U(2) MONOPOLES AS FUNDAMENTAL CONSTITUENTS

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ABSTRACT

Extending Dirac's arguments, which yield an electron and a magnetic monopole as fundamental charges in the U(1) theory of electromagnetism, to a theory combining the electromagnetic and colour symmetries in the gauge group U(3), we deduce that the fundamental charges of the theory consist of two colour triplets with electric charges $2e/3$, $-e/3$, plus their magnetic counterparts with magnetic charges $1/e$, $-1/2e$, but all carrying a further multiplicative colour—magnetic quantum number (trichrocity). It is suggested that these electric charges behave as quarks under ordinary circumstances but, by virtue of their "trichrocity", are capable also of forming leptons as bound states via their colour—magnetic interactions at ultra—short distances.
In a gauge theory, the gauge boson occurs naturally as a connection in a fibre bundle whose properties are completely determined by the gauge group. In other words, it is a geometrical object. However, in current gauge theories of particles, the existence of matter fields, such as quarks in chromodynamics, has to be separately postulated. It would be much more satisfying if one could find a theory in which the matter fields also occurred as a consequence of the geometry.

For electromagnetism, such a theory has already been proposed by Dirac\textsuperscript{1} in his fundamental paper of 1931, which we may paraphrase for our purpose here as follows. Given the usual 'local' (in group space) form of electrodynamics, in terms of a Lagrangian say, the electric charge can in principle take any value. It is then an assumption to insist that we have electrons only with a certain fundamental charge \( e \). However, if we stipulate that the gauge group is compact, namely \( U(1) \), then the electric charge \( q \) is quantised,\textsuperscript{2} or in other words all charges in the theory can be generated from some fundamental charge \( q_0 \), which may then be designated as the charge of the electron \( q_0 = e \). On the other hand, the fact that the gauge group is \( U(1) \) leads also to the prediction that the magnetic charge \( q^* \) is quantised, namely:
\[ 2 q^* q = n, \]
or that all magnetic charges in the theory can be generated from some fundamental magnetic charge \( q_0^* = 1/2e \). For consistency, therefore, one should postulate the existence also of a magnetic monopole with this charge value. (In Dirac's own words: "Under these circumstances one would be surprised if Nature had made no use of it.") In other words, in a \( U(1) \) theory, one is led to consider a system consisting of a gauge boson and two fundamental charges, namely a photon (\( \gamma \)), an electron (\( e \)) and a magnetic monopole (\( g \)). The dynamics of this e-g-\( \gamma \) system has been investigated recently in a series of beautiful papers by Yang and co-workers which is now nearly complete.\textsuperscript{3-6} Presumably, if Dirac's theory is correct, Yang's equations will one day replace the current electrodynamics, yielding the latter as an approximation. We note that in this theory both electric and magnetic charges appear as Chern classes or topological invariants of fibre bundles, and are therefore geometrical.

Now the discoveries of the last decade suggest the existence of
another exact symmetry in nature, namely $\text{su}(3)$ for colour. Our aim here is to repeat Dirac's considerations when this colour symmetry is combined with electromagnetism. Indeed, we shall show that one is led to the conclusion that the fundamental charges of the theory are now 'quarks' with fractional (electric) charges, $2e/3$ and $-e/3$, plus their magnetic counterparts, but all carrying a further multiplicative colour magnetic quantum number which is expected to give them unfamiliar properties at ultra-short distances.

Following Dirac then, we stipulate that the gauge group is compact. In this case, however, there are three compact groups which share the same Lie algebra $\text{su}(3) + \text{u}(1)$, namely:

(i) $\text{SU}(3) \times \text{U}(1)$,

(ii) $(\text{SU}(3)/\mathbb{Z}_3) \times \text{U}(1)$,

(iii) $\text{U}(3) = (\text{SU}(3) \times \text{U}(1))/\mathbb{Z}_3$.

