ELECTRON DYNAMICS WITH RADIATION
AND NONLINEAR WIGGLERS

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ABSTRACT: The physics of electron motion in storage rings is described by supplementing the Hamiltonian equations of motion with fluctuating radiation reaction forces to describe the effects of synchrotron radiation. This leads to a description of radiation damping and quantum diffusion in single-particle phase-space by means of Fokker-Planck equations. For practical purposes, most storage rings remain in the regime of linear damping and diffusion; this is discussed in some detail with examples, concentrating on longitudinal phase space. However special devices such as nonlinear wiggler may permit the new generation of very large rings to go beyond this into regimes of nonlinear damping. It is shown how a special combined-function wiggler can be used to modify the energy distribution and current profile of electron bunches.

1. INTRODUCTION

In this lecture we shall present some mathematical tools which are particularly useful in the study of electron or positron dynamics in storage rings and apply them to some important problems. However the emphasis is on understanding the physical content rather than the mathematics itself. Accordingly, some more technical material has been placed in appendices.

It is assumed that the reader has some familiarity with general accelerator theory and has, in particular, been introduced to the phenomenon of synchrotron radiation and how it affects electron motion. Excellent introductions to these topics were given in the first School of this series. Among other introductions, the classic lectures of Sands are especially worth reading. Texts in Hamiltonian dynamics, classical and quantum electrodynamics provide the physical background while books on the theory of stochastic processes will give more details of some of the techniques employed here. Since a rigorous mathematical discussion of the latter would get us irrecoverably sidetracked, we shall adopt a formal approach, trusting to intuition for the meaning of words like "random" and "noise".

The last part of this lecture is devoted to the special topic of nonlinear wiggler which give rise to new dynamical phenomena amenable to description in terms of a Fokker-Planck equation and ideas from the stability theory of dissipative systems. Their use is limited to very large storage rings such as LEP and no such wiggler has yet been operated. However they open up an interesting new range of possibilities for controlling the parameters of the beams.

Radiation effects on betatron motion are not discussed in detail here because most of the mathematical techniques can be illustrated in connexion with synchrotron motion and the most interesting effects of nonlinear wigglers are longitudinal. We have taken the opportunity to provide a Hamiltonian formulation of synchrotron motion, specially geared for electron machines with localised RF cavities. This framework allows a natural development of longitudinal chromatic effects and the important notion of the damping aperture. Betatron motion was treated in Refs. 2, 10, 11 and, with the approach used here, in Ref. 13.
2. THE DYNAMICS OF ELECTRONS IN A STORAGE RING

We shall formulate the equations of motion for electrons (or positrons) in a storage ring. The Hamiltonian description of particle motion in a circular accelerator, regarded as a special configuration of external electric and magnetic fields, is familiar to the reader from other lectures in this School. Since it provides the shortest route from the general equations of motion of a charged particle in an electromagnetic field to the specific forms which these equations take in a storage ring (Hill equations for betatron motion etc.), we shall employ it freely to avoid a long recapitulation of basic accelerator physics.

On the other hand, our main interest here is the effect of synchrotron radiation on electron dynamics. And this cannot be described solely in the context of Hamilton's equations. We must add dissipative terms to describe the energy loss through radiation. Moreover, these terms must somehow include the essentially random nature of the photon emission process. Appropriate mathematical tools are found in the theory of stochastic processes, notably in stochastic differential equations and the associated Fokker-Planck equations. Here we shall follow the treatment outlined in Refs. 15 and 13 although the applications will be somewhat different; some alternative approaches to the mathematical side of the problem may be found in Refs. 16, 17 and 18.

2.1 Coordinate system and Hamiltonian

We shall use the curvilinear coordinate system of Courant and Snyder and follow the conventions of most of the standard optics programs (MAD, TRANSPORT, etc.).

![Fig. 1 The reference orbit and Courant-Snyder coordinate system](image)

For simplicity, we assume that the magnets are perfectly aligned in the sense that there exists a closed planar reference curve \( r_0(s) \), passing through all their centres which is also the closed orbit for a hypothetical reference particle of constant momentum \( p_0 \) which neither radiates nor couples to the RF...
accelerating fields. The position of a real particle of kinematic momentum \( p \) is then described by giving the azimuthal position \( s \) of the closest point on this curve and its radial and vertical deviations \( x \) and \( y \) from that point, as shown in Fig. 1.

If we neglect edge effects in the magnets and use the Coulomb gauge then, in many important cases, the fields can be described in the electromagnetic fields can be derived from a single scalar function, \(^9\) the canonical vector potential\(^\dagger\)

\[
A_s(x, y, t, s) = A_s \cdot e_s (1 + G(s)x) \\
= -\frac{p_0 c}{e} \left\{ xG(s) \left( 1 + G(s) \frac{x}{2} \right) + \frac{1}{2} K_1(s)(x^2 - y^2) + \frac{1}{6} K_2(s)(x^3 - 3xy^2) + \cdots \right\} \\
+ \sum_k \frac{e_0}{\omega_k} \delta_C(s - s_k) \cos(\omega_k t + \phi_k).
\] (2.1)

This includes only the fundamental accelerating mode of a set of RF cavities with peak voltages \( V_k \) located at positions \( s_k \); \( \delta_C \) is a \( \delta \)-function, periodic on the circumference \( C = 2\pi R \). Our assumption about the closed orbit of the reference particle can only hold if the normalised dipole field strength is equal to the curvature of the reference orbit:

\[
G(s) \equiv \frac{K''(s) \cdot X_0(s)}{|X_0(s)|}
\] (2.2)

(where primes denote derivatives with respect to \( s \)). Following Courant and Snyder,\(^9,3\) we take \( s \) as an independent variable so that \( (x, y, t) \) may be taken as canonical coordinates and \( (p_x, p_y, -E) \) as canonical momenta. The Hamiltonian of a particle with kinematic momentum \( p \) is

\[
H_s(x, y, t, p_x, p_y, -E; s) = (p + (e/c)A) \cdot e_s (1 + G(s)x) \\
= -\left( \frac{e}{c} \right) A_s(x, y, t, s) \\
- (1 + G(s)x) \sqrt{E^2/c^2 - m^2c^2 - p_x^2 - p_y^2}.
\] (2.3)

Before proceeding further, it is convenient to replace the energy, \( E \), by the magnitude of the total momentum \( p \); this requires a canonical transformation of one pair of variables,

\[
(t, -E) \mapsto (t, p),
\] (2.4)

effected by means of the generating function

\[
F_2(p, t) = -ct \sqrt{p^2 + m^2 c^2},
\] (2.5)

and resulting in the new Hamiltonian

\[
H(x, y, zt, p_x, p_y, p; s) = -\frac{e}{c} A_s(x, y, t(zt, p), s) - (1 + G(s)x) \sqrt{p^2 - p_x^2 - p_y^2}.
\] (2.6)

\(^\dagger\) This recently introduced\(^9\) term saves a good deal of circumlocution.
The new canonical variables are related to the old by
\[ p = \sqrt{E^2/c^2 - m^2c^2}, \]
\[ z_t = -ct\sqrt{1 - m^2c^2/E^2} \]
\[ = -(\text{instantaneous velocity}) \times (\text{time particle passes } s). \]
\[ (2.7) \]

\textit{i.e. } \text{\textit{z}}t \text{ \textit{is not} immediately related to the path length, except while the energy is constant. The explicit form of Hamilton’s equations in these variables is}
\[ x' = (1 + Gz)\frac{p_x}{\sqrt{p^2 - p_z^2 - p_y^2}} \approx (1 + Gz)\frac{p_x}{p} \]
\[ y' = (1 + Gz)\frac{p_y}{\sqrt{p^2 - p_z^2 - p_y^2}} \approx (1 + Gz)\frac{p_y}{p} \]
\[ z_t' = -(1 + Gz)\frac{p}{\sqrt{p^2 - p_z^2 - p_y^2}} \approx -(1 + Gz) \]
\[ p_z' = -G(p - p_0) - p_0(G^2 + K_1)x - \frac{1}{2}p_0K_2(x^2 - y^2) + \cdots \]
\[ p_y' = p_0K_1y + \frac{1}{2}p_0K_2xy + \cdots \]
\[ p' = -\sum_k \frac{e\hat{V}_k}{c}\delta_c(s - s_k)\sin(\omega_tz_t/c + \phi_k). \]
\[ (2.8) \]

With these variables, the reference momentum \( p_0 \) factors out of all terms in \( H \) except those describing the cavities. Thus, particle motion in magnets will be “geometric”, depending only on \( \delta = (p - p_0)/p_0 \) and not on the mass or absolute value of \( p \). The price paid comes through the more complicated expression of the time-dependence of \( A_s \) in (2.6). There, \( t(z_t, p) \) denotes the solution of (2.7) for \( t \) in terms of \( z_t \) and \( p \). This does not matter much since the motion of high energy electrons is extreme-relativistic and the third argument of \( A_s \) in (2.6) may be set equal to \(-z_t/c\). From this also follows the excellent approximation
\[ t' \approx (1 + Gz)/c \]
\[ (2.9) \]

For later convenience, let us define the normalised magnetic field strength \( b \) through
\[ B(x, y, s) = \nabla \times A = \frac{p_0c}{e}b(x, y, s) \]
\[ (2.10) \]
and also write \( b(x, y, s) \) for \( |b(x, y, s)| \).

\subsection{2.2 Statistical properties of incoherent synchrotron radiation}

We review the essential facts about incoherent synchrotron radiation\(^{10,2,5,6} \) and recast them in a notation suited for our present purposes.

As a particle is accelerated transversely in a magnetic field, it emits photons. Because this is a quantum-mechanical phenomenon, the emission times and the quanta of energy carried away by the
photons are random quantities. However certain average quantities such as the mean emission rate and the mean radiated power may be calculated with good accuracy\footnote{Provided the energies and magnetic fields are not too high.} within classical electrodynamics. In the classical picture, the accelerated particle emits a continuous beam of radiation in a narrow cone around its momentum vector. Quantum mechanically, the momentum vector of each photon is almost collinear with the particle's momentum.

Since the orbital quantum numbers of electrons in typical storage rings are very large, we may use classical arguments to construct the equations of motion provided we do not attempt to describe the emission process itself. In fact it will be represented simply as an instantaneous jump in energy. This is acceptable since, for an electron of energy $E = \gamma mc^2$ in a magnetic field $B = E/e\rho$, photon emission occurs within a time

$$\tau_\gamma \simeq \frac{\rho}{\gamma\epsilon} \ll \frac{1}{\Omega},$$

(2.11)

where $\Omega$ is a frequency characteristic of betatron or synchrotron oscillations. In addition, the fact that

$$\frac{1}{\tau_\gamma} \ll \omega_c,$$

(2.12)

where $\omega_c$ is the frequency corresponding to the critical energy (defined below), means that the frequency (or energy) spectrum of the photons is locally well-defined.$^2$

Let us now develop these ideas formally.

An individual photon emission event, in which a photon of energy $u_j$ is emitted at $s = s_j$, is specified by the ordered pair of random variables $(u_j, s_j)$. The distribution function of $(u_j, s_j)$ depends only on the local magnetic field and the particle's momentum. Since these conditions vary as the particle moves, there is no very meaningful way of relating time averages (along the trajectory of a given particle) and ensemble averages (over many hypothetical particles experiencing the same conditions).

For definiteness, let us define the expectation value of a dynamical variable $\hat{A}(x, y, z, p_x, p_y, p_z; s)$, associated with the instantaneous state of the particle, to be the average of $\hat{A}$ over all possible realisations of $(u_j, s_j)$, that is to say, all the ways in which the particle's photon emission history might occur, weighted appropriately. We denote such ensemble averages by

$$\langle A(x, y, z, p_x, p_y, p_z; s) \rangle \quad \text{or, more briefly and generally,} \quad \langle \hat{A} \rangle_{X_s}$$

(2.13)

as it suits us; here $X$ is a shorthand notation for the set of (usually canonical) variables describing the instantaneous state of the particle in whatever representation we happen to be using. The averaging is understood to be taken while the azimuthal position $s$ and the phase-space coordinates $X$—and thereby the magnetic field felt by the particle—are supposed fixed. Parameters characterising the synchrotron radiation may be regarded as dynamical variables of the particle since they too are determined by $X$ and $s$. For example, the critical energy,

$$u_c \overset{\text{def}}{=} \frac{3}{2} \frac{\hbar c p_0}{(mc)^3} p^2 b(x, y, s),$$

(2.14)

may be thought of as a parameter determining the overall scale of the distribution in energy of the photons which the particle has a propensity to emit.
The exact density of a given realisation (holding \( X \) fixed) in \((u, s)\) space is

\[
\Omega_X(u, s) = \sum_j \delta(s - s_j)\delta(u - u_j),
\]

(2.15)

where the sum is taken over all events which actually take place. Its expectation value is the distribution function of \((u_j, s_j)\) (more correctly termed the probability density function)

\[
(\Omega_X(u, s)) = N_X(s) f_X(u; s)/c,
\]

(2.16)

which factorises, reflecting the statistical independence of \(s_j\) and \(u_j\). Here,

\[
N_X(s) = \frac{5\sqrt{3}}{6} \frac{e^3}{mc^2\hbar} |B(x, y, s)| = \frac{5\sqrt{3}}{6} \frac{e^3 r_p c}{\hbar} b(x, y, s)
\]

(2.17)

is the distribution function of \(s_j\), or the average photon emission rate, and is independent of the particle's momentum; \(r_p = e^2/mc^2\) is the classical electron radius.

