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Implementing Spectral Methods for Partial Differential Equations

Algorithms for Scientists and Engineers

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Contents

Preface	vii
Part I Approximating Functions, Derivatives and Integrals	
1 Spectral Approximation	3
1.1 Preamble: Series Solution of PDEs	3
1.2 The Fourier Basis Functions and Fourier Series	4
1.3 Series Truncation	6
1.4 Modal vs. Nodal Approximation	11
1.5 Discrete Orthogonality and Quadrature	11
1.6 Fourier Interpolation	14
1.6.1 Direct Computation of the Fourier Interpolation	17
1.6.2 Error of the Fourier Interpolation	19
1.7 The Derivative of the Fourier Interpolant	21
1.8 Polynomial Basis Functions	23
1.8.1 The Legendre Polynomials	24
1.8.2 The Chebyshev Polynomials	25
1.9 Polynomial Series	26
1.10 Polynomial Series Truncation	28
1.10.1 Derivatives of Truncated Series	30
1.11 Polynomial Quadrature	31
1.12 Orthogonal Polynomial Interpolation	35
2 Algorithms for Periodic Functions	39
2.1 How to Compute the Discrete Fourier Transform	39
2.1.1 Fourier Transforms of Complex Sequences	40
2.1.2 Fourier Transforms of Real Sequences	43
2.1.3 The Fourier Transform in Two Space Variables	48
2.2 The Real Fourier Transform	50
2.3 How to Evaluate the Fourier Interpolation Derivative by FFT	53
2.4 How to Compute Derivatives by Matrix Multiplication	54
3 Algorithms for Non-Periodic Functions	59
3.1 How to Compute the Legendre and Chebyshev Polynomials	59
3.2 How to Compute the Gauss Quadrature Nodes and Weights	62
3.2.1 Legendre Gauss Quadrature	62
3.2.2 Legendre Gauss-Lobatto Quadrature	64
3.2.3 Chebyshev Gauss Quadratures	67
3.3 How to Evaluate Chebyshev Interpolants via the FFT	67
3.3.1 The Fast Chebyshev Transform	68
3.4 How to Evaluate Polynomial Interpolants in Lagrange Form	73

3.5	How to Evaluate Polynomial Derivatives	78
3.5.1	Direct Evaluation of the Derivative	79
3.5.2	Evaluation of Derivatives by Matrix Multiplication	81
3.5.3	Even-Odd Decomposition	82
3.5.4	Evaluation by Transform Methods	84
3.5.5	Performance of Various Polynomial Derivative Algorithms	84

Part II Approximating Solutions of PDEs

4	Survey of Spectral Approximations	91
4.1	The Fourier Collocation Method	94
4.1.1	How to Implement the Fourier Collocation Method	96
4.1.2	Benchmark Solution	99
4.2	The Fourier Galerkin Method	101
4.2.1	How to Implement the Fourier Galerkin Method	103
4.2.2	Benchmark Solution	106
4.3	Nonlinear and Product Terms	107
4.3.1	The Galerkin Approximation	107
4.3.2	How to Compute the Convolution Sum	109
4.3.3	The Collocation Approximation	112
4.4	Polynomial Collocation Methods	115
4.4.1	Approximation of the Diffusion Equation	115
4.4.2	How to Implement the Methods	117
4.4.3	Benchmark Solution	119
4.4.4	Approximation of Scalar Advection	120
4.5	The Legendre Galerkin Method	123
4.5.1	How to Implement the Method	127
4.6	The Nodal Continuous Galerkin Method	129
4.6.1	How to Implement the Method	133
4.6.2	Benchmark Solution	134
4.7	The Nodal Discontinuous Galerkin Method	134
4.7.1	How to Implement the Method	138
4.7.2	Benchmark Solution	143
4.8	Summary and Some Broad Generalizations	144
5	Spectral Approximation on the Square	149
5.1	Approximation of Functions in Multiple Space Dimensions	149
5.2	Potential Problems on the Square	151
5.2.1	The Collocation Approximation	152
5.2.2	The Nodal Galerkin Approximation	173
5.3	Approximation of Time Dependent Advection-Diffusion	188
5.3.1	The Collocation Approximation	188
5.3.2	The Nodal Galerkin Approximation	189
5.3.3	Time Integration	191
5.3.4	How to Implement the Approximations	193
5.3.5	Benchmark Solution: Advection and Diffusion of a Spot in a Uniform Flow	200