The alternative (ii) is uninteresting since it cannot have colour triplets and therefore contradicts the basic idea of 'quarks' in the colour theory. The group (i) on the other hand has representations with any colour, and any electric charge whose value is an integral multiple of a certain minimal charge $q_0$ which by present convention is $q_0 = e/3$. It permits therefore in particular the existence of colour singlets with $q = e/3$, which being unconfined by current interpretation of colour, can exist free in nature and contradict the near-absolute non-observation of fractional charges in experiment. In any case, although in principle still admissible, we find its loose structure rather unattractive. This leaves us then with only the gauge group $\text{U}(3)$, which we shall now examine in detail.

We first prove the following statement:

**Theorem 1**: $\text{U}(3)$ admits those and only those representations which can be built from the following fundamental charges or their conjugates:

(a) colour triplet, $q = 2e/3$,

(b) colour triplet, $q = -e/3$.

\[\footnote{We adopt the convention of denoting a Lie algebra by small letters and a group by capitals, thus e.g. $\text{su}(3)$ and $\text{SU}(3)$ respectively.}\]
Proof.- Since by definition the minimal charge $q_0$ is $e/3$, the period of $\alpha$ in $\exp i \alpha q$ belonging to $U(1)$ is $6\pi/e$. Hence,

$$dr = \exp i 2\pi r \frac{Q}{e}, \quad r = 0, 1, 2$$

represent three distinct elements of the electromagnetic $U(1)$ group. Similarly,

$$c_r = \exp (i 2\pi r/3) I_3, \quad r = 0, 1, 2$$

are three distinct elements of the colour group $SU(3)$. The couples $(c_r, d_r)$ therefore represent distinct elements of $SU(3) \times U(1)$ which are however identified in $U(3)$.

Let $\Psi$ be built from $n_1$ colour triplets with $q = 2e/3$ and $n_2$ colour triplets with $q = -e/3$. ($n_1$ and $n_2$ are not necessarily positive. A negative $n_1$ is meant to represent $n_1$ factors of anti-triplets $\bar{\mathbf{3}}$ with electric charges $-q_1$). Then when operated on $\Psi (c_r, d_r)$ for $r = 0, 1, 2$ all give the same result:

$$(c_r, d_r) \Psi = \exp \left[ i \frac{2\pi r}{3} (n_1 + n_2) + i 2\pi r \frac{2n_1}{3} - i 2\pi r \frac{n_2}{3} \right]$$

$$= \exp (i 2\pi r n_1) \Psi = \Psi,$$

i.e. $\Psi$ is a representation of $U(3)$.

Conversely, let $\tilde{\Psi}$ belong to $3 \times 3 \times \ldots \times 3$ (m factors) $\times 3 \times 3 \times \ldots \times 3$ ($\xi$ factors). Then,

$$(c_r, d_r) \tilde{\Psi} = \exp \left[ i 2\pi r \left( \frac{m-1}{3} + \frac{\xi}{e} \right) \right] \tilde{\Psi} = \tilde{\Psi}$$

implies that
\[
\frac{\ell}{3} + \frac{\ell}{d} = n, \quad \text{i.e.} \quad \frac{\ell}{2} = n - \frac{\ell - \ell}{3} = \frac{-\ell - n}{3} + \frac{2n}{3}.
\]

Hence, \( \tilde{u} \) can be built from \( n_1 = n \) triplets with \( q = 2e/3 \) and \( n_2 = m - \ell - n \) triplets with \( q = -e/3 \). \( \Box \)

Notice that the occurrence of the two 'flavours' (a) with \( q = 2e/3 \) and (b) with \( q = -e/3 \) appears here as a consequence and not as a separate assumption.