![Fig. 2 Distribution of photons in energy](image)

The distribution of photon energies, \(f_X(u; s)\), is closely related to the classical frequency spectrum of synchrotron radiation

\[
f_X(u; s) = \frac{5\sqrt{3}}{8} \frac{(mc)^3}{\hbar c p_0} \frac{F(u/u_c)}{p^2 b(x, y, s)},
\]

(2.18)

where we follow the standard definition\(^2\)

\[
F(\xi) \stackrel{\text{def}}{=} \frac{S(\xi)}{\xi}, \quad S(\xi) \stackrel{\text{def}}{=} \frac{9\sqrt{3}}{8\pi} \int_0^\infty K_{5/3}(\zeta) d\zeta
\]

(2.19)

and \(K_{5/3}\) is a modified Bessel function (see e.g. Refs. 2, 5, 10 for derivations of (2.17) and (2.18)). The universal functions \(F(\xi)\) and \(S(\xi)\) are plotted in Fig. 2; note that a non-algebraic dependence of \(f_X(u; s)\) on the momentum and magnetic field arises through the factor \(u_c\) in the argument of \(F\).
The instantaneous radiated power is

$$P_X(s) = c \int_0^\infty u \Omega_X(u, s) \, du = c \sum_j u_j \delta(s - s_j). \quad (2.20)$$

A typical realisation of $P_X(s)$, obtained by simulation, is shown in Fig. 3; the parameters are such that the expectation value of the number of photons emitted in one revolution period is

$$N_X(s)T_0 = 1000 \quad (2.21)$$

Counting the peaks, we find 1049, a 1.55σ deviation. In preparing the figure, $\tau_\gamma$ was been taken to be fixed and equal to the width of a line on the printer so that the distribution of the heights of the peaks, displayed in units of the critical energy $u_c$, is given by the function $f_X(u; s)$ defined in (2.18). It is worth remarking that, although half the energy is carried away by photons with energies greater than $u_c$, there are few such photons. In fact 91% of the photons have $u < u_c$ and 50% have $u < 0.1 u_c$.

The expectation value of $P_X(s)$ is the classical power, given by the relativistic Larmor formula. Using the Lorentz force equation in 4-vector form ($p^\mu = (E/c, p)$ and proper time $\tau$), for a particle in a purely magnetic field with $p \cdot B = 0$

$$\frac{dp^\mu}{d\tau} = \frac{e}{mc} (0, p \times B), \quad (2.22)$$

we express this in terms of the canonical variables

$$\langle P_X(s) \rangle = \frac{2}{3} \frac{e^2}{m^2 c^3} \frac{dp^\mu dp_\mu}{d\tau} = \frac{2}{3} \frac{e^4}{m^4 c^3} (p \times B)^2$$

$$= \frac{2}{3} \frac{e^2 r_0^2 p^2}{(mc)^3} |B(x, y, s)|^2 = \frac{2 r_0^2 p^2}{3 m^2 c^2} b(x, y, s)^2 \quad (2.23)$$

$$= N_X(s)(u) \chi$$
The last form uses the mean photon energy

$$
(u)_{\chi s} = \int_0^\infty u f_X(u; s) du = \frac{4}{5\sqrt{3}} \frac{\hbar c p_0}{(mc)^3} p^2 b(x, y, s) = \frac{8}{15\sqrt{3}} u_c.
$$

(2.24)

Similarly, the mean-square quantum energy is the second moment of $f_X(u; s)$:

$$
\langle u^2 \rangle_{\chi s} = \int_0^\infty u^2 f_X(u; s) du = \frac{11}{12} \frac{(\hbar c p_0)^2}{(mc)^6} p^4 b(x, y, s)^2.
$$

(2.25)

The exact power (2.20) may be split into its mean and fluctuating parts:

$$
P_X(s) = \langle P_X(s) \rangle + \hat{P}_X(s),
$$

(2.26)

where $\hat{P}_X(s)$ is just the difference between the classical power (2.23) and the instantaneous power in a given realisation.

The two-time correlation function of such a quantity is given by (a generalised version of) Campbell’s Theorem,\(^{21,9}\) closely related to the well-known Schottky formula,

$$
\langle \hat{P}_X(s) \hat{P}_X(s') \rangle = c N_X(s) (u^2)_{\chi s} \delta(s - s')
$$

$$
= \frac{55}{24\sqrt{3}} \frac{r_r \hbar c p_0^3}{(mc)^6} p^4 b(x, y, s)^3 \delta(s - s').
$$

(2.27)

The $\delta$-function expresses the fact that $\hat{P}_X(s)$ and $\hat{P}_X(s')$ are uncorrelated when $s \neq s'$. This really means $|s - s'| \gg c r_\gamma$ with $r_\gamma$ as defined in (2.11).

Let us introduce a unit noise source, $\xi(s)$, known technically\(^{8,7}\) as a centred, Gaussian Markov process. It is defined to have the formal properties

$$
\langle \xi(s) \rangle = 0, \quad \langle \xi(s) \xi(s') \rangle = \delta(s - s').
$$

(2.28)

with respect to our ensemble-averaging operation $\langle \ldots \rangle$. With this, we can correct a formal representation of the stochastic power which reproduces the essential properties derived above, namely (2.23) and (2.27),

$$
P_X(s) = p^2 c^2 [c_1 b(x, y, s)^2 + \sqrt{c_2} |b(x, y, s)|^{5/2} \xi(s)]
$$

(2.29)

where the constants\(^*\) $c_1$ and $c_2$ are

$$
c_1 = \frac{2r_r \hbar c p_0^3}{3(mc)^5}, \quad c_2 = \frac{55r_r \hbar c p_0^3}{24\sqrt{3}(mc)^6}.
$$

(2.30)

Statistically, there is no way of distinguishing our conceptual model of discrete random photon emission and this formal representation.

Noting that $c_2 \propto \hbar$, we see that, in this formalism, the classical radiation power has been corrected by a stochastic term of order $\sqrt{\hbar}$. We also observe that, in general, the average radiation power and its quantum fluctuations depend nonlinearly on the particle’s coordinates through the spatial dependences of the magnetic field.

\(^*\) These are not “fundamental” constants because they still depend on the absolute value of the bending field in the ring through the reference momentum $p_0$. 


2.3 Radiation reaction forces

Now that we know the distribution of photons, we can include their effect on the motion of the electrons by adding radiation reaction forces to Hamilton's equations (2.8).

A single photon emission of energy \( u_j \) at azimuth \( s = s_j \) (when \( t = t_j \), say) will produce an abrupt (since \( r_\gamma \) is short) change in the momentum but leave the spatial position of the particle unchanged. At high energy the opening angle of the beam of radiation is

\[
\theta_{\text{max}} \approx \frac{1}{2\gamma} \approx 0.26 \text{ mrad} \quad \text{at E}_0 = 1 \text{ GeV}. \tag{2.31}
\]

It is therefore an excellent approximation to take the photon's 3-momentum vector \( u_j / c = u_j p / pc \) to be collinear with the momentum \( p \) and apply momentum conservation to evaluate the changes in the canonical momenta

\[
\begin{align*}
    p & \rightarrow p - u_j / c, \\
    p_x & \rightarrow p_x - \frac{u_j}{c} p_x = p_x - \frac{u_j}{c} \frac{x'}{\sqrt{1 + (x')^2 + (y')^2}} = p_x - \frac{u_j x'}{c^2 t'}, \\
    p_y & \rightarrow p_y - \frac{u_j y'}{c^2 t'}.
\end{align*} \tag{2.32}
\]

If we consider a time-interval surrounding the moment of photon emission, which is so short that the probability of more than one photon being emitted can be neglected, then we know that the energy of the emitted photon is equal to the time-integral of the fluctuating radiation power (2.20),

\[
    u_j = \int_{s_j - \varepsilon}^{s_j + \varepsilon} P_X(s) \, ds / c \quad \text{(with probability } \rightarrow 1 \text{ as } \varepsilon \rightarrow 0^+) \tag{2.33}
\]

and we may reinterpret (2.32) as stochastic differential equations

\[
\begin{align*}
    dp &= -P_X(s) \, dt / c = -P_X(s) t' \, ds / c = -P_X(s) x'_t \, ds / c^2 + O(\gamma^{-2}) \, ds, \\
    dp_x &= -P_X(s)(x'/t') \, dt / c^2 = -P_X(s) x' \, ds / c^2, \\
    dp_y &= -P_X(s)(y'/t') \, dt / c^2 = -P_X(s) y' \, ds / c^2.
\end{align*} \tag{2.34}
\]

To complete the equations of motion, we must restore the forces due to the direct action of the external fields given by Hamilton's equations, \( x' = \partial H / \partial p_z \), etc. In this way we find

\[
\begin{align*}
    x' &= \frac{\partial H}{\partial p_z}, & p'_t &= -\frac{\partial H}{\partial x} - \frac{P_X(s) \, \partial H}{c^2 \, \partial p_z}, \\
    y' &= \frac{\partial H}{\partial p_y}, & p'_t &= -\frac{\partial H}{\partial y} - \frac{P_X(s) \, \partial H}{c^2 \, \partial p_y}, \\
    z'_t &= \frac{\partial H}{\partial p}, & p'_t &= -\frac{\partial H}{\partial z_t} + \frac{P_X(s) \, \partial H}{c^2 \, \partial p}. \tag{2.35}
\end{align*}
\]

The Hamiltonian part of these equations has already been written out explicitly in (2.8), but it is instruc-

\* We are neglecting a very, very small increase in \( z_t \) due to the small reduction in the velocity of the electron; see (2.7). It is easy to check that this is utterly negligible.
tive to write out the radiation terms in detail:

\[ p'_x = -\frac{\partial H}{\partial x} - (1 + Gz)p_p [c_1 b(x, y, s)^2 + \sqrt{c_2} b(x, y, s)^{3/2} \xi(s)] \stackrel{\text{def}}{=} -\frac{\partial H}{\partial x} + \frac{p_0}{c} \Pi_x, \]

\[ p'_y = -\frac{\partial H}{\partial y} - (1 + Gz)p_p [c_1 b(x, y, s)^2 + \sqrt{c_2} b(x, y, s)^{3/2} \xi(s)] \stackrel{\text{def}}{=} -\frac{\partial H}{\partial y} + \frac{p_0}{c} \Pi_y, \]

\[ p' = -\frac{\partial H}{\partial z_1} - (1 + Gz)p_p [c_1 b(x, y, s)^2 + \sqrt{c_2} b(x, y, s)^{3/2} \xi(s)] \stackrel{\text{def}}{=} -\frac{\partial H}{\partial y} - \frac{p_0}{c} \Pi_1, \] (2.36)

This also serves to define the radiation coupling functions $\Pi_x$, $\Pi_y$ and $\Pi_1$.

Notice the dependence on the canonical momenta—this is at the root of Robinson's Theorem on the damping partition numbers. \(^{22,2}\)

### 3. NORMAL MODES AND OPTICAL FUNCTIONS

In principle, the equations (2.35) completely describe electron motion under the combined influences of the applied electromagnetic fields and synchrotron radiation. At a fundamental level their physical content is manifest but they are not in a form suitable for many practical calculations. Other lectures in this School have shown how useful it is to describe particle trajectories first of all in terms of the three normal modes of linearised motion around the closed orbit in the 6-dimensional phase space and then in terms of the optical functions which characterise the storage ring lattice and determine the frequencies of these modes. In a planar ring with $x$-$y$ coupling terms, such as we have assumed, these are the familiar modes of linearised betatron and synchrotron motion.

In the coordinates $(x, y, z_1)$, the Hamiltonian contains linear coupling terms between $x$ and $p$ due to the spectrometer effect of the (horizontal) bending magnets, but these may be eliminated by introducing the dispersion functions.

The remainder of this section may be skimmed by the reader who does not wish to be convinced of each step in the introduction of the dispersion, functions and the concepts of synchrotron and betatron motion as they emerge in the Hamiltonian formulation used here. He will be familiar with these notions from other lectures in the School. The following sections are included principally to cover certain aspects peculiar to electron machines.

Behind the formalism, however, there lie a few key physical ideas which are essential to the understanding of what follows. In particular the reader should be aware of the distinction between the two components of the momentum deviation $\delta_x$ and $\delta$ (to be introduced) and the way in which the energy loss by synchrotron radiation is coupled into the transverse oscillations through the dispersion functions.