5.4	Approximation of Wave Propagation Problems	202
5.4.1	The Nodal Discontinuous Galerkin Approximation	204
5.4.2	How to Implement the Nodal Discontinuous Galerkin Approximation	212
5.4.3	Benchmark Solution: Plane Wave Propagation	216
5.4.4	Benchmark Solution: Propagation of a Circular Sound Wave	217
6	Transformation Methods from Square to Non-Square Geometries . .	223
6.1	Mappings and Coordinate Transformations	223
6.1.1	Mapping a Straight Sided Quadrilateral	224
6.1.2	How to Approximate Curved Boundaries	225
6.1.3	How to Map the Reference Square to a Curved-Sided Quadrilateral	229
6.2	Transformation of Equations under Mappings	231
6.2.1	Two-Dimensional Forms	238
6.3	How to Approximate the Metric Terms	240
6.4	How to Compute the Metric Terms	242
7	Spectral Methods in Non-Square Geometries	247
7.1	Steady Potentials in a Quadrilateral Domain	247
7.1.1	The Collocation Approximation	247
7.1.2	The Nodal Galerkin Approximation	252
7.1.3	Solution of the Linear Systems	254
7.1.4	Benchmark Solution: Potential in Non-Square Domains . .	259
7.1.5	Benchmark Solution: Incompressible Flow over a Circular Obstacle	261
7.2	Steady Potentials in an Annulus	264
7.2.1	Benchmark Solution: Potential in an Annulus with a Source	271
7.3	Advection and Diffusion in Quadrilateral Domains	272
7.3.1	Transformation of the Advection-Diffusion Equation	272
7.3.2	The Collocation Approximation	273
7.3.3	The Nodal Galerkin Approximation	274
7.3.4	How to Implement the Approximations	275
7.3.5	Benchmark Solution: Advection and Diffusion in a Non-Square Geometry	276
7.3.6	Benchmark Solution: Advection and Diffusion of a Pollutant in a Curved Channel	277
7.4	Conservation Laws in Quadrilateral Domains	279
7.4.1	The Nodal Discontinuous Galerkin Approximation	280
7.4.2	How to Implement the Nodal Discontinuous Galerkin Approximation	282
7.4.3	Benchmark Solution: Acoustic Scattering off a Cylinder . .	285
8	Spectral Element Methods	293
8.1	Spectral Element Methods in One Space Dimension	296
8.1.1	The Continuous Galerkin Spectral Element Method	297

8.1.2	How to Implement the Continuous Galerkin Spectral Element Method	301
8.1.3	Benchmark Solution: Cooling of a Temperature Spot	305
8.1.4	The Discontinuous Galerkin Spectral Element Method . . .	308
8.1.5	How to Implement the Discontinuous Galerkin Spectral Element Method	310
8.1.6	Benchmark Solution: Wave Propagation and Reflection . .	315
8.2	The Two-Dimensional Mesh and Its Specification	317
8.2.1	How to Construct a Two-Dimensional Mesh	321
8.2.2	Benchmark Solution: A Spectral Element Mesh for a Disk .	326
8.3	The Spectral Element Method in Two Space Dimensions	326
8.3.1	How to Implement the Spectral Element Method	331
8.3.2	Benchmark Solution: Steady Temperatures in a Long Cylindrical Rod	340
8.4	The Discontinuous Galerkin Spectral Element Method	341
8.4.1	How to Implement the Discontinuous Galerkin Spectral Element Method	343
8.4.2	Benchmark Solution: Propagation of a Circular Wave in a Circular Domain	344
8.4.3	Benchmark Solution: Transmission and Reflection from a Material Interface	347
A	Pseudocode Conventions	355
B	Floating Point Arithmetic	359
C	Basic Linear Algebra Subroutines (BLAS)	361
D	Linear Solvers	363
D.1	Direct Solvers	363
D.1.1	Tri-Diagonal Solver	363
D.1.2	LU Factorization	364
D.2	Iterative Solvers	368
E	Data Structures	373
E.1	Linked Lists	373
E.1.1	Example: Elements that Share a Node	376
E.2	Hash Tables	377
E.2.1	Example: Avoiding Duplicate Edges in a Mesh	381
	References	385
	Index of Algorithms	387
	Subject Index	389