Now \( U(3) \) being infinitely-connected, a theory with \( U(3) \) as gauge group admits monopoles, whose charges have already been analysed in ref. 7. It was found that the magnetic charge is quantised in units of \( 1/2e \), namely:

\[
q^m = \frac{n}{2e},
\]

but that the monopoles carry in addition a multiplicative colour-magnetic quantum number:

\[
\omega_c = \exp(-i \frac{2\pi}{3} n).
\]

Equivalently, one may regard the logarithm of \( \omega_c \) as being, like triality, additive modulo \( 3 \). We shall refer henceforth to \( \omega_c \) (or its logarithm) as 'trichrocity' \( \xi \) to emphasize its significance as a topological invariant in the colour gauge theory which is not shared by the term 'triality' under normal usage.

Next, the dual of a \( U(1) \) gauge field \( f_{\mu \nu} \) defined as:

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\( \xi \): Oxford Dictionary: trichroic a. showing three colours (f. Gk TRI (khroos f. khrôs colour) + ic).
\[ f_{\mu \nu}^{\ast} = \frac{1}{2} \varepsilon_{\mu \nu \sigma \rho} f^{\sigma} \]  

(3)

is also a gauge field, meaning that there exists a potential \( A'_{\mu} \) for which:

\[ f_{\mu \nu}^{\ast} = \partial_{\mu} A'_{\nu} - \partial_{\nu} A'_{\mu}. \]  

(4)

Now the Dirac quantisation condition (1) means that \( A_{\mu} \) and \( A'_{\mu} \) have the same electromagnetic gauge group \( U(1) \). We may therefore enquire as to what gauge group is obtained by combining the \( U(1) \) symmetry corresponding to \( A''_{\mu} \) with the colour symmetry \( SU(3) \). Again, there are three possibilities:

(i) \( SU(3) \times U(1) \),

(ii) \( (SU(3)/\mathbb{Z}_3) \times U(1) \),

(iii) \( U(3) = (SU(3) \times U(1))/\mathbb{Z}_3 \).

All three cases admit monopoles which however are now electric monopoles. We need therefore first to check whether the electric charges so obtained are consistent with the representations of the \( U(3) \) group as detailed above in Theorem 1.

The charges of monopoles for all cases (i) - (iii) can be deduced from the analysis of ref. 7) and 9) by replacing the minimal electric charge \( q_0 = e/3 \) there by the minimal magnetic charge \( q_0^* = 1/2e \) deduced from (1) above. The result is as follows:

(i) \( q = ne, \omega_C = 1 \);  

(5)

(ii) \( q = ne, \omega_C = \exp i2\pi r/3, r = 0, 1, 2 \);  

(6)

(iii) \( q = ne/3, \omega_C = \exp (-i2\pi r/3) \);  

(7)

of which (i) and (ii) are inconsistent with the statement that the minimal electric charge is in fact \( e/3 \). By insisting therefore that all permissible charges be realized in our theory, we are left with only the choice (iii).
From (7) it follows that the fundamental triplets which have electric charges \(2e/3\) and \(-e/3\) must both carry unit trichrois, i.e. \(\omega_C = \exp\,i2\pi/3\). From (iii)', by replacing \(q_0 = e/3\) with \(q_0' = 1/2e\) in Theorem 1, it follows that the fundamental magnetic charges are:

(a) colour triplet, \(q^* = 1/e\),
(b) colour triplet, \(q^* = -1/2e\),

and that both carry, according to (2), unit trichrois: \(\omega_C = \exp\,i2\pi/3\).

Following Dirac's line of reasoning then, we are led to postulate the existence of the following matter fields:

\[
\begin{align*}
\tau^e_M & : \text{colour } 3, \omega_C = \exp\,i2\pi/3, q = 2e/3; \\
\tau^e_F & : \text{colour } 3, \omega_C = \exp\,i2\pi/3, q = -e/3; \\
\tau^m_M & : \text{colour } 3, \omega_C = \exp\,i2\pi/3, q^* = 1/e; \\
\tau^m_F & : \text{colour } 3, \omega_C = \exp\,i2\pi/3, q^* = -1/2e;
\end{align*}
\]

from which all charges permissible in the theory can be generated.