#### 3.1 Synchrotron motion

Let us simplify the Hamiltonian (2.6) by neglecting higher order kinematic terms in the transverse momenta and in the momentum deviation $\delta$. Accordingly, we approximate the square root term by

\[
(1 + Gz)\sqrt{p^2 - p_z^2 - p_p^2} = (1 + Gz)p - \frac{p_z^2 + p_p^2}{2p} + \cdots \tag{3.1}
\]

Before writing down the Hamiltonian, we perform a simple rescaling of variables which makes all the
momenta dimensionless:

\[ H \mapsto H_1 = H/p_0, \quad p_x \mapsto p_x = p_x/p_0, \quad p_y \mapsto p_y = p_y/p_0, \quad p \mapsto P = p/p_0. \tag{3.2} \]

The canonical coordinates and the independent variable all have dimensions of length and the Hamiltonian is

\[ H_1(x, y, z_1, p_x, p_y, P; s) \simeq -G x (P - 1) + \frac{p_x^2 + p_y^2}{2P} \]
\[ + \frac{C^2 x^2}{2} + \frac{1}{2} K_1(x^2 - y^2) + \frac{1}{6} K_2(x^3 - 3xy^2) + \cdots \tag{3.3} \]
\[ - \sum_k \frac{eV_k}{\hbar \omega_k} \delta_C(s - s_k) \cos(\omega_k z_1/c + \phi_k). \]

The dispersion functions \( \eta \) and \( \zeta \) are designed to eliminate the linear coupling appearing in the first term. Some higher-order couplings can be eliminated at the same time by allowing these functions to depend on momentum\(^{20}\) and this approach is often used in nonlinear optics studies when synchrotron motion is neglected.\(^*\) On the other hand, for electron rings, where the value of \( p \) oscillates relatively rapidly and certainly must be included as a dynamical variable, it appears at first sight that the simplest approach would consist in defining \( \eta \) and \( \zeta \) with respect to the reference momentum \( p_0 \). This avoids having a Hamiltonian which depends on a canonical momentum through functions which have to be calculated (and, eventually, differentiated) numerically.\(^\dagger\)

However we can do a little better than this if we recognize that, depending on the precise value of the RF frequency, the equilibrium momentum of the beam may not be equal to \( p_0 \); synchrotron oscillations will then take place around a slightly different value of the momentum which we shall denote as \( p_0(1 + \delta_s) \), with \( \delta_s \ll 1 \). With this in mind, the dispersions may be introduced by means of a canonical transformation

\[ (x, y, z_1, p_x, p_y, P) \mapsto (x, y, z, p_x, p_y, \varepsilon) \tag{3.4} \]

whose generating function is

\[ F_S(p_x, p_y, \varepsilon, x, y, z_1) = p_x[x - \eta(\delta_s, s)(\delta_s + \varepsilon)] + x\zeta(\delta_s, s)(\delta_s + \varepsilon) + p_y y \]
\[ + (1 + \delta_s + \varepsilon)[z_1 + Z_0(s)]. \tag{3.5} \]

In this expression, \( Z_0(s), \eta(\delta_s, s) \) and \( \zeta(\delta_s, s) \) are as yet unspecified functions of \( s \); natural choices for them will emerge in the following. In order to take proper account of chromatic effects in cases where the equilibrium value of \( p \) is other than \( p_0 \), they have also been allowed to depend parametrically on the constant \( \delta_s \). Later we shall show how the value of \( \delta_s \) is determined naturally by the RF frequency. When the equations of motion are constructed from the new Hamiltonian there is no need to differentiate \( \eta \) or \( \zeta \) with respect to \( \delta_s \).

\* E.g. for hadron colliders where the synchrotron oscillation frequency is very low.

\† Despite what is said in the following paragraphs, it may yet prove convenient to define the dispersion in this way because it provides a measure of by how much the equilibrium orbit differs from the reference orbit, presumed to pass through the reference points of the beam position monitors at the centres of the magnet apertures. In practice a value quoted for the dispersion will almost always be this one, denoted below as \( \eta(0, s) \).
The new coordinates and momenta are given by
\[
\begin{align*}
x'_\beta &= \frac{\partial F_2}{\partial p_\beta} = x - \eta(\delta_s, s)(\delta_s + \epsilon), & P_x &= \frac{\partial F_1}{\partial \epsilon} = p_\beta + \zeta(\delta_s + \epsilon), \\
y' &= \frac{\partial F_2}{\partial p_y} = y, & P_y &= \frac{\partial F_1}{\partial \gamma} = p_y, \\
z' &= \frac{\partial F_2}{\partial \epsilon} = z + Z_0(s) - \eta p_\beta + \zeta x_\beta + \eta \gamma(\delta_s + \epsilon), & P &= \frac{\partial F_1}{\partial \zeta} = 1 + \delta_s + \epsilon, \\
\end{align*}
\]  
(3.6)

\[
H_2 = H_1 + \frac{\partial F_2}{\partial \delta_s} = H - p_\beta(\delta_s + \epsilon) + z' + \eta(\delta_s + \epsilon) \zeta' \quad (3.7)
\]

The splitting of \( x \) into its betatron and "energy" components should be familiar. It is perhaps less well-known that, in order to preserve the canonical structure (symplecticity), one must also use a new longitudinal coordinate \( z \) which takes account of local changes in the length of the particle's orbit due to its betatron oscillations. Since there is no vertical bending in our perfect machine, there is no vertical dispersion and the \( y \) transformations are trivial.

Expressing the new Hamiltonian \( H_2 \) in terms of the new coordinates, we can eliminate coupling terms linear in \( x_\beta \) by imposing the conditions
\[
\frac{\partial H_2}{\partial x_\beta} = \frac{\partial H_2}{\partial p_\beta} = 0 \quad \text{for} \quad x_\beta = p_\beta = y = p_y = z = \epsilon = 0. \quad (3.8)
\]

Writing these out explicitly, we find that \( \eta \) and \( \zeta \) must satisfy first-order differential equations and a periodicity condition,
\[
\eta' = \frac{\zeta}{1 + \delta_s}, \quad \zeta' = G - (K_1 + G^2)\eta - \frac{1}{2}K_2\eta^2\delta_s, \quad \eta(\delta_s, s + 2\pi R) = \eta(\delta_s, s), \quad (3.9)
\]

which are nothing but the familiar equations defining the dispersion function; we emphasise again that \( \delta_s \) appears as a parameter and that primes denote differentiation with respect to \( s \). A common practical means of determining these functions for a range of values of \( \delta_s \) is to expand them as
\[
\eta(\delta_s, s) = \eta_0(s) + \eta_1(s)\delta_s + \ldots, \quad \zeta(\delta_s, s) = \zeta_0(s) + \zeta_1(s)\delta_s + \ldots, \quad (3.10)
\]

and equate coefficients of \( \delta_s \) in the equations (3.9). Then each function may be evaluated once and for all, independently of \( \delta_s \).

From (2.36) and (3.6) it is straightforward to work out the new equations of motion
\[
\begin{align*}
x'_\beta &= \frac{\partial H_2}{\partial p_\beta} + \Pi_\zeta \eta/c, & p'_\beta &= -\frac{\partial H_2}{\partial x_\beta} + (\Pi_x + \Pi_\zeta \zeta)/c, \\
y' &= \frac{\partial H_2}{\partial p_y}, & p'_y &= -\frac{\partial H_2}{\partial y} + \Pi_y/c, \\
z' &= \frac{\partial H_2}{\partial \epsilon} - \Pi_\zeta \eta/c - \Pi_\zeta \zeta \eta/c, & \epsilon' &= -\frac{\partial H_2}{\partial \zeta} - \Pi_\zeta/c. \quad (3.11)
\end{align*}
\]

Defining an effective quadrupole gradient for a particle with the reference momentum which happens

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** Sometimes \( \eta \) is denoted \( D, \psi \) or \( \omega_\eta(s) \) but there are no other notations for the conjugate function \( \zeta(\delta_s, s) \); we have introduced it to exhibit the fact that (3.9) can themselves be derived from a Hamiltonian.

† Alternatively we could use the transformation theory of Appendix B.

‡ Had we included octupole fields, it would also have been natural to define an effective sextupole gradient in a similar fashion.
to be at the position of the off-momentum orbit\textsuperscript{5}
\begin{equation}
k_1(\delta_s, s) = K_1(s) + \frac{1}{2} K_2(s) \eta(\delta_s, s) \delta_s
\end{equation}
we find, after a good deal of algebra, exploiting the cancellations implied by (3.8) and dropping several
constant terms, that the Hamiltonian is
\begin{equation}
H_2(x_\beta, y, z, p_\beta, p_y, \epsilon; s) = \frac{1}{2} \left[ \frac{s^2}{1 + \delta_s} - \left( G^2 + k_1 \eta^2 \right) \right] \left( \delta_s + \epsilon \right)^2 + \left[ Z_0(\delta_s) - 1 \right] \epsilon
+ \frac{1}{6} K_2(\epsilon - \delta_s/2) (\delta_s + \epsilon)^2
+ \frac{p_\beta^2 + p_y^2}{2(1 + \delta_s)} + \frac{G^2 x_\beta^2}{2} + \frac{1}{2} k_1 (x_\beta^2 - y^2) + \frac{1}{6} K_2 (x_\beta^2 - 3 \epsilon \delta_s y^2)
- \sum_k \frac{\epsilon V_k}{p_0 \omega ft} \delta_C(s - s_k) \cos \left\{ \frac{\omega ft}{c} \left[ z - Z_0(s) + \eta p_\beta - \zeta p_y - \xi \delta_s + \phi_k \right] \right\}.
\end{equation}
Although the terms describing betatron motion are simplified, the local formulation of synchrotron motion
appears fairly complicated. However we recall that, in order to avoid dangerous synchro-betatron coupling
effects, storage rings are almost always designed so that the dispersion functions vanish at the locations
of the RF cavities:
\begin{equation}
\eta(s_k) = \zeta(s_k) = 0, \quad \text{for each } k.
\end{equation}
[In practice, of course, imperfections will usually create some horizontal and vertical dispersion in the
cavities.] Then, thanks to the \( \delta \)-functions, the phase of the cosine describing the RF waveform simplifies to
\begin{equation}
\Phi(z, s) = \frac{\omega ft}{c} \left[ z - Z_0(s) + \phi_k \right],
\end{equation}
and all coupling effects between the longitudinal and transverse motions have been eliminated. We remain
free to choose the function \( Z_0(s) \) to our best advantage. A formal analogy with (3.8) prompts us to demand that
\begin{equation}
\frac{\partial H_2}{\partial \epsilon} = 0 \quad \text{for} \quad x_\beta = p_\beta = y = p_y = z = \epsilon = 0.
\end{equation}
The physical interpretation of this condition is clear if we notice that (in the absence of radiation effects)
the change in \( z \) around an orbit will be
\begin{equation}
z(2\pi R) - z(0) = \int_0^{2\pi R} z' \, ds = \int_0^{2\pi R} \frac{\partial H}{\partial \epsilon} \, ds,
\end{equation}
and the condition (3.16) determines a shift in the origin of phase space to a fixed point of the mapping
which describes the evolution of the phase space coordinates over one turn. Moreover we are insisting
that this hold true at every point on the circumference. Alternatively, we can describe this as a canonical
transformation to a reference frame moving with the synchronous particle.

\textsuperscript{5} A further step might be to divide the Hamiltonian and all magnetic field terms by \((1 + \delta_s)\) so that the rôle of the original reference momentum \( p_0 \) would be taken over by \( p_0(1 + \delta_s) \). This is sometimes convenient. However since it is often useful
to take advantage of the properties of the separated-function lattice which is used for most electron machines, we shall
refrain from taking this step. The interested reader may consult Ref. 20.
Working out (3.16) explicitly, we find that the effects of the sextupole terms in the longitudinal part of the Hamiltonian cancel, leaving us with

$$\left\{ \frac{s^2}{1 + \delta_s} - (G^2 + k_1) \eta^2 \right\} \delta_s + Z_0(s) - 1 = 0,$$

which is straightforwardly integrated to yield

$$Z_0(s) = -z_s + s - \delta_s \int_0^s \Gamma(\delta_s, s) \, ds,$$

where $z_s$ is a constant, related to the stable phase angle and we defined the local path length slippage function by

$$\Gamma(\delta_s, s) \overset{\text{def}}{=} \frac{s^2}{1 + \delta_s} - (G^2 + k_1) \eta^2.$$

In Appendix C we give the details of the Fourier analysis of this function, showing that its average value is just the negative of the momentum compaction factor, $\alpha_c$, and how one can construct the smooth approximation to synchrotron motion from this local description.

