SCATTERING OF PARTICLES BY PHASE DISPLACEMENT
ACCELERATION IN STORAGE RINGS

by

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1. INTRODUCTION

When a frequency modulated radio frequency (r.f.) voltage is applied to an accelerating gap of a storage ring, there are regions of phase stable oscillations (buckets) in which charged particles may be accelerated in synchronism with the radio frequency. Particles initially not in synchronism with the r.f. voltage will suffer from an energy displacement associated with a scattering, when they have been traversed by the buckets. Considering a beam of particles as an incompressible fluid, on the average, the displacement equals the mean height $\delta E_b$ in energy of the moving bucket:

$$<\Delta E> = \delta E_b.$$  \hspace{1cm} (1)

This is a consequence of Liouville's theorem, which allows the acceleration or deceleration of a stored particle beam.

The total energy displacement of an individual particle can be calculated from equation (85) of Symon and Sessler\textsuperscript{1)}, which yields in units of the mean energy height $\delta E_{st}$ of a stationary bucket

$$S = \frac{\Delta E}{\delta E_{st}} = \frac{\pi}{4} \int_{\phi_0}^{\infty} \frac{\sin \phi \cdot d\phi}{y(\phi, \phi_0, \Gamma)}.$$  \hspace{1cm} (2)

The variable $y$ is proportional to the energy difference between a particle sitting at phase $\phi$ and the moving buckets; it is a function of the phase $\phi_0$ at which the particle passes by the buckets and of

$$\Gamma = \sin \phi_s,$$

where $\phi_s$ is the phase of a particle in synchronism with the radio frequency.
The scattering of particles after r.f. modulation obviously corre-
sponds to the statistical distribution of $S$ defined by equation (2).
Since the distribution of the phase $\phi_0$ is determined by the parameter $\Gamma$, as is shown below, the scattering depends exclusively on that parameter. Now the question arises whether the scattering in phase displacement ac-
celeration can be minimized by using an appropriate $\Gamma$, as to some extent this quantity can be chosen freely. Another reason why it is interesting to know the dependence of the dispersion on $\Gamma$ is perhaps to have the most easily available means to make a known amount of longitudinal stirring in a stack of particles by phase displacement acceleration$^1$.

The integral in formula (2) has already been evaluated by Jones et al.$^3$, but unfortunately, there are no results given for the dispersion of particles as function of $\Gamma$. In this respect the only available results are in reference 4, where the scattering has been determined from sim-
mulating the phase displacement process on a computer for $\Gamma$'s smaller than one. The result given there is that the increase of r.m.s. energy spread is proportional to $\Gamma$. However, unexpected results in the range $0.8 < \Gamma < 1.0$ showed that the simulation method seems to be less reliable for $\Gamma$'s of that interval. Therefore, the analytic method of Jones et al. to calculate the dispersion of particles for a large set of $\Gamma$'s was again studied. In addition, some computations using the simulation method were carried out for parameters of $\Gamma$ larger than one.

2. REVIEW OF THE ANALYTIC METHOD

In order to define quantities which will be used later, an extract of the phase displacement theory is given. Details may be found in refer-
ences 1 and 3.

The phase displacement process can be described with the aid of va-
riables of the synchrotron phase space. The canonical coordinates of this
space are the phase \( \phi \) of a particle relative to the r.f. voltage and the variable

\[
W = \int \frac{dE}{f(E)},
\]

where \( f(E) \) is the frequency of revolution at energy \( E \). For convenience, usually instead of \( W \) the dimensionless variable \( y \) is taken which may be defined as

\[
y = \frac{8}{\pi} \frac{f_s}{\delta E_{st}} (W - W_s),
\]

where the subscript refers to the synchronous particle. Then the trajectories of particles above transition energy are given by \(^4\)

\[
- \frac{1}{2} y^2 + \cos \phi + \Gamma \phi = \text{constant.} \tag{3}
\]

Figure 1 shows a plot of the curves of constant hamiltonian in \((\phi, y)\) phase space. The separatrix, which is the phase curve containing the unstable fixed point at \( \phi_1 = \pi - \arcsin \Gamma \), is indicated by a dashed line.

Considering an ensemble of particles uniformly distributed in energy between two subsequent separatrices, it is easily verified with formula (3) that the width \( \Delta y \) corresponding to the spread of these test particles for a large \( y \) is

\[
\Delta y = \frac{2\pi \Gamma}{y}. \tag{4}
\]

The phase plane point \((\phi_1, y_1)\) representing a particle in the interval \( \Delta y \) (see figure 1) will slide along the phase curve and pass the bucket at \((\phi_0, 0)\). Assuming the \( y_1 \) to be uniformly distributed, from formulae (3)...