It is indeed interesting that the arguments of Dirac as we have applied them here are so remarkably stringent. Starting from the very basic premises simply of combining the electromagnetic and colour symmetries, we had no other freedom besides that of choosing one gauge group out of three in which we were guided by experiment. \(\%\) Once we had decided on \(U(3)\) as the gauge group, we were then driven inexorably to the conclusion that our fundamental matter fields must have the charges listed in (8). Moreover, as in the Dirac theory of electrons and monopoles, the matter fields here have all a topological significance. In other words, they enter not as a separate postulate as in usual gauge theories, but as a consequence of the geometry.

\(\%\) Similar considerations to those above when applied to the group \(SU(3) \times U(1)\) yields a theory with 'factorisable' quantum numbers, similar to the preon model of Pati and Salam,\(^{10}\) which we find less attractive than our \(U(3)\) scheme here. The group \((SU(3)/Z_3) \times U(1)\) was already said to be unacceptable.
Does the scheme make physical sense? Any answer to this question is necessarily speculative at present especially since we do not yet know anything about the dynamics governing the unfamiliar multiplicative charges $\omega_C$. In principle, since trichroisity has geometrical significance in a gauge theory, one may be able to deduce the dynamics by following a program similar to what Yang and co-workers did for the magnetic monopole, but we have not done so. We shall therefore limit ourselves at present only to some observations.

(I) Electro-taons as hadron constituents.

Suppose first that taons carry spin $1/2$ so that in addition to generating all permissible charges in the theory, they can generate also all permissible spins. We notice then that the electro-taons $\tau^e$ in (8) have the same quantum numbers as Gell-Mann's quarks apart from their nonzero trichroisity, i.e. $\omega_C \neq 1$. Now the presence of $\omega_C$ means presumably that the equations of motion of ordinary QCD will have to be modified in much the same way that the existence of magnetic monopoles modifies ordinary electrodynamics to Yang's equations.\(^4\),\(^6\) We argue, however, that at the distances explored so far by experiment, this colour-magnetic interaction is unlikely to make much difference. The strength of the interaction is characterised by a colour-magnetic charge $g^s = 1/2g$ where $g$ is the usual (colour-electric) coupling constant of QCD. According to current phenomenology, $\alpha_s = g^2/4\pi \sim 1$ at distances of hadronic dimensions. In other words, $g^2 \alpha_s \sim 1/6\pi$ which is clearly too small to be detected by current phenomenological 'tests' of QCD. Therefore we expect electro-taons to behave under normal circumstances essentially as the quarks of present QCD theory, and to be confined, presumably by colour-electric forces, to form hadrons. Further, these hadrons, being colour singlets, must have trichroisity zero, and so can have themselves no colour-magnetic interactions.

(II) Leptons as colour-magnetic bound states.

The colour-electric coupling $\alpha_s$ is believed to decrease with decreasing distance,\(\ddagger\) so that eventually colour-magnetic interactions

\(\ddagger\) This assertion in the usual form of 'asymptotic freedom' is based on perturbation theory. In the presence of colour-magnetic charges whose strength $g^s = 1/2g$ increases as $g$ decreases, it is not obvious that any perturbation theory will apply. However, for lack of anything better, we shall assume that the conclusions obtained there is still valid in some range where $g^2/4\pi$ is reasonably small, but $g^s$ is not yet of order 1.
between taons will become strong. Presumably then at these ultra-short distances they will be capable of forming tightly bound states with zero trichrocity. In particular, there can be bound states of three s-wave electro-taons with total colour 1 and total spin 1/2. Now, the group theory involved here being the same as that in combining three (u, d) quarks to form s-wave baryons, it follows that there are just two such states with electric charges \( q = 1 \) and 0, corresponding exactly to the leptons \( e^+ \) and \( \nu \). Assume next that this colour-magnetic binding will become effective only at distances where the coupling strength \( g^* \) is of order 1. Then, from the formula:

\[
\alpha_s = \frac{g^4}{4\pi} \approx \frac{12\pi}{3\cdot2N_f} \left[ \ln \left( \frac{q^2}{\Lambda^2} \right) \right]^{-1}
\]