Neglecting unimportant sextupole terms, the Hamiltonian for local synchrotron motion is now

$$H_s(x, \varepsilon, s) = \frac{1}{2} \Gamma(\delta_s, s) \varepsilon^2 - \sum_k \frac{e V_k}{p_0 c \omega_{rf}} \delta C(s - s_k) \cos \left\{ \frac{\omega_{rf}}{c} (z + z_s) - \frac{\omega_{rf}}{c} s + \frac{\omega_{rf} \delta_s}{c} \int_0^s \Gamma(\delta_s, \sigma) \, d\sigma + \phi_k \right\}.$$

Finally the requirement that this Hamiltonian be periodic in $s$,

$$H_s(x, \varepsilon, s + 2\pi R) = H_s(x, \varepsilon, s),$$

means that the argument of the cosine must advance by an integer multiple of $2\pi$ per revolution. In the limit $\delta_s \to 0$, the RF frequency has to be an integer multiple of the revolution frequency on the reference orbit:

$$\omega_{rf} = 2\pi f_{rf} = 2\pi h f_0 = \frac{hc}{R}, \quad (\delta_s = 0),$$

where $h$ is called the harmonic number. From this it follows that (3.22) is equivalent to

$$- \frac{\omega_{rf}}{c} (2\pi R) + \frac{\omega_{rf} \delta_s}{c} \int_0^{2\pi R} \Gamma(\delta_s, s) \, ds = -2\pi h$$

$$\Leftrightarrow \quad \delta_s = \frac{1}{\alpha_c(\delta_s)} \left( 1 - \frac{hc}{\omega_{rf} R} \right)$$

The equilibrium momentum of the beam may be determined by small shifts of the RF frequency. Writing
\[ f_{\text{f}} = f_0 + \Delta f_{\text{f}}, \] we may write the familiar linearised version of this relationship,

\[ \delta s \approx -\frac{1}{\alpha_c(0)} \frac{\Delta f_{\text{f}}}{f_{\text{f}}}, \] (3.25)

and exhibit the dependence of the average radius of the equilibrium orbit on the momentum compaction factor:

\[ R(\delta s) \overset{\text{def}}{=} \frac{hc}{\omega_{\text{f}}} = R(1 + \alpha_c(\delta s) \delta s), \quad \frac{1}{R} \frac{dR}{d\delta s} = \alpha_c(\delta s). \] (3.26)

In Appendix C, the details of the phasing of the RF cavities are worked out and it is shown that, in smooth approximation, one may replace (3.13) with the simplified Hamiltonian

\[
H_3(x, y, z, p_x, p_y, \epsilon; s) = -\frac{\alpha_c e^2}{2} - \frac{e\gamma}{2\pi \rho_c c h} \cos(h(z + z_s)/R) \\
+ \frac{p_x^2 + p_y^2}{2(1 + \delta)} + \frac{\gamma^2 x^2}{2} + \frac{1}{2} k_1(x^3_y - y^3) + \frac{1}{6} k_2(x^3 - 3x^2 y^2).
\] (3.27)

Including the radiation reaction effects, the equations of motion are given by (3.11). To make the radiation terms explicit, we have to work forward through the chain of variable substitutions from the original forms of \( \Pi_x \) and \( \Pi_t \) as functions of \( (x, p, s) \). These operations are deferred to the next section.

In this formulation of synchrotron motion it might appear that we are always above transition energy and have somehow neglected the possibility of the revolution frequency’s increasing with momentum as it does below transition. This appearance is only a consequence of our having used \( s \), and not time \( t \), as independent variable. Transition energy does indeed occur when the increase in time taken to cover a greater orbit length due to a momentum deviation is exactly compensated by the greater velocity of the particle on that orbit. If the transformation of independent variable is made, and if higher order terms are included in (2.9), the velocity (and hence the familiar \( \gamma^{-2} \) factor) enters explicitly. A canonical transformation then restores variable sign in the coefficient of \( \epsilon^2 \).

4. RADIATION DAMPING

The deterministic parts of the equations (3.11) show how the transverse momenta are damped directly by photon emission and how, moreover, the dispersion function couples the longitudinal damping into the radial phase space. As mentioned in the Introduction, we shall not discuss betatron motion any further than this.

4.1 Damping in longitudinal phase space

Let us set \( x_\theta = p_\theta = y = p_y = 0 \) and study the effects of radiation on the dynamics of longitudinal phase space in smooth approximation. Carrying through the changes of variables, we find from (2.36),
(3.6), (3.11) and (3.27) that the deterministic parts of the equations of synchrotron motion are

\[ e' = \frac{e\hat{V}}{2\pi R(\delta_s)p_0c} \sin\left(h(z + z_s)/R\right) - \frac{1}{2\pi R} \int_0^{2\pi R} ds \left(\Pi_t/c\right), \]

\[ z' = -\alpha_c e - \frac{1}{2\pi R} \int_0^{2\pi R} ds \left(\Pi_\eta/c - \Pi_\zeta\eta/c\right). \]

(4.1)

The only dissipative term which does not average out is that in the equation for \( e' \) and is simply related to the average energy loss. Integrating the definition of \( \Pi_t \) contained in (2.35) and (2.36) we find that

\[ p_0 \int_0^{2\pi R} \Pi_t ds = \int_0^{2\pi R} P_X(s) t' ds = U(\delta_s + \varepsilon) \]

(4.2)

is the energy loss per turn of a particle with total momentum \( p_0(1 + \delta_s + \varepsilon) \).

The normalised magnetic field strength at a displacement \( z \) in the median plane is

\[ b(x, 0, s) = G(s) + K_1(s)x + \frac{1}{2}K_2(s)x^2 + \cdots. \]

(4.3)

Taking into account the energy lost in the lattice dipoles and quadrupoles, we can write out the first few terms in the expansion of the expectation value in powers of \( \delta_s \) and \( \varepsilon \):

\[ U(\delta_s + \varepsilon) \approx p_0^2 c \int_0^{2\pi R} ds \left\{ 1 + (2 + G(s)\eta)(\delta_s + \varepsilon) \right\} c_1 b(\eta(\delta_s + \varepsilon), 0, s)^2 \]

(4.4)

\[ = c_1 p_0^2 c \left\{ I_2 + \delta_s(2I_2 + I_4) + \varepsilon(2I_2 + I_4) + (\delta_s^2 + 2\varepsilon\delta_s)(I_2 + 2I_4 + I_6) + O(\varepsilon^2) \right\}. \]

The arguments of \( \eta \) have been suppressed and the definitions of the synchrotron radiation integrals \( I_2, I_4 \) and \( I_6 \) will be found in Appendix D.

A fixed point of the equations (4.1) is a point where \( e' = z' = 0 \) and it is easy to see that one exists on the line \( e = 0 \) in the phase plane. The still undetermined constant \( z_s \) can now be chosen according to

\[ e\hat{V} \sin(hz_s/R) = U(\delta_s) = c_1 p_0^2 c \left\{ I_2 + (2I_2 + I_4)\delta_s + \delta_s^2(2I_4 + I_6) \right\} \]

(4.5)

so as to move the origin to this natural position.

In the case of a stable fixed point, \( \phi_s = hz_s/R \) is called the stable phase angle. A particle which maintains this phase relationship with the RF wave will find that its acceleration just balances its average energy loss by synchrotron radiation.
Introducing the damping partition number \cite{2,22,25},

\[
J_\varepsilon(\delta_s) = \frac{d \log U(\delta_s)}{d\delta_s} = 2 + \frac{I_4}{I_2} + \delta_s \left( \frac{2 I_8}{I_2} - \frac{I_8^2}{I_4^2} - 2 \right) + O(\delta_s^2),
\]

we can linearise (4.1) (still neglecting the fluctuation terms) to find

\[
z' = -\alpha_c \varepsilon, \quad \varepsilon' = -\frac{eV \cos \phi_s}{2\pi R^2 h p_0 c} z - J_\varepsilon(\delta_s) \frac{U(\delta_s)}{p_0 c^2} f_0 \varepsilon.
\]

These, of course, are the equations of a damped linear oscillator with natural frequency \( \Omega_s \) given by

\[
e^2 \Omega_s^2 = \frac{\alpha_c eV}{2\pi R^2 h p_0 c} \cos \phi_s,
\]

and are equivalent to a single second-order differential equation

\[
\ddot{z} + 2\alpha_c \dot{z} + \Omega_s^2 z = 0,
\]

where the damping rate \( \alpha_c \) and the damping time \( \tau_c \) are defined by

\[
\alpha_c \overset{\text{def}}{=} \frac{1}{\tau_c} = \frac{J_\varepsilon}{2\tau_c} = \frac{J_\varepsilon(\delta_s) \alpha_c}{2} \overset{\text{def}}{=} \frac{J_\varepsilon(\delta_s) U(\delta_s)}{2 p_0 c} f_0 = \frac{r_c f_0}{3} \left( \frac{p_0}{mc} \right)^3 (2I_2 + I_4).
\]

The damping rate \( \alpha_c \) coincides with the quantity \( \alpha_\varepsilon \) when \( J_\varepsilon = 2 \). This case is a useful reference point, as will be explained below.

### 4.2 Damping partition numbers and damping aperture

Although we have not given the derivations here, damping partition numbers analogous to (4.6) also exist for the radial and vertical betatron oscillations.\cite{10} While the damping is linear, they satisfy the sum rule known as Robinson's Theorem:

\[
J_x(\delta_s) + J_y(\delta_s) + J_z(\delta_s) = 4.
\]

for all values of \( \delta_s \) such that an off-momentum closed orbit given by \( \eta(\delta_s, a) \) exists. This holds even if one of the partition numbers is negative. In most lattice designs, \( J_y = 1 \).

By varying the RF frequency, and thereby \( \delta_s \), it is therefore possible to redistribute the damping between the longitudinal and radial modes. In a storage ring, one must ensure that each damping partition number remains positive. The range of values of \( \delta_s \) in which this is true is called the damping aperture, and is determined by the values of the synchrotron integrals \( I_2, I_4 \) and \( I_8 \):

\[
-\frac{2I_2 + I_4}{2I_8} < \delta_s < \frac{I_2 - I_4}{2I_8}.
\]

Together with (3.25), this translates directly into an allowable range of variation of RF frequencies or an allowable displacement of the equilibrium orbit. For further details, including the use of Robinson wigglers to shift the damping aperture, the reader may consult Ref. 10.

In small storage rings, the physical aperture is usually smaller than the damping aperture.

The damping aperture is easily measured by varying the RF frequency and watching for beam blow-up on a synchrotron light monitor.
5. QUANTUM FLUCTUATIONS AND FOKKER-PLANCK EQUATIONS

When certain conditions are satisfied, sets of stochastic ordinary differential equations can be replaced by a partial differential equation for a distribution function on phase space. In the limit of vanishing correlation time of the random terms, this partial differential equation takes the form of a Fokker-Planck equation. Its physical meaning and precise relation to the stochastic equations are discussed in Appendix D. Fokker-Planck equations have been applied to several problems in accelerator physics; for some examples, see Refs. 12 (several articles), 14, 17, 23 and the reference lists which they contain.

5.1 Quantum fluctuations in longitudinal phase space

We now reintroduce the fluctuating part of the radiation power into the longitudinal equations of motion. To prepare the ground for writing down the Fokker-Planck equation, we write them in the form

\[ z' = K_z(z, \varepsilon) + Q_z(z, \varepsilon)\xi(s), \quad \varepsilon' = K_\varepsilon(z, \varepsilon) + Q_\varepsilon(z, \varepsilon)\xi(s) \]  

(5.1)

where the \( K_\cdot \) and \( Q_\cdot \) functions are

\[ K_z(z, \varepsilon) = -\alpha_z \varepsilon, \quad K_\varepsilon(z, \varepsilon) = (\Omega_z/c)^2 z - \frac{J_\varepsilon}{c} \varepsilon \]
\[ Q_z(z, \varepsilon) = 0, \quad Q_\varepsilon(z, \varepsilon) = -p_0 \sqrt{c_0 G(s)^3} \]  

(5.2)

Strictly speaking, these are in something of a hybrid form since the smooth approximation has not yet been applied to the fluctuation terms. These, by their nature, must be approximated in a root-mean-square fashion, rather than directly. This is easier to understand in terms of the Fokker-Planck equation. Since

\[ \frac{\partial Q_\varepsilon}{\partial \varepsilon} = \frac{\partial Q_z}{\partial z} = 0 \]  

(5.3)

the recipe for writing it down (see Appendix A) simplifies by virtue of the lack of “spurious drift” terms and we find

\[ \frac{\partial F(z, \varepsilon, s)}{\partial s} = \alpha_z \varepsilon \frac{\partial F(z, \varepsilon, s)}{\partial z} - (\Omega_z/c)^2 z \frac{\partial F(z, \varepsilon, s)}{\partial \varepsilon} + \frac{J_\varepsilon}{c} \alpha_z \frac{\partial}{\partial \varepsilon} [\varepsilon F(z, \varepsilon, s)] \]  
\[ + \frac{c_0 p_0 J_3}{2(2\pi R)} \frac{\partial^2 F(z, \varepsilon, s)}{\partial \varepsilon^2} \]  

(5.4)

where \( F(z, \varepsilon, s) \) is the distribution function in longitudinal phase space and we have made the smooth approximation of the diffusion term (i.e. the one with second derivatives) in terms of the synchrotron integral \( J_3 \) defined in Appendix D.