\(^4\) below transition: the (-) sign in equation (3) has to be replaced by a (+) sign which will have no influence on the results.
and (4) the normalized probability distribution $P(\phi_o)$ for the phase $\phi_o$ can be derived:

$$P(\phi_o) \, d\phi_o = \frac{\Gamma - \sin \phi_o}{2\pi \Gamma} \, d\phi_o .$$

Then the definition of the variable

$$q = \int_{-\pi}^{\phi_o} P(\phi_o) \, d\phi_o = \frac{\phi_o + \pi}{2\pi} + \frac{\cos \phi_o + 1}{2\pi \Gamma}$$

(5)

yields the proper distribution of the phase $\phi_o$ in the interval $[-\pi, +\pi]$ in respect to a uniformly distributed $q$ running from 0 to 1.

Figure 2 shows $\phi_o$ as function of $q$. The jump for $\Gamma < 1$ corresponds to the interval between $\phi_2$ and $\phi_1$ of figure 1. This range is inaccessible for particles initially outside the buckets and therefore has to be excluded from the integral (5). For particles running through $(\phi_o, 0)$ from equation (3) follows

$$y = \sqrt{2} \cdot \sqrt{\cos \phi - \cos \phi_o + \Gamma(\phi - \phi_o)}$$

and hence from equation (2)

$$S = \frac{\pi}{4\sqrt{2}} \int_{\phi_o}^{\infty} \frac{\sin \phi \, d\phi}{\sqrt{\cos \phi - \cos \phi_o + \Gamma(\phi - \phi_o)}} = \frac{\pi}{4\sqrt{2}} I(\phi_o, \Gamma),$$

(6)

or

$$S = \frac{\pi}{4\sqrt{2}} \int_{0}^{\infty} \frac{\sin(\phi_o + x) \, dx}{\left[\cos(\phi_o + x) - \cos \phi_o + \Gamma x\right]^{\frac{1}{2}}} = \frac{\pi}{4\sqrt{2}} I .$$

(7)

The integral $I(\phi_o, \Gamma)$ must be evaluated numerically. For a large $\Gamma$ an approximation can be derived.

...
3. APPROXIMATION FOR LARGE \( \Gamma \)'s

In the situation where \( \Gamma \) is large:

\[
\Gamma \gg \frac{\cos(\phi_0 + x) - \cos \phi_0}{x}, \quad x > 0;
\]

the integral (7) can be written as

\[
I \approx \frac{\gamma}{\Gamma} = \frac{1}{2 \Gamma} \int_0^\infty \frac{\sin \phi_0 \cos x + \cos \phi_0 \sin x}{\sqrt{x}} \, dx.
\]

Noting that

\[
\int_0^\infty \frac{\cos x}{\sqrt{x}} \, dx = \int_0^\infty \frac{\sin x}{\sqrt{x}} \, dx = \sqrt{\frac{\pi}{2}};
\]

the integral \( I \) becomes approximately

\[
I = \left( \frac{\pi}{2} \right)^{\frac{1}{2}} \left( \frac{1}{\Gamma} \right)^{\frac{1}{2}} (\sin \phi_0 + \cos \phi_0).
\]

Since equal intervals in \( q \), and therefore in \( \phi_0 \) for a large \( \Gamma \), are equally likely (see formula (5) with the corresponding figure 2), i.e.

\[
dq = \frac{1}{2\pi} \, d\phi_0,
\]

the distribution in energy of an initially monoenergetic beam (\( \delta y \ll 1 \)) after phase displacement will be as shown in figure 3. From equation (8) follows

\[
\gamma_S = \left( \frac{\pi}{4} \right)^{\frac{3}{2}} \Gamma^{-\frac{1}{2}} (\sin \phi_0 + \cos \phi_0)
\]

and therefore

\[
|\gamma_S| \leq \sqrt{\frac{\pi^3}{32 \Gamma}}.
\]
As expected, in the approximation, where $\Gamma$ certainly must be larger than one, it follows

$$<S> = \int_{-\pi}^{\pi} d\phi_0 \cdot \frac{1}{2\pi} \cdot S(\phi_0) = 0.$$  

Thus for the increase of r.m.s. energy spread

$$<S>_{\text{rms}} = \frac{\delta E_{\text{rms}}}{\delta E_{\text{st}}} = \left[ \int_{-\pi}^{\pi} d\phi_0 \cdot \frac{1}{2\pi} \cdot S^2(\phi_0) \right]^{\frac{1}{2}}$$

$$= \left( \frac{\pi}{4} \right)^{\frac{3}{2}} \Gamma^{-\frac{1}{2}}$$

is obtained. This result has already been found by H.G. Hereward\(^2\).