(9)

of ordinary QCD, where \( N_f \) is the number of flavours and \( \Lambda \) is a cut-off parameter \( \sim 0.5 \text{ GeV} \) determined from experiment, one obtains that these forces are effective only when \( |q| \sim 10^{16} \text{ GeV} \), corresponding to distances of the order \( 10^{-17} \text{ fermi or } 10^{-20} \text{ cm} \). This is well within the present experimental limit of \( 10^{-16} \text{ cm} \) for the absence of structure in leptons. Further, although the assumption that leptons and hadrons share the same constituents implies that only \( B - L \) (baryon number minus lepton number) is conserved and that protons will decay, there is so little probability of finding all three taons of a proton inside a region of dimension \( 10^{-17} \text{ fermi} \), that the proton will remain very stable. The current limit of \( 10^{-29} \text{ cm} \) on the range of proton instability obtained from the proton lifetime through simple dimensional arguments is very similar to our estimated value.

(III) Other lepton-like states.

As in all composite models of leptons, we have to worry about the proliferation of leptonic states. Consider first still just s-wave colour singlets. We may have spin \( 3/2 \) objects with \( q = 2, 1, 0, -1 \), corresponding to \( \Delta \) in the quark hadron model.\(^{13}\) If they exist, they are presumably heavy leptons which will one day be discovered. Besides, there

\(^{13}\) J.C. Pati\(^{13}\) has given an argument why composite leptons with spin \( 3/2 \) need not exist.
can be s-wave coloured bound states, since colour singlet implies zero
trichrocity, but the converse is not true: \( 3 \times 3 \times 3 = 1 + 8 + 8 + 10 \).
These are not leptons, being coloured, but must remain confined and appear
as hadron constituents. Their quantum numbers are the same as qqq
'chromions' in multiquark spectroscopy which have been listed in e.g.
ref. 14). Indeed, with the colour-magnetic interaction here playing the
role of nuclear forces in atoms, these new objects would behave in hadrons
in much the same way as nuclei behave in atoms, and make hadron spectro-
copy into an even closer analogue to ordinary chemistry than was previously
realized.\(^{15}\) Fortunately, it seems that this is about as far as the pro-
liferation of leptons will go. The interaction, being by assumption
effective over only such short distances, is unlikely to sustain any
stable radial excitations or \( \lambda \)-excitations. One can of course also
consider zero trichrocity combinations of electro-taons other than \( \tau \tau \tau \).
There is however only one other combination of interest, namely \( \tau^e \tau^e \)
in s-wave, giving \( J^P = 1^- \), \( 0^- \) and \( q = 1, 0, 0, -1 \) (W-bosons and Higgses?).
Any state containing more taons will simply dissociate, e.g. \( (\tau \tau \tau \tau \tau \tau ) \)
\( \rightarrow (\tau \tau \tau \tau ) + (\tau \tau \tau ) \), since the interaction is automatically saturated by
trichrocity zero.

Like all other current sub-constituent models of quarks and leptons,
our present scheme poses many more questions than it can yet answer. We
shall list here only the most outstanding:

(A) All our previous remarks for electro-taons apply to magneto-taons
also, and one should be able to combine magneto-taons with one another
and with their electric counterparts to form leptons, hadrons and 'hadronic
nuclei' which carry now magnetic, as well as electric, charges. As in the
Dirac theory therefore, one would need to explain why these magnetic mono-
poles are not yet observed.

(B) There is yet no explanation for the generation index.

(C) It is left open whether weak interactions is to be introduced as a
further fundamental gauge theory, or as an effective gauge theory for
composites to be deduced from electro-chromodynamics.

Unlike other sub-constituent models, however, our choice of consti-
tuents is based not on phenomenology but on the geometrical structure of
the gauge theory, and their charges, being topological invariants, have
therefore interactions, which though unknown at present, may in principle one day be deduced.

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