This equation can be solved completely in terms of its Green function\(^{23,0}\) or by eigenfunction expansions\(^8\) but we can simplify it further by making a phase-mixing assumption

\[ \langle z \rangle = \int_{-\infty}^{\infty} dz \int_{-\infty}^{\infty} d\varepsilon z F(z, \varepsilon, s) = 0 \]  

(5.5)

which will be true in many situations, including that of equilibrium. Then we can integrate (5.4) over \( z \)
to get an equation for the reduced distribution function

\[ F(\varepsilon, s) = \int_{-\infty}^{\infty} F(x, \varepsilon, s) \, dx, \]

namely

\[ \frac{\partial F(\varepsilon, s)}{\partial s} = \frac{J_x \alpha_e}{\varepsilon} \frac{\partial}{\partial \varepsilon} \left[ \varepsilon F(\varepsilon, s) \right] + \frac{c^2 p_0^2 I_3}{2(2 \pi R)} \frac{\partial^3 F(\varepsilon, s)}{\partial \varepsilon^3}. \]

To find the equilibrium solution \( F_0(\varepsilon) \), we simply set \( \partial_s F = 0 \) and integrate once to find

\[ -\varepsilon F_0(\varepsilon) = \frac{c^2 p_0^2 I_3 f_0}{2 J_x \alpha_e} F_0'(\varepsilon). \]

When we integrate this again, choosing the constant of integration to normalise the distribution to unity, we find the familiar gaussian distribution of momentum (or energy deviations)

\[ F_0(\varepsilon) = \frac{1}{\sqrt{2\pi \sigma^2}} \exp \left( -\frac{\varepsilon^2}{2\sigma^2} \right), \]

where \( \sigma \) is the r.m.s. energy spread in the beam for a linear damping rate determined by the value of \( J_x \):

\[ \langle \varepsilon^2 \rangle \equiv \sigma^2 = \left( \frac{2\sigma^2}{J_x} \right) = \frac{55}{32\sqrt{3} mc} \left( \frac{p_0}{mc} \right)^2 \frac{I_3}{J_x J_2}. \]

The quantity \( \sigma \), which can be regarded as a measure of the strength of quantum excitation, is defined to be the energy spread for the reference case \( J_x = 2^+ \).

Since \( I_2/I_3 \approx \rho \), the bending radius, in an isomagnetic ring there is very little which can be done, beyond varying \( J_x \), to reduce the energy spread of an electron storage ring. Moreover, the energy spread is directly proportional to \( E \). A very small decrease of \( \sigma \) can in principle be achieved with wiggler magnets but their usual effect is to increase it.

This is an important limitation since it determines the energy resolution of particle physics experiments which may be trying, for example, to detect, or measure the widths of, narrow resonances in the mass spectrum. In fact, since design considerations for colliding beam rings usually imply \( \rho \propto E_0^2 \), it almost always turns out that \( \sigma \approx 0.1\% \) at the top energy of a given ring. Nevertheless, \( e^+ e^- \) rings still provide a much finer energy resolution than any foreseeable linear collider or hadron collider and there remains the possibility of enhancing it still further, with the so-called "monochromator" insertions.

The gaussian distribution of energy deviation is by no means inevitable—nonlinear terms (dissipative or conservative) may well change it, especially in the tails (as we shall see shortly). Arguments based on the "Central Limit Theorem" should only be applied in linear approximation and the analogy with the Maxwell-Boltzmann velocity distribution in a gas is not a complete one.

* Often no notational distinction is made between \( \sigma \) and \( \sigma_e \), so one should always be careful to understand which is meant.
5.2 Fokker-Planck equation in action-angle variables

We transform to action-angle variables of linearized synchrotron motion and make a rescaling to variables \((\chi, I)\) with

\[
\begin{align*}
\varepsilon &= -\frac{1}{\kappa_z} \sqrt{2I} \cos(\kappa_z \chi), & \quad \varphi &= \sqrt{2I} \sin(\kappa_z \chi), & \quad \kappa_z &= \frac{\Omega_z}{\alpha_c c}.
\end{align*}
\]

(5.11)

The constant \(\kappa_z\) can be thought of as a conversion factor between energy deviation and length units:

\[
\sqrt{\langle \varepsilon^2 \rangle} = \kappa_z \sqrt{\langle \varphi^2 \rangle} = \frac{Q_s}{\alpha_c R} \sqrt{\langle \varphi^2 \rangle}
\]

(5.12)

and \(Q_s = \Omega_z/2\pi f_0\) is the synchrotron tune. With these variables, the longitudinal Hamiltonian reduces to

\[
H_z = -\alpha_c I
\]

(5.13)

and, by applying the results of Appendix B (or otherwise), we can derive stochastic equations of motion equivalent to (5.4)

\[
\begin{align*}
\chi' &= K_x(\chi, I) + Q_x(\chi, I) \xi(s), & \quad I' &= K_I(\chi, I) + Q_I(\chi, I) \xi(s)
\end{align*}
\]

(5.14)

where

\[
\begin{align*}
K_x(\chi, I) &= -\alpha_c - \frac{J_c(\beta_s) \alpha_c}{2c \kappa_z} \sin(2\kappa_z \chi), & \quad K_I(\chi, I) &= -\frac{J_c(\beta_s) \alpha_c}{c} \left[1 - \cos(2\kappa_z \chi)\right],
\end{align*}
\]

(5.15)

\[
\begin{align*}
Q_x(\chi, I) &= -\frac{2\alpha_c}{\kappa_z} \sqrt{\frac{\alpha_c}{2c I}} \cos(\kappa_z \chi), & \quad Q_I(\chi, I) &= -2\alpha_c \sqrt{\frac{\alpha_c}{2c I}} \sin(\kappa_z \chi).
\end{align*}
\]

Now \(Q_I\) and \(Q_x\) depend on \(I\) and \(\chi\) and their derivatives will contribute spurious drift terms and some algebraic complications to the Fokker-Planck equation

\[
\frac{\partial F(I, X, s)}{\partial s} = -\frac{\partial}{\partial I} [D_I F(I, X, s)] - \frac{\partial}{\partial X} [D_X F(I, X, s)]
\]

\[
+ \frac{1}{2} \frac{\partial^2}{\partial I^2} [Q_I^2 F(I, X, s)] + \frac{\partial^2}{\partial I \partial X} [Q_I Q_X F(I, X, s)] + \frac{1}{2} \frac{\partial^2}{\partial X^2} [Q_X^2 F(I, X, s)].
\]

(5.16)

Here, the complete drift terms are

\[
\begin{align*}
D_I &= K_I + \frac{1}{2} \frac{\partial Q_I}{\partial I} Q_I + \frac{1}{2} \frac{\partial Q_I}{\partial X} Q_X = -\frac{\alpha_c}{c} (J_c I - 2 \sigma_c^2) + \frac{J_c \alpha_c I}{c} \cos(2\kappa_z \chi),
\end{align*}
\]

(5.17)

\[
\begin{align*}
D_X &= K_X + \frac{1}{2} \frac{\partial Q_X}{\partial I} Q_I + \frac{1}{2} \frac{\partial Q_X}{\partial X} Q_X = -\alpha_c - \frac{\alpha_c}{\kappa_z c} \left(J_c + \frac{\sigma_c^2}{I}\right) \sin(2\kappa_z \chi),
\end{align*}
\]

and the diffusion terms are

\[
\begin{align*}
Q_I^2 &= \frac{4\alpha_c \sigma_c^2 I}{c} \left[1 - \cos(2\kappa_z \chi)\right], & \quad Q_I Q_X &= \frac{2\alpha_c \sigma_c^2}{\kappa_z c} \sin(2\kappa_z \chi), & \quad Q_X^2 &= \frac{\alpha_c \sigma_c^2}{\kappa_z^2 c} \left[1 + \cos(2\kappa_z \chi)\right].
\end{align*}
\]

(5.18)

In action-angle variables, it is of course easy to apply the averaging method\(^*\) and write down an averaged Fokker-Planck equation for the action variable which will be valid on time-scales longer than that of a

\(^*\) This is a step beyond the smooth approximation.
synchrotron oscillation, notably the damping time scale:

\[
\tau_\varepsilon \frac{\partial F}{\partial t} = - \frac{\partial F}{\partial I} \left[ (-J_\varepsilon I + 2\sigma_\varepsilon^2 F) \right] + \frac{\partial F}{\partial \chi} + 2\sigma_\varepsilon \frac{\partial^2 F}{\partial I^2} (IF) + \frac{\sigma_\varepsilon^2}{2\kappa^2 I} \frac{\partial^2 F}{\partial \chi^2}.
\]

(5.19)

The absence of a damping term for the phase $\chi$ guarantees that the phase diffusion term superposed on the rapid oscillatory phase advance will lead to a uniform distribution in $\chi$ on $[0, 2\pi]$. Moreover, we have not included the dependence of the synchrotron frequency on amplitude which produces a filamentation effect, further accelerating the phase-mixing.

We can solve (5.19) for the stationary distribution

\[
F_0(I) = \frac{1}{\sigma_z^2} \exp \left( -\frac{I}{\sigma_z^2} \right)
\]

(5.20)

from which we can evaluate the longitudinal emittance (in these units)

\[
\langle I \rangle = \int_0^\infty I F_0(I) dI = \sigma_z^2 = \left( \frac{\varepsilon^2 + \kappa^2 \sigma_\varepsilon^2}{2} \right).
\]

(5.21)

Fig. 4  Gaussian longitudinal phase space distribution and its projection

The distribution (5.20) is equivalent to a joint gaussian in $z$ and $\varepsilon$:

\[
F_0(z, \varepsilon) = F_0(\kappa^2 z^2 / 2 + \varepsilon^2 / 2) = \frac{1}{2\pi \sigma_z \sigma_\varepsilon} \exp \left( -\frac{\varepsilon^2}{2\sigma_\varepsilon^2} - \frac{z^2}{2\sigma_z^2} \right),
\]

(5.22)

where the natural (or zero-current) bunch length is

\[
\sigma_z = \sigma_\varepsilon / \kappa_z.
\]

(5.23)

This distribution is shown in Fig. 4 for some typical values of $J_\varepsilon$ and $\sigma_z$; the meaning of the parameters $b$ and $R$ will be explained in a later section. In this, and similar plots to be shown later, we also show the
projection of the action distribution along one axis, given by the integral

$$\lambda(\kappa_z z) = \int_{-\infty}^{\infty} F_0(z, \varepsilon) \, d\varepsilon = \int_{0}^{\infty} \frac{F_0(I)}{\sqrt{2I - \kappa_z^2 z^2}} \, dI. \quad (5.24)$$

By virtue of the special properties of the gaussian, this is just the same as (5.9). In general it need not be, as we shall see later. Such a projection corresponds to the energy distribution or the longitudinal current density profile of the bunch. It is not the shadow of the phase space distribution. The second equality in (5.24) holds only for rotationally symmetric distributions.

6. NONLINEAR WIGGLERS

In small and medium-sized $e^+e^-$ storage rings, it is an excellent approximation to assume that the radiation damping and quantum excitation effects remain linear to large amplitudes. On the other hand, the new generation of large rings (such as TRISTAN, LEP or the HERA electron ring) begin to enter a regime where these effects can develop amplitude-dependences which may have to be included in a realistic calculation. Such nonlinear effects will tend to produce equilibrium distributions whose cores are fatter and whose tails decay more slowly\textsuperscript{13} than predicted by the linear theory of the previous sections. Such effects may generally be expected to be detrimental to beam stability and lifetime.

More optimistically, we might regard the existence of such effects as an opportunity to favourably influence the distribution function by means of intentional dissipative nonlinearities. In this section, we shall introduce the idea of a nonlinear wiggler\textsuperscript{25} which allows one some freedom\textsuperscript{*} to shape the energy distribution in an $e^+e^-$ ring. Such wigglers have been studied in the context of the LEP design as a means of reducing the severity of certain collective instabilities and, possibly, depolarizing effects.

Nonlinear wigglers are special combined-function magnets which modify the low-intensity particle distribution\textsuperscript{28,14,29,30} in longitudinal phase space. The original idea\textsuperscript{25} for a nonlinear wiggler in LEP was a combined function dipole-octupole magnet. Although this works, it produces a large additional energy loss through its contribution to the integral $I_2$. One, and only one, other multipole combination exists which produces the same nonlinear damping effect with negligible additional energy loss.

6.1 Quadrupole-sextupole wiggler

We shall consider the quadrupole-sextupole wiggler in which, as the name suggests, there are superposed quadrupole and sextupole fields. The vertical component of $B$ in the $y = 0$ plane is

$$B_w(x) = \pm \frac{p_0 c}{e} (K_{1w} x + K_{2w} x^2 / 2), \quad (6.1)$$

alternating in sign between adjacent blocks of the device, so that the integrated quadrupole and sextupole field components vanish. For simplicity, let us assume that $K_{1w}$ and $K_{2w}$ have constant magnitude in the blocks, whose total length is $L_w$. To correct this approximation for a real wiggler, one must first find

\textsuperscript{*} Beyond the variation of damping partition numbers.
the correct gradient profiles, either numerically or by measurements on real magnets. Each term in the energy loss given below is of the form of a product of $L_w$ and powers of field-gradients and dispersion functions and simply has to be replaced by the corresponding synchrotron radiation integral.

### 6.2 Nonlinear damping

Particles with a momentum deviation $\rho_0 \varepsilon$ from the synchronous value $\rho_0 (1 + \delta_\varepsilon)$ will pass through the wiggler with a horizontal displacement $\eta_w(\delta_\varepsilon + \varepsilon)$ off the axis. The additional contribution to the total energy loss of such particles due to the wiggler is equal to

$$U_w(\delta_\varepsilon + \varepsilon) = c_1 \rho_0^2 c L_w \left[ \frac{K_{1w}^2 \eta_w^2 \delta_\varepsilon^2 + (2K_{1w}^2 \eta_w^2 + K_{1w} K_{2w} \eta_w^3) \delta_\varepsilon^3}{(i)} \right]$$

$$\left\{ \frac{2K_{1w}^2 \eta_w^2 \delta_\varepsilon + 3(2K_{1w}^2 \eta_w^2 + K_{1w} K_{2w} \eta_w^3) \delta_\varepsilon^3}{(ii)} \right\} \varepsilon^2$$

$$\left\{ \frac{K_{1w}^2 \eta_w^2 + 3(2K_{1w}^2 \eta_w^2 + K_{1w} K_{2w} \eta_w^3) \delta_\varepsilon^3}{(iii)} \right\} \varepsilon^3 + \ldots$$

(6.2)

Each term in this expression has a different physical significance and must be examined in its turn:

(i) These terms are independent of $\varepsilon$ and simply add to the total value of $U(\delta_\varepsilon)$; they include a contribution to the integral $I_8$, reducing the damping aperture somewhat.

