENTS

	<u>Page</u>
1. INTRODUCTION	1
2. INTUITIVE PICTURE OF JETS	5
Jet Evolution as a Branching Process	5
QCD Jets as Fractals	7
3. JET CALCULUS -- EVOLUTION OF JETS IN LLA	11
Description in (x,y) Variables --	
Parton Picture	12
Analogy with the Markov Branching Process --	
Kolmogorov Equations	14
Formalism in the Moments -- Old-Fashioned	
Perturbation Theory	15
Jet Field Theory	16
4. MEASURING QUARKS AND GLUONS	17
Two-Particle Energy Correlation	
at Fixed Relative Angle (or p_T)	17
Jet Energy Spread -- New Measurement of	
QCD Anomalous Dimensions	22
Words of Caution on $\langle p_T^2 \rangle$ Inside a Jet	25
Initial Parton Jets ("Spectator" Jets)	26
Effective Quark Form Factors and Distribution	
at $0 \leq p_T \leq Q^2$	27
5. MULTIPLICITY DISTRIBUTIONS IN QCD	28
6. MOMENTUM, COLOUR, AND SPACE-TIME EVOLUTION OF JETS	33
Preconfinement	33
Generalizing the Jet Calculus	35
A Note on the Use of Light-like Axial Gauges	37
Jet Evolution in Real Space-Time	37
"Reversing" the Rapidity Ordering	39
7. THE JET CALCULUS BEYOND LEADING LOGS	41
8. CONCLUSION AND OUTLOOK	46
REFERENCES	48