4. SOME REMARKS ON THE METHODS OF INTEGRATION

The numerical solution of the integral $I(\phi_0, \Gamma)$ of formula (6) requires an integration method which disregards the indefinite integrand at the lower boundary. For that reason, the CERN library routine D 103 was used which complies with this condition. As the integrand is an oscillating function which tends steadily to zero as the upper limit goes to infinity, the integration can be stopped early. Let

$$k_{\ell}^m = \left| \int_{2\pi}^{m\pi} \frac{\sin\phi d\phi}{\sqrt{\cos\phi - \cos\phi_0 + \Gamma(\phi_0 - \phi_0)}} \right|,$$

and let\(^4\)

$$I_n = K_0^1 - K_1^2 + K_2^3 - \ldots = (-1)^{n-1} K_n^1 + (-1)^n K_{n+\frac{1}{2}}.$$  

(9)

\(^{4}\) \(\ell = 0\) refers to the phase $\phi_0$
Thus computations for $\Gamma > 0$ show

$$I_{2n+2} - I_{2n} > 0, \quad I_{2n+1} - I_{2n-1} < 0; \quad n = 1, 2, 3, \ldots$$

Hence $I_2, I_4, I_6, \ldots, I_{2n}, \ldots$ is an increasing sequence and $I_1, I_3, I_5, \ldots, I_{2n-1}, \ldots$ is a decreasing sequence. As

$$I_{2n} < I_{2n+1} \quad \text{and} \quad \lim_{n \to \infty} (I_{2n} - I_{2n+1}) = 0,$$

both sequences must have the same limit which is the integral $I$. Therefore, the sum of the series (9) lies between $I_n$ and $I_{n+1}$. The error committed by taking the sum of the first $n$ terms instead of the sum of the infinite series is not greater than the modulus of the $(n+1)$th term $\delta$.

The summation of the integrals $I_i$ calculated at a $\phi_0$ corresponding to

$$q_i = \frac{i}{N}, \quad i = 0, 1, 2, \ldots, N-1$$

has been carried out by "Simpson's Rule" integration. It yields the average displacement $<S>$ and the increase of r.m.s. energy spread as

$$<S>_{\text{rms}} = \left[ \int_0^1 (S - <S>)^2 \, dq \right]^{\frac{1}{2}}.$$

Unfortunately the only CERN library routine available for this integration method, D 100, does not have an error estimation. However, from formula (1) it is known that

$$\frac{<\Delta E>}{\delta E_{st}} = \frac{\delta E_h}{\delta E_{st}} = \alpha(\Gamma),$$

where the bucket parameter $\alpha(\Gamma)$ is available in tabulated form $\delta$). Therefore the quantity $^4$)

$$\epsilon = \alpha(\Gamma) - <S> \quad \ldots$$

$^4) \ \alpha(\Gamma): = 0 \quad \text{for} \ \Gamma \geq 1$
is a measure of the total accuracy of the computations, and since the error of the individual integrals $I$ is known, the accuracy of Simpson's integration method may be estimated.

5. **RESULTS**

a) Analytical method

For each value of $\Gamma$, 100 points $q_i$ were taken, from which an attempt to obtain the integrals $S$ with an accuracy of better than $10^{-3}$ was made. Figure 4 shows the displacement $S$ as a function of $q$ for several values of $\Gamma$. The results, such as mean displacement $<S>$, maximum displacement $S_{\text{max}}$ and the r.m.s. spread $<S>_{\text{rms}}$, are given in table 1. From the last column it can be seen that the total error of computation, expressed by the quantity $\epsilon$, is smaller than $10^{-2}$. The r.m.s. energy spread $<S>_{\text{rms}}$ (circlets) and the approximation $<S>_{\text{rms}}$ (smooth line) are plotted into figure 5 as function of $\Gamma$.

b) Additional results from the simulation method

To check our computer programmes for r.f. studies for parameters of $\Gamma > 1$ rather than to check the results above, the phase displacement process was simulated with the aid of the computer. The input parameters except $\Gamma$ and the statistics have been the same as in the previous computations which are described in more detail in reference 4. The results are given in table 2. Apart from little differences the results agree well with those of table 1. The comparison of the corresponding $\epsilon$ proves the analytical method to be more accurate.