(ii) The coefficient of $\varepsilon^1$ is related to (i) and adds to the linear damping rate.

(iii) Being proportional to $\varepsilon^2$ these average out over the phase of the oscillation.

(iv) With these terms we find a qualitatively new effect; being proportional to $\varepsilon^3$, they provide a damping proportional to $I^2$. The most significant contribution comes from the quadrupole-sextupole cross term which only exists by virtue of the fact that the quadrupole and sextupole fields are spatially superposed. Building a wiggler with separate quadrupole and sextupole blocks will not produce the same effects, no matter how closely the blocks are spaced.

One can verify numerically that the nonlinear quantum excitation due to the wiggler is much less important than the nonlinear damping.

Taking (6.2) into account, the equation of motion for $I$ is given by (5.14) where $Q_f$ is given by (5.15) but now

$$K_f(\chi_, I) = -\frac{J_\varepsilon(\delta_\varepsilon)c}{c} \left[ 1 - \cos(2\kappa_\varepsilon \chi) \right] + \left( \frac{b}{2} I^2 \right) \frac{8}{3} \sin^2(\kappa_\varepsilon \chi)^4,$$

(6.3)

where the nonlinear damping coefficient for a quadrupole-sextupole wiggler is defined by

$$b \overset{\text{def}}{=} \frac{3}{I_2} \int_{L_w} (2K_{1w}^2 \eta_w^2 + K_{1w} K_{2w} \eta_w^3) \, ds \approx \frac{3L_w}{I_2} (2K_{1w}^2 \eta_w^2 + K_{1w} K_{2w} \eta_w^3).$$

(6.4)

Since the average of $(\sin \theta)^4$ is $3/8$ the nonlinear term makes an important contribution after phase-averaging.
6.3 Negative $J_e$ and birth of a limit cycle

From (6.3) we can see that, even when $J_e$ is negative, so that small amplitude synchrotron oscillations are anti-damped, the nonlinear terms generated by the quadrupole-sextupole wiggler can restore positive damping at larger amplitudes. It is possible to choose $\delta_2$ so that the central momentum of the beam lies outside the damping aperture on the side of negative $J_e$.

Returning to cartesian coordinates in phase space, it is easy to show that the deterministic equation (4.9) should be replaced by a van der Pol equation

$$\ddot{e} + \alpha_3 (J_e + b\epsilon^2\mathbf{z}^2)\dot{e} + \Omega_2^2 e = 0.$$  

(6.5)

![Figure 5: Approximate solutions of the Van der Pol equation](image)

In Fig. 5, we show solutions of this equation which are obtained analytically (for $\alpha_3/\Omega_2 \ll 1$) by an application of the averaging method. They are equivalent to integrating (6.3) for two qualitatively distinct cases. For clarity the damping time has been artificially shortened to a few times the synchrotron period. Two cases are plotted:

(a) Here $J_e = 2$, $b = 0$ and the origin of phase space is a simple attracting point. All orbits within the separatrix of the RF bucket (not shown here) are attracted to it. If $b$ is given a positive value no qualitative change occurs but particles with large amplitudes are damped more rapidly.

(b) Now $J_e = -1$ and $b = 5 \times 10^5$; small amplitudes are anti-damped but positive damping is restored at larger amplitudes. The linear anti-damping and nonlinear damping balance at the value

$$I = \dot{I} = -2J_e/b,$$

(6.6)

corresponding to a limit cycle of (6.5), clearly visible as a periodic orbit which attracts particles from both larger and smaller amplitudes. The fixed point at the origin has become unstable via the so-called Hopf bifurcation.
6.4 Fokker-Planck equation with nonlinear wiggler

To see how this new phase space structure affects the distribution function, we construct the averaged Fokker-Planck equation in the action variable by generalizing (5.19) and integrating over phase

\[
\tau_x \frac{\partial F(I,t)}{\partial t} = -\frac{\partial}{\partial I} \left\{ - \left( J_x I + \frac{b}{2} I^2 - 2\sigma_x^2 \right) F(I,s) \right\} + 2\sigma_x^2 \frac{\partial^2}{\partial I^2} [IF(I,s)]. \tag{6.7}
\]

6.5 Equilibrium solution

Integrating (6.7), we find a non-gaussian equilibrium distribution,

\[
F_0(I) = Z(J_x, b, \sigma_x)^{-1} \exp \left( -\frac{J_x}{2\sigma_x^2} I - \frac{b}{8\sigma_x^2} I^2 \right). \tag{6.8}
\]

where the normalisation constant \(Z(J_x, b, \sigma_x)\) will be discussed in detail below.

For \(b > 0\), the tails of this distribution decay very much faster than gaussian ones \(| \exp(-x^4)\) rather than \(\exp(-x^2)\) | and this can considerably improve the lifetime and stability of the particle beam. Even when \(J_x\) is made negative, the balance between linear anti-damping at small amplitudes and nonlinear damping at larger amplitudes results in a stable distribution, i.e., one which can be normalised.

By adjusting the two free parameters \(J_x\) and \(b\) we find that we have an additional degree of freedom in moulding the longitudinal profile of the bunch. In addition, our freedom to vary \(J_x\) is extended by the possibility of moving it onto the negative real axis.

It can be shown\(^{14}\) that distributions with the same value of the dimensionless parameter

\[
R = -\frac{J_x}{\sqrt{2b\sigma_x}} \tag{6.9}
\]

are geometrically similar.
$J_z = 1$
$b = 5.5E5$
$\sigma_z = 0.7E-3$
$R = -1.429$
$\sqrt{\langle \epsilon^2 \rangle} = 0.86E-3$

Fig. 7 $J_z$ reduced but still positive

$J_z = 0$
$b = 5.5E5$
$\sigma_z = 0.7E-3$
$R = 0$
$\sqrt{\langle \epsilon^2 \rangle} = 1.26E-3$

Fig. 8 The critical value of $J_z$ on the bifurcation.

Each of the Figs. 6–11 is analogous to Fig. 4 but includes the influence of a nonlinear wiggler of a certain strength. To ease comparison, all three scales of Figs. 6–11 are the same as in Fig. 4.

In this sequence of figures, we can follow what happens as a single parameter, $J_z$, is varied from positive to negative values. All other relevant parameters, namely the wiggler strength $b$ and the quantum excitation $\sigma_z$ are held constant.

In Figs. 6 and 7 the distribution is similar in form to the gaussian Fig. 4 when $b = 0$ except that the tails decay faster. Few particles lie in the region of phase space where the wiggler has much influence and therefore the r.m.s. energy spread is only very slightly reduced. However if off-momentum particles are responsible for unwanted effects (e.g. depolarization), such a distribution may be very beneficial.
When \( J_\varepsilon \) is decreased to zero, Fig. 8 there is no quadratic term in the exponent of (6.8); the distribution remains stable but has spread out considerably since small amplitude particles are hardly damped at all.

When \( J_\varepsilon \) goes negative (Figs. 9–11) the peak at the centre of the phase space distribution becomes a crater. The maximum density then occurs approximately above the attracting limit cycle of the deterministic equations of motion (4.7). For sufficiently small negative values of \( J_\varepsilon \), (Fig. 9) the profile of the bunch in real space still contains only one hump because the crater is so shallow that it is wiped out in the integration across phase space.

This state of affairs persists until \( J_\varepsilon \) has passed through a special value: \(^{14}\)

\[
J_\varepsilon < -0.5409 \sqrt{25 \sigma_\varepsilon}.
\]  

(6.10)
\[ J_\varepsilon = -0.6 \\
b = 5.5E5 \\
\sigma_\varepsilon = 0.7E-3 \\
R = 0.8571 \\
\sqrt{\langle \varepsilon^8 \rangle} = 1.68E-3 \]

Fig. 11  Two peaks appear in current profile

Around this transition value (Fig. 10), the profile is very flat. Beyond it two humps appear in the current profile (Fig. 11).

Such a flattening-out of the current distribution can be of considerable utility in the attempts to increase the amount of stored current in a storage ring. The peak current in the bunch is lowered and the energy spread and bunch length are increased, tending to reduce the wake-fields.

At first sight, it may seem that a much larger RF voltage would be necessary to accommodate the larger energy spread with reasonable quantum lifetime. Generally however this is not a serious problem for two reasons:

(i)  In the case of a storage ring which is also an accelerator, the bunch-lengthening effect would be needed mainly in the lower part of the energy range of the storage ring, notably at injection energy and, there, there ought to be RF voltage to spare.

(ii) For a given energy spread and RF voltage, the quantum lifetime for a distribution such as (6.8) is much longer than that of (5.20) because of the much faster decay of its tails. In other words one can fill up a much larger proportion of the RF bucket with particles without increasing the loss rate across the separatrix.

6.6  Partition function and moments

The distribution (6.8) is normalised to unity by means of the partition function

\[ Z(J_\varepsilon, b, \sigma_\varepsilon) = \int_0^\infty \exp \left( - \frac{J_\varepsilon I}{2\sigma_\varepsilon^2} - \frac{b}{8\sigma_\varepsilon^2} I^2 \right) \, dI = 2\sigma_\varepsilon \sqrt{\frac{\pi}{2b}} w \left( \frac{iJ_\varepsilon}{\sqrt{2b\sigma_\varepsilon}} \right), \tag{6.11} \]

where \( w \) is the error function for complex arguments (sometimes known as the plasma dispersion function). Considered as a function of the parameters \( J_\varepsilon, b \) and \( \sigma_\varepsilon \), the partition function contains a lot of information
about the global properties of the distribution and is a convenient tool for calculation. In this respect, it is analogous to the partition functions of equilibrium statistical mechanics.

As an aid to physical understanding, it is particularly useful to make two distinct asymptotic expansions of the reciprocal:

\[
Z(J_e, b, \sigma_e)^{-1} \sim \frac{J_e}{4\pi \sigma_e^2} + \frac{b}{4\pi J_e} - \frac{b^2 \sigma_e^2}{2\pi J_e^2} + \ldots \quad (b \to 0^+, \ J_e > 0),
\]

(6.12)

which is useful as we consider the transition from positive linear damping to nonlinear damping (small values of \(b\)), and

\[
Z(J_e, b, \sigma_e)^{-1} \sim \sqrt{\frac{b}{2\pi^3 \sigma_e^2}} + \frac{J_e}{2\pi^2 \sigma_e^2} + \left(1 - \frac{1}{4}\right) \frac{J_e^2}{\pi \sqrt{2\pi} b \sigma_e^2} + \ldots \quad (J_e \to 0, \ b > 0),
\]

(6.13)

which describes the neighbourhood of the bifurcation as \(J_e\) changes sign. For comparison with (5.21), the longitudinal emittance is given by

\[
\langle I \rangle = \int_0^\infty I F_0(I) \, dI = (-2\sigma_e^2) \frac{\partial \log Z}{\partial J_e} = \frac{8\pi \sigma_e^2}{b Z(J_e, b, \sigma_e)} - \frac{2Z(J_e, b, \sigma_e)}{b}.
\]

(6.14)

Evaluating this in the limit of small \(b\), with the help of (6.12), shows that it does indeed reduce to (5.21) when the nonlinear wiggler is turned off

\[
\langle I \rangle \sim \frac{2\sigma_e^2}{J_e} \left(1 - \frac{2b \sigma_e^2}{J_e}\right), \quad (b \to 0^+, \ J_e > 0).
\]

(6.15)

More generally, all the moments of the equilibrium distribution can be found from

\[
\langle I^n \rangle = \int_0^\infty I^n F_0(I) \, dI = (-2\sigma_e^2)^n Z^{-1} \frac{\partial^n Z}{\partial J_e^n}.
\]

(6.16)

6.7 Quantum lifetime

The longitudinal quantum lifetime of the beam is the inverse of the loss rate of particles across the separatrix due to quantum fluctuations. Other loss mechanisms may also contribute to determine the net lifetime of the beam.

To calculate the quantum lifetime, one interprets the Fokker-Planck equation as a continuity equation in phase space and identifies the diffusive component of the particle flux across the separatrix. This component constitutes the loss rate. At the separatrix, it is not balanced by a flux of particles damping down from larger amplitudes. The details of such a calculation for the gaussian distribution were given
in Ref. 11 and can be generalized for (6.8); the details of this part may be found in Ref. 14. The result is

\[ r_q = \frac{\tau_c Z(J_e, b, \sigma_e) \exp \left( \frac{\xi^2 - b}{8} + \frac{\xi^2}{2} \right)}{\pi \xi \sigma_e^2 (\xi \sigma_e^2 b + 2J_e)} \]  

(6.17)

where the parameter \( \xi \) is half the squared bucket half-height in units of \( \sigma_e \):

\[ \xi = \frac{\sigma_z^2}{2 \sigma_e^2} \]  

(6.18)

Except for the fact that we have taken \( J_e = 2 \) as a reference case, this is precisely the same definition as introduced in Ref. 2. The energy aperture \( \epsilon_{\text{max}} \) is usually thought of as arising from the RF voltage limitation but it may also arise from a limitation of the vacuum chamber aperture in a dispersive region or even a reduced dynamic aperture at large momentum deviation.

The formula (6.17) includes the result for a gaussian distribution\(^{2,11}\) as a special case.

6.8 Practical aspects

A study of the available gradients and apertures of combined-function quadrupole-sextupole magnets would be out of place here. We only mention that, to make the nonlinear damping effect noticeable, we need \( 2b \sigma_z^2 \sim 1 \). From (6.4), we see that, for given gradients, \( b \propto p_0^{-2} \) while from (6.10), we have \( \sigma_z^2 \propto p_0^2 \), so that \( b \sigma_z^2 \) is independent of energy for a given set of wigglers and storage ring. In addition the dependence on the bending radius cancels out from the product. Because the dispersion function is of the order of 1 m in almost all lattice designs, it follows that the length of wiggler required is roughly independent of the size of the storage ring and the energy at which it is operated. Since several tens of metres of wiggler are required, we can contemplate installing them only in the largest rings.

Taking the example of LEP, we find that if a set of quadrupole-sextupole wigglers were installed with effective parameters

\[ \frac{\partial B_z}{\partial x} = 10 \text{ tesla m}^{-1}, \quad \frac{\partial^2 B_z}{\partial x^2} = 400 \text{ tesla m}^{-2}, \quad L_w = 40 \text{ m}, \quad \eta_w = 1.8 \text{ m}. \]  

(6.19)

then, at a beam energy of 20 GeV, where \( \sigma_z = 0.3 \times 10^{-3} \) we would have \( b \approx 5 \times 10^5 \), the value we used in Figs. 6–11. The energy spread used in the figures would correspond to a beam energy of around 45 GeV in LEP.

The only serious adverse effect of a quadrupole-sextupole wiggler seems to be the reduction in damping aperture due to its contribution to the integral \( J_e \). This requires a slightly more elaborate control of the RF frequency to achieve the desired value of \( J_e \) since the variation of \( J_e \) will be coupled with the excitation of the wiggler.
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* * *

REFERENCES


3. J.S. Bell, Hamiltonian mechanics, these proceedings.


11. A. Piwinski, Beam losses and lifetime in Reference 1.


27. A. Renieri, *Possibility of achieving very high energy resolution in electron-positron storage rings*, Frascati Preprint /INF-75/6(R), (1975).


31. C.N. Lashmore-Davies, *Kinetic theory and the Vlasov equation, these proceedings*.

APPENDIX A: Physical meaning of the Fokker-Planck equation

After some discussion of its relationship to Liouville's Theorem, we state a recipe for writing down the Fokker-Planck equation corresponding to a given set of stochastic differential equations. A variety of derivations of this relationship may be found in the literature on stochastic processes and one tailored to the present problems and notations has been given previously.

Consider some vector of coordinates \( \mathbf{X}(t) = (X_1, \ldots, X_N) \) (e.g. in a near-Hamiltonian system, the canonical coordinates and momenta \((q_1, \ldots, q_n, p_1, \ldots, p_n)\), where \( N = 2n \)). It evolves in time according to a set of first-order stochastic differential equations with a single noise source \( \xi(t) \) satisfying

\[
\langle \xi(t) \rangle = 0, \quad \langle \xi(t) \xi(t') \rangle = \delta(t - t').
\]

(A1)

The stochastic differential equations are taken to be

\[
\dot{\mathbf{X}}(t) = \mathbf{K}(\mathbf{X}, t) + \mathbf{Q}(\mathbf{X}, t) \xi(t).
\]

(A2)

Note that putting (2.35) in this form requires \( \mathbf{K} \) to include the terms in \( \langle F_X(s) \rangle \).

When we speak of a realisation of \( \xi(t) \) we mean just one of the ensemble of possible histories of the stochastic process. Of course this ensemble would contain much more information than we could possibly cope with so we bear in mind that we must afterwards average over all realisations with an appropriate weighting.

To each realisation of \( \xi(t) \) there corresponds a realisation of the solution \( \mathbf{X}(t) \). Let us introduce the exact phase-space density for this solution

\[
F(\mathbf{X}, t) = \delta(\mathbf{X} - \mathbf{X}(t)) = \prod_i \delta(X_i - X_i(t))
\]

(A3)

where \( \mathbf{X} \) is a free variable and \( \mathbf{X}(t) \) is the realisation of the solution of (A2) starting from initial conditions \( \mathbf{X}(0) \). The continuity equation for \( F \) is

\[
\partial_t F(\mathbf{X}, t) + \nabla_X \cdot [\dot{\mathbf{X}}(t) F(\mathbf{X}, t)] = 0.
\]

(A4)

or, less ambiguously,

\[
\partial_t F(\mathbf{X}, t) + \nabla_X \cdot [\mathbf{K}(\mathbf{X}, t) F(\mathbf{X}, t)] + \nabla_X \cdot [\mathbf{Q}(\mathbf{X}, t) \xi(t) F(\mathbf{X}, t)] = 0.
\]

(A5)

and is completely equivalent to (A2). Since \( \mathbf{X}(t) \) depends on the values of the fluctuations \( \xi(t) \), \( F(\mathbf{X}, t) \) is also a fluctuating quantity:

\[
F(\mathbf{X}, t) = \langle F(\mathbf{X}, t) \rangle + \tilde{F}(\mathbf{X}, t).
\]

(A6)

and it would be just as difficult to deal with—let alone find—the ensemble of its solutions as it would be to deal with that of (A2).

* The generalisation to several noise sources is easy; we only need one in our application since the changes in the three canonical momenta of the electron due to a particular photon emission are correlated.
In the particular case of a Hamiltonian system, where

\[
\mathbf{K} = \left( \frac{\partial H}{\partial p_1}, \ldots, \frac{\partial H}{\partial p_n}, \frac{\partial H}{\partial q_1}, \ldots, \frac{\partial H}{\partial q_n} \right), \quad Q = 0,
\]  

(A7)

the gradient operator \( \nabla_{\mathbf{X}} = (\partial/\partial q, \partial/\partial p) \) in (A5) may be commuted to the right, past the \( \dot{X}(t) \), giving Liouville's theorem.

To derive the Fokker-Planck equation describing the time evolution of the average phase-space density of a system, subject to deterministic drift \( \mathbf{K} \) and diffusion \( Q \), one must average (A5) but take account of the way in which the fluctuations make the sharply-peaked phase space density (A3) spread out to become, upon averaging, a smooth function.\(^{13}\) In this process, information about the fine details of the distribution function is lost and we make the transition to an irreversible description of the time-evolution. Reinterpreting \( F'(X, t) \) to mean this average or smoothed-out function, the final result is

\[
\partial_t F = -\sum_i \frac{\partial}{\partial X_i} \left\{ \left[ K_i + \frac{1}{2} \sum_j \frac{\partial Q_i}{\partial X_j} Q_j \right] F \right\} + \frac{1}{2} \sum_i \sum_j \frac{\partial}{\partial X_i} \frac{\partial}{\partial X_j} \left[ Q_i Q_j F \right].
\]  

(A8)

The second term inside the curly brackets is known as the "spurious drift" and is absent when the diffusion \( Q(X, t) = Q(t) \) independently of amplitude. Because it often vanishes, this term is sometimes overlooked although it is essential for a complete description—see e.g. its rôle in section 5.

It is possible to avoid this apparent complication by writing stochastic differential equations which incorporate the spurious drift terms into the definition of \( \mathbf{K} \). One must then work with the so-called \( \dot{\text{I}} \)to calculus, in which the ordinary rules of differentiation are replaced by rules which bring in extra terms designed precisely to maintain the simple relationship between the differential equations and the Fokker-Planck equation. A change of variables is then rather more work than it is when one works in the Stratonovich interpretation as we have done in this lecture. What you gain on the swings of Fokker-Planck equations, you lose on the roundabouts of variable transformations. We spent more time on the roundabouts so the Stratonovich interpretation was the better choice. In this scheme, Markovian stochastic differential equations are written down as limits of equations governing real processes, as was implicit in the formulation of the electron equations of motion (2.35). \( \dot{\text{I}} \)to equations of motion can be written down to describe the same physical system but, since the Fokker-Planck equation has to be the same, they look different from (2.35).

For a system, like the electron in a storage ring, which is close to being Hamiltonian, (A2) would take the form

\[
\begin{align*}
\dot{q} &= \frac{\partial H(q, p, t)}{\partial p} + \epsilon K_q(q, p, t) + \sqrt{\epsilon} Q_q(q, p, t) \xi(t), \\
\dot{p} &= -\frac{\partial H(q, p, t)}{\partial q} + \epsilon K_p(q, p, t) + \sqrt{\epsilon} Q_p(q, p, t) \xi(t).
\end{align*}
\]  

(A9)

Here the ordering of terms with respect to the small parameter \( \epsilon \) has been chosen so that the dissipative terms have an effect proportional to \( \epsilon \Delta t \) in a short time interval \( \Delta t \). That this is so is clearer when,
following (A8), we write down the Fokker-Planck equation corresponding to (A9)

\[
\partial_t F = -\frac{\partial H}{\partial p_i} \frac{\partial F}{\partial q_i} + \frac{\partial H}{\partial q_i} \frac{\partial F}{\partial p_i} - \epsilon \frac{\partial}{\partial q_i} \left\{ k_{ij} + \frac{1}{2} \frac{\partial Q_{ij}}{\partial q_j} Q_{ij} + \frac{1}{2} \frac{\partial Q_{ij}}{\partial p_j} Q_{ij} \right\} F
\]

\[
\text{(A10)}
\]

\[
- \epsilon \frac{\partial}{\partial p_i} \left\{ k_{pi} + \frac{1}{2} \frac{\partial Q_{pi}}{\partial q_j} Q_{pi} + \frac{1}{2} \frac{\partial Q_{pi}}{\partial p_j} Q_{pi} \right\} F
\]

\[
+ \frac{\epsilon}{2} \frac{\partial}{\partial q_i} \frac{\partial}{\partial q_j} [Q_{ij} Q_{ij} F] + \epsilon \frac{\partial}{\partial q_i} \frac{\partial}{\partial p_j} [Q_{ij} Q_{ij} F] + \frac{\epsilon}{2} \frac{\partial}{\partial q_i} \frac{\partial}{\partial q_j} [Q_{ij} Q_{ij} F]
\]

where summation over repeated indices is to be understood. The first two terms represent the incompressible flow in phase space described by Liouville's Theorem; the second pair represents an at least partially irreversible drift (e.g. damping); finally the last three terms describe diffusion.

Since this School has included a course on plasma kinetic theory, it may be helpful to briefly discuss the relationship of the Fokker-Planck equation described there to that given here.

Mathematically speaking, these equations are not of the same form, but they are nevertheless related. Physically, of course, they describe quite different phenomena—this is one resemblance between high energy electron bunches and plasmas which turns out to be rather superficial.

Here, the most notable difference is that (A8) is always linear, while in plasma physics, the name is commonly applied to an equation (sometimes also called the Landau equation) in which the distribution function appears quadratically in order to describe collisions between particles of the same, or different, species. Thus, while we have been exclusively concerned with single particles subject, in effect, to random external forces, the collision terms of the "plasma" Fokker-Planck equation describe forces acting between pairs of particles. This richer physical content makes it considerably more difficult to solve. Yet, when the distribution is linearised about the equilibrium Maxwell-Boltzmann form, an equation of the form (A8) is recovered. This, of course, is not surprising since, in that case, we may think of the equation as describing the distribution function of a single test particle subjected to random forces arising from the thermal background plasma. In this limit, the Rosenbluth potentials become independent of the perturbation of the distribution function and are related to the functions K and Q used here.
APPENDIX B: Canonical transformations for dissipative systems

The title of this appendix verges on the self-contradictory. Of course, we do not suggest that dissipative systems can be made canonical, only that, when their equations of motion contain a Hamiltonian part, upon which it is convenient to make canonical transformations, we may extend the formalism of generating functions to transform the dissipative parts too. This is of obvious utility in electron storage ring theory because the radiation terms in the equations of motion are small on average compared with those describing the applied electromagnetic fields.

Let us consider the case of a free (i.e. such that old and new coordinates are independent) transformation from \((q,p) = (q_1, \ldots, q_n, p_1, \ldots, p_n)\) to new variables \((Q,P) = (Q_1, \ldots, Q_n, P_1, \ldots, P_n)\).