6. **DISCUSSION**

The results for the mean and the maximum displacement agree well with the results of Jones et al.. This, of course, was to be expected, as only the method of numerical evaluation of the integrals might have been different.
The initially uniform distribution of particles between two separatrices provides us with a proper statistical sample of a practically monoenergetic beam. As can be derived from figure 4, for parameters of \( \Gamma \) smaller than one, the particle distribution after r.f. modulation starts abruptly at energies corresponding to the maximum displacement and decreases monotonously into the direction of the bucket traverse \(^*\). For large \( \Gamma \)'s the particles are distributed as shown in figure 3. Though for these distributions unlike a Gaussian distribution the r.m.s. energy spread is a rather incomplete measure of dispersion, it describes best the scattering of the particles by means of a single number. This quantity gives the increase of dispersion of particles due to the modulation of the radio frequency.

For parameters of \( \Gamma \) smaller than 0.8 the results confirm those of the previous computations \(^4\) based on the simulation method. Also, some scattering of results for the range around \( \Gamma = 1.0 \) is found which might come from the skew character of the displacement distribution. However, it is not as pronounced as in the results obtained from the simulation method, where either a too small number of particles being processed or a premature switch off of the r.f. modulation \(^7\) might have led to the unexpected results for \( 0.8 < \Gamma < 1.0 \).

Figure 5 shows that the approximation \( <S>_{\text{rms}} \) describes the r.m.s. energy spread very well for \( \Gamma \)'s larger than 1.5. Hence

\[
<S>_{\text{rms}} \approx \begin{cases} 
\Gamma & \text{for } \Gamma < 1 \\
(m/4)^{3/2} \cdot \Gamma^{-1/2} & \text{for } \Gamma > 1.5
\end{cases}
\]

where the approximation for small \( \Gamma \)'s is simply derived from figure 5 (broken line).

The consequence of the results for \( \Gamma < 1 \) is that fortunately a de-

\(^*\) see figure 5 of the Jones et al. paper \(^3\).
crease of $\Gamma$ provides both a decrease of r.m.s. energy spread and an increase of mean energy displacement. Therefore it is suggested that small values of $\Gamma$ be used in phase displacement acceleration.

Finally, for large $\Gamma$'s the r.m.s. energy spread is proportional to $\Gamma^{-\frac{1}{2}}$. Thus by a suitable choice of $\Gamma$ a predetermined longitudinal stirring of a stack of particles is possible by modulation of the radio frequency. After a number $n$ of modulations the final r.m.s. width $\langle \Delta E_{\text{rms}} \rangle_f$ of a particle beam of initial r.m.s. width $\langle \Delta E_{\text{rms}} \rangle_{\text{in}}$ becomes

$$\langle \Delta E_{\text{rms}} \rangle_f = \left[ \left( \langle \Delta E_{\text{rms}} \rangle_{\text{in}}^2 + \frac{n}{\Gamma} \cdot \left( \frac{n}{4} \right)^3 \langle \Delta E_{\text{st}} \rangle^2 \right) \right]^{\frac{1}{2}}.$$  

For Gaussian-like distributions it may be written approximately

$$\delta E_f = \left[ \left( \delta E_{\text{in}} \right)^2 + \frac{n}{\Gamma} \cdot \left( \frac{n}{8} \right) \ln 2 \left( \delta E_{\text{st}} \right)^2 \right]^{\frac{1}{2}},$$

where $\delta E_{\text{in}}$ and $\delta E_f$ are the full widths at half maximum of the initial and the final particle distribution respectively.

7. ACKNOWLEDGEMENT

The author is very much indebted to Mr. H.G. Hereward for helpful remarks and for the reading of a preliminary version of the manuscript.

8. REFERENCES


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4) E.W. MESSERSCHMID, "Dispersion of stacked protons in synchrotron phase space by a modulated radio frequency voltage", CERN - ISR - RF/72 - 28 (1972).


### Table 1

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Figure 1 A moving bucket for $\Gamma = 0.5$
Figure 2  Distribution of $\phi_0$ for different $\Gamma$'s

Figure 3  Energy distribution after one phase modulation for large $\Gamma$'s
Figure 4  Phase displacement $S$ as function of $q$ for several parameters of $T$. A single dash indicates the mean displacement $<S>$. 