Call the old Hamiltonian \(H(q,p,s)\) and the new \(K(Q,P,s)\). Both in the old and new variables, Hamilton's equations have to be supplemented by terms \((a,b)\) and \((A,B)\) which describe the dissipation:

\[
\begin{align*}
\text{OLD} & \quad q' = \frac{\partial H}{\partial p} + a(q,p,s) \\
p' = -\frac{\partial H}{\partial q} + b(q,p,s)
\end{align*} \quad \iff \quad \begin{align*}
\text{NEW} & \quad Q' = \frac{\partial K}{\partial P} + A(Q,P,s) \\
P' = -\frac{\partial K}{\partial Q} + B(Q,P,s).
\end{align*} \tag{B1}
\]

Since the transformation is canonical, there exists a generating function \(S(Q,q,s)\), depending on the old and new coordinates and no momentum, such that the relationship between old and new coordinates and momenta is obtained by solving

\[
p = \frac{\partial S}{\partial q}, \quad P = -\frac{\partial S}{\partial Q}, \quad K = H + \frac{\partial S}{\partial s}. \tag{B2}
\]

In a system with 3 degrees of freedom, 6 different types of generating function are necessary to generate all possible canonical transformations, and in practice each of them comes up sooner or later. The other cases can be worked by analogy to this one; for a different example, see Ref. 13.

The equivalence of the two descriptions of the Hamiltonian system is guaranteed by these relationships. Given that the two sets of equations in (B1) are supposed to describe the same dissipative system, we need to know how to calculate the new dissipative terms \((A,B)\).

[As in Appendix A, we must now acknowledge that any stochastic terms are to be interpreted in the Stratonovich, not the Itô, sense. The following results would be much more complicated in the Itô calculus.]

To avoid cumbersome notations, let us denote partial derivatives by subscripts where convenient; thus, for example, we may write a vector or a matrix

\[
S_q = \begin{pmatrix}
\frac{\partial S}{\partial q_1} \\
\frac{\partial S}{\partial q_2} \\
\vdots \\
\frac{\partial S}{\partial q_n}
\end{pmatrix}, \quad S_{QQ} = \begin{pmatrix}
\frac{\partial^2 S}{\partial Q_1 \partial q_1} & \frac{\partial^2 S}{\partial Q_1 \partial q_2} & \cdots & \frac{\partial^2 S}{\partial Q_1 \partial q_n} \\
\frac{\partial^2 S}{\partial Q_2 \partial q_1} & \frac{\partial^2 S}{\partial Q_2 \partial q_2} & \cdots & \frac{\partial^2 S}{\partial Q_2 \partial q_n} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial^2 S}{\partial Q_n \partial q_1} & \frac{\partial^2 S}{\partial Q_n \partial q_2} & \cdots & \frac{\partial^2 S}{\partial Q_n \partial q_n}
\end{pmatrix}. \tag{B3}
\]

\[\text{†} \quad \text{This type of generating function, which is useful in the transformation to action-angle variables, is sometimes denoted } F_1(Q,q,s).\]
By hypothesis, the arguments of the generating function provide a unique labelling of points in extended phase space. Hence, their differentials form a basis for the vector space of differentials of dynamical variables and any such differential can be expressed as a linear combination of them. In particular, from (B2), we may use the ordinary chain rule to calculate

\[
dp = S_{qQ} dq + S_{qQ} dQ + S_{qQ} ds.
\]

(B4)

When the postulated equations of motion (B1), are used to project this identity along the local direction of time-evolution of the system, we find

\[
\begin{align*}
dp &= S_{qQ}(H_p + a)ds + S_{qQ}(K_p + A)ds + S_{qQ}ds \\
&= -H_q ds + b ds
\end{align*}
\]

\[
\begin{cases}
S_{qQ} H_p + S_{qQ} K_p + S_{qQ} = -H_q \\
S_{qQ} a + S_{qQ} A = b.
\end{cases}
\]

(B5)

where we separately identified the Hamiltonian and dissipative parts of the equality.

In an analogous way, we can calculate \(dP\) and conclude

\[
S_{qQ} H_p + S_{qQ} K_p + S_{qQ} = K_p,
\]

\[
-S_{qQ} a - S_{qQ} A = B.
\]

(B6)

Since the canonical transformation is free, the Jacobian determinants of either set of canonical variables with respect to the independent coordinates \((Q, q)\) cannot vanish, i.e.

\[
\frac{\partial (p, q)}{\partial (Q, q)} = \left| \begin{array}{cc} \frac{\partial p}{\partial Q} & \frac{\partial p}{\partial q} \\ \frac{\partial q}{\partial Q} & \frac{\partial q}{\partial q} \end{array} \right| = \det \frac{\partial p}{\partial Q} = \det S_{qQ} \neq 0.
\]

(B7)

Hence the matrix \(S_{qQ}\) is invertible and it follows that the last members of (B5) and (B6) may be solved simultaneously to yield a linear relationship between the dissipative terms in the old and new representations:

\[
\begin{align*}
A &= [S_{qQ}]^{-1}(b - S_{qQ} a) \\
B &= -S_{qQ} a - S_{qQ} [S_{qQ}]^{-1}(b - S_{qQ} a).
\end{align*}
\]

(B8)

By means of these formulae, the dissipative terms can be transformed in parallel with the canonical transformation; (B8) may be regarded as a convenient recipe for carrying out complicated transformations. It should be remembered that, in the course of this work, all derivatives of the generating function must be expressed in terms of the variables \((Q, q)\) and the expressions for \(q(Q, P)\) and \(p(Q, P)\) should only be substituted afterwards. Of course, exactly the same constraint applies while transforming the Hamiltonian.

---

* When these terms have fluctuating parts this property is not at all trivial. It applies only as a consequence of our use of the Stratonovich interpretation.
APPENDIX C: Local synchrotron motion and smooth approximation

A single synchrotron oscillation takes many turns of the machine; in fact the largest synchrotron tunes are realised in large machines like LEP and are of the order of 0.1. It is therefore natural to simplify the description of this motion by making a Fourier decomposition of the Hamiltonian on the circumference and looking only at the most slowly-varying terms.

Let us consider the terms in (3.21) individually.

C1 Momentum compaction

From (3.21) the "kinetic energy" of synchrotron oscillations is given by the term

\[
\frac{1}{2} \Gamma(\delta_s, s) e^2 = \frac{1}{2} \left\{ \frac{\epsilon^2}{1 + \delta_s} - (G^2 + k) \eta^2 \right\} e^2 = \frac{1 + \delta_s}{2} \{ (\eta \eta')' - G \eta \} e^2,
\]

where we used (3.9). We make a Fourier analysis of the function

\[
\Gamma(\delta_s, s) = \sum_{n=-\infty}^{\infty} \Gamma_n(\delta_s) e^{ins/R},
\]

where the coefficients will be given by

\[
\Gamma_n(\delta_s) = \frac{1}{2\pi R} \int_{0}^{2\pi R} ds \Gamma(\delta_s, s) e^{-ins/R}.
\]

Integrating by parts twice and using periodicity arguments, it is not difficult to show that

\[
\Gamma_n(\delta_s) = \Gamma^*_{-n}(\delta_s) = -\frac{1 + \delta_s}{2\pi R} \int_{0}^{2\pi R} \eta(\delta_s, s) \left[ \frac{n^2 \eta(\delta_s, s)}{2R^2} + G(s) \right] e^{ins/R} ds.
\]

In particular, the constant term is just

\[
\Gamma_0(\delta_s) = \frac{1 + \delta_s}{2\pi R} \int_{0}^{2\pi R} \{ \frac{\epsilon^2}{1 + \delta_s} - (G^2 + K_1) \eta^2 \} ds = -\frac{1 + \delta_s}{2\pi R} \int_{0}^{2\pi R} G \eta \ ds = -\alpha_c(\delta_s),
\]

where \(\alpha_c(\delta_s)\) is called the momentum compaction factor which, like the dispersion functions may be expanded in powers of \(\delta_s\). Generally, \(\alpha_c \ll 1\).

Neglect of the higher harmonics of \(\Gamma(\delta_s, s)\) (and the RF voltage terms) gives the smooth approximation. In addition, simple scaling arguments show that the first term in the integral (C4) is much smaller than the second.

To simplify the analysis of the RF voltage distribution we define a function

\[
\Sigma(\delta_s, s) = -\int_{0}^{s} \Gamma(\delta_s, \sigma) d\sigma,
\]

which gives the increase in path length per unit momentum deviation \(\delta_s\) in the sector of the ring between azimuths 0 and \(s\).
C2 Effective RF voltage

A similar Fourier analysis of the RF voltage term in (3.21), may be effected by substituting the identity

\[ \delta_C(s) = \frac{1}{2\pi R} \sum_{n=-\infty}^{\infty} e^{ins/R} \]  

and expanding the cosine into complex exponentials. The result is

\[
-\frac{1}{2\pi R} \sum_{k} \frac{e^{\hat{\nu}_k}}{p_{0Wt}} \frac{1}{2} \sum_{n=-\infty}^{\infty} \left\{ \exp \left[ \frac{in(s-s_k)}{R} + \frac{h(z+x_z)}{K(d_z)} - \frac{h}{K(d_z)} [s + \Sigma(d_z, s)] - \phi_k \right] \right. \\
\left. \exp \left[ \frac{in(s-s_k)}{R} - \frac{h(z+x_z)}{K(d_z)} + \frac{h}{K(d_z)} [s + \Sigma(d_z, s)] - \phi_k \right] \right\}. \tag{C8} 
\]

Bearing in mind the rôle of the RF harmonic number, we recognize that nearly all the terms in the expansion are rapidly oscillating functions of the independent variable \( s \) and will not produce significant average effects on the beam. The terms which do count are those in which the \( s \)-dependence can be made to cancel from the arguments of the exponentials. From (3.23), it follows that there are precisely two of these, namely the term with \( n = +h \) in the first group and that with \( n = -h \) in the second. Combining them, we can reassemble a slowly-varying cosine function

\[
-\frac{1}{2\pi R} \sum_{k} \frac{e^{\hat{\nu}_k}}{p_{0Wt}} \cos \left( \frac{h}{R} \left( \frac{s}{R} - \frac{s}{K(d_z)} - \frac{\Sigma(d_z, s)}{K(d_z)} \right) - \frac{hs_k}{R} + \frac{h(z+x_z)}{K(d_z)} + \phi_k \right) \tag{C9} 
\]

We separate the function \( \Sigma \) into a contribution from the \( n = 0 \) term in (C2) and a remainder:

\[
\Sigma(d_z, s) = \alpha_c(d_z)s + \tilde{\Sigma}(d_z, s) \tag{C10} 
\]

where, by virtue of (3.26), (C1) and (C2),

\[
\tilde{\Sigma}(d_z, s) = \sum_{n=1}^{\infty} \frac{1}{n\pi R} \int_{0}^{2\pi R} \Gamma(d_z, \sigma) \sin \left[ \frac{n(s-\sigma)}{R} \right] d\sigma. \tag{C11} 
\]

Then (treating the two terms in (C10) differently with respect to the original \( \delta \)-functions) the \( s \)-dependence cancels and the argument of the cosine becomes

\[
-\frac{hs_k}{R} + \frac{h(z+x_z)}{K(d_z)} - \frac{h\delta_s}{K(d_z)} \tilde{\Sigma}(d_z, s_k) + \phi_k. \tag{C12} 
\]

It is clear that for the most efficient use of the RF system\(^*\) one should choose the relative phases of the

\(^*\) We are ignoring all collective effects here.
APPENDIX D: Common synchrotron radiation integrals

In the following table we collect the definitions of some of the more important synchrotron radiation integrals together with indications of the contexts in which they occur.

<table>
<thead>
<tr>
<th>Definition</th>
<th>Uses</th>
</tr>
</thead>
<tbody>
<tr>
<td>$I_1 = \int_0^{2\pi} R G \eta , ds$</td>
<td>momentum compaction factor</td>
</tr>
<tr>
<td>$I_2 = \int_0^{2\pi} R G^2 , ds$</td>
<td>energy loss, energy spread, damping times, emittances, damping partition numbers</td>
</tr>
<tr>
<td>$I_3 = \int_0^{2\pi} R</td>
<td>G^2</td>
</tr>
<tr>
<td>$I_{3a} = \int_0^{2\pi} R G^3 , ds$</td>
<td>polarization level</td>
</tr>
<tr>
<td>$I_4 = \int_0^{2\pi} R (G^2 + 2K_1)G \eta , ds$</td>
<td>energy spread, emittances, damping partition numbers</td>
</tr>
<tr>
<td>$I_5 = \int_0^{2\pi} R</td>
<td>G^3</td>
</tr>
<tr>
<td>$I_{6a} = \int_0^{2\pi} K_1^2 \beta_z , ds$</td>
<td>energy loss in quadrupoles, nonlinear radiation damping</td>
</tr>
<tr>
<td>$I_8 = \int_0^{2\pi} K_1^2 \eta^2 , ds$</td>
<td>damping partition number variation</td>
</tr>
</tbody>
</table>

The function $\eta$ is defined\textsuperscript{2,11} by

$$
\eta(s) = \{\eta(s)^2 + (\beta(s)\epsilon(s) + \alpha(s)\eta(s))^2\}/\beta(s).
$$

Integrals 1–5 were defined in Ref. 32 which also describes useful algorithms for evaluating them. These are implemented in several computer programs (e.g., BEAMPARAM, COMFORT, PATRICIA …). Further information on the use of these integrals will be found in Refs. 2, 10, 11, 13, 17, 25, 26 and, especially